Exact WKB Solutions:
Their Borel Summability and Relationship to Abelianisation of $\hbar$-Connections

Nikita Nikolaev

School of Mathematics
University of Birmingham

30 January 2023

Workshop on Quantization and Resurgence

SwissMAP Research Station
Les Diablerets, Switzerland

§0. Setting

- Start with singularly perturbed linear ODE in a domain $\mathsf{X} \subset \mathbb{C}_x$:

$$(\hbar\partial_x)^n\psi + p_1(\hbar\partial_x)^{n-1}\psi + \ldots = \left(\sum_{k=0}^{n} p_{n-k}\hbar^k\partial_x^k\right)\psi(x,\hbar) = 0 \qquad (\bigstar)$$

where $p_k(x,\hbar) \in \mathcal{O}_{\mathsf{X}}[\hbar]$ or $\mathcal{O}_{\mathsf{X}}(\mathsf{D})[\hbar]$

§0. Setting

- Start with singularly perturbed linear ODE in a domain $\mathsf{X} \subset \mathbb{C}_x$:

$$(\hbar\partial_x)^n\psi + p_1(\hbar\partial_x)^{n-1}\psi + \ldots = \left(\sum_{k=0}^{n} p_{n-k}\hbar^k\partial_x^k\right)\psi(x,\hbar) = 0 \qquad (\bigstar)$$

where $p_k(x,\hbar) \in \mathcal{O}_{\mathsf{X}}[\hbar]$ or $\mathcal{O}_{\mathsf{X}}(\mathsf{D})[\hbar]$

- **Examples**: **(1)** $\big(\hbar^2\partial_x^2 + q(x,\hbar)\big)\psi = 0$ Schrödinger equation
(2) $\big(\hbar^3\partial_x^3 + 3\hbar\partial_x + 2ix\big)\psi = 0$ Berk-Nevins-Roberts equation

§0. Setting

- Start with singularly perturbed linear ODE in a domain $\mathsf{X} \subset \mathbb{C}_x$:

$$(\hbar\partial_x)^n\psi + p_1(\hbar\partial_x)^{n-1}\psi + \ldots = \left(\sum_{k=0}^{n} p_{n-k}\hbar^k\partial_x^k\right)\psi(x,\hbar) = 0 \qquad (\bigstar)$$

  where $p_k(x,\hbar) \in \mathcal{O}_{\mathsf{X}}[\hbar]$ or $\mathcal{O}_{\mathsf{X}}(\mathsf{D})[\hbar]$

- **Examples:** **(1)** $\big(\hbar^2\partial_x^2 + q(x,\hbar)\big)\psi = 0$ Schrödinger equation
  (2) $\big(\hbar^3\partial_x^3 + 3\hbar\partial_x + 2ix\big)\psi = 0$ Berk-Nevins-Roberts equation

- More generally: $\hbar$-differential operator on a line bundle $\mathcal{L}$ over a curve (X,D):

$$P : \mathcal{L} \to \mathcal{L} \otimes \mathsf{Sym}^n\,\Omega^1_{\mathsf{X},\mathsf{D}}[\hbar] \qquad \text{such that} \qquad P\big|_{\hbar=0}(fe) = fP\big|_{\hbar=0}(e)$$

§0. Setting

- Start with singularly perturbed linear ODE in a domain $\mathsf{X} \subset \mathbb{C}_x$:

$$(\hbar\partial_x)^n\psi + p_1(\hbar\partial_x)^{n-1}\psi + \ldots = \left(\sum_{k=0}^{n} p_{n-k}\hbar^k\partial_x^k\right)\psi(x,\hbar) = 0 \qquad (\bigstar)$$

  where $p_k(x,\hbar) \in \mathcal{O}_{\mathsf{X}}[\hbar]$ or $\mathcal{O}_{\mathsf{X}}(\mathsf{D})[\hbar]$

- **Examples:** **(1)** $\big(\hbar^2\partial_x^2 + q(x,\hbar)\big)\psi = 0$ Schrödinger equation
  (2) $\big(\hbar^3\partial_x^3 + 3\hbar\partial_x + 2ix\big)\psi = 0$ Berk-Nevins-Roberts equation

- More generally: $\hbar$-differential operator on a line bundle $\mathcal{L}$ over a curve (X,D):

$$P : \mathcal{L} \to \mathcal{L} \otimes \mathsf{Sym}^n\,\Omega^1_{\mathsf{X},\mathsf{D}}[\hbar] \qquad \text{such that} \qquad P\big|_{\hbar=0}(fe) = fP\big|_{\hbar=0}(e)$$

- Even more generally: $\hbar$-connection on a vector bundle $\mathcal{E}$ over a curve (X,D):

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{\mathsf{X},\mathsf{D}}[\hbar] \qquad \text{such that} \qquad \nabla(fe) = f\nabla e + \hbar\,\mathrm{d}f \otimes e$$

§0. Setting

- Start with singularly perturbed linear ODE in a domain $\mathsf{X} \subset \mathbb{C}_x$:

$$(\hbar\partial_x)^n\psi + p_1(\hbar\partial_x)^{n-1}\psi + \ldots = \left(\sum_{k=0}^{n} p_{n-k}\hbar^k\partial_x^k\right)\psi(x,\hbar) = 0 \qquad (\bigstar)$$

  where $p_k(x,\hbar) \in \mathcal{O}_{\mathsf{X}}[\hbar]$ or $\mathcal{O}_{\mathsf{X}}(\mathsf{D})[\hbar]$

- **Examples:** **(1)** $\left(\hbar^2\partial_x^2 + q(x,\hbar)\right)\psi = 0$ Schrödinger equation
  (2) $\left(\hbar^3\partial_x^3 + 3\hbar\partial_x + 2ix\right)\psi = 0$ Berk-Nevins-Roberts equation

- More generally: $\hbar$-differential operator on a line bundle $\mathcal{L}$ over a curve (X, D):

$$P : \mathcal{L} \to \mathcal{L} \otimes \operatorname{Sym}^n \Omega^1_{\mathsf{X},\mathsf{D}}[\hbar] \qquad \text{such that} \qquad P\big|_{\hbar=0}(fe) = fP\big|_{\hbar=0}(e)$$

- Even more generally: $\hbar$-connection on a vector bundle $\mathcal{E}$ over a curve (X, D):

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{\mathsf{X},\mathsf{D}}[\hbar] \qquad \text{such that} \qquad \nabla(fe) = f\nabla e + \hbar\, \mathrm{d}f \otimes e$$

- ***WKB method***: solve $(\bigstar)$ using the ***WKB ansatz*** $\quad \psi(x,\hbar) = \exp\left(\frac{1}{\hbar}\int_{x_0}^{x} s(x,\hbar)\,\mathrm{d}x\right)$

§0. Setting

- Start with singularly perturbed linear ODE in a domain $\mathsf{X} \subset \mathbb{C}_x$:

$$(\hbar\partial_x)^n\psi + p_1(\hbar\partial_x)^{n-1}\psi + \ldots = \left(\sum_{k=0}^{n} p_{n-k}\hbar^k\partial_x^k\right)\psi(x,\hbar) = 0 \qquad (\bigstar)$$

where $p_k(x,\hbar) \in \mathcal{O}_{\mathsf{X}}[\hbar]$ or $\mathcal{O}_{\mathsf{X}}(\mathsf{D})[\hbar]$

- **Examples:** **(1)** $\big(\hbar^2\partial_x^2 + q(x,\hbar)\big)\psi = 0$ Schrödinger equation
  (2) $\big(\hbar^3\partial_x^3 + 3\hbar\partial_x + 2ix\big)\psi = 0$ Berk-Nevins-Roberts equation

- More generally: $\hbar$-differential operator on a line bundle $\mathcal{L}$ over a curve (X,D):

$$P : \mathcal{L} \to \mathcal{L} \otimes \mathsf{Sym}^n\,\Omega^1_{\mathsf{X},\mathsf{D}}[\hbar] \qquad \text{such that} \qquad P\big|_{\hbar=0}(fe) = fP\big|_{\hbar=0}(e)$$

- Even more generally: $\hbar$-connection on a vector bundle $\mathcal{E}$ over a curve (X,D):

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{\mathsf{X},\mathsf{D}}[\hbar] \qquad \text{such that} \qquad \nabla(fe) = f\nabla e + \hbar\,\mathrm{d}f \otimes e$$

- ***WKB method***: solve $(\bigstar)$ using the ***WKB ansatz*** $\psi(x,\hbar) = \exp\left(\dfrac{1}{\hbar}\displaystyle\int_{x_0}^{x} s(x,\hbar)\,\mathrm{d}x\right)$

Two Questions Addressed Today

1. When does the WKB method lead to solutions of $(\bigstar)$ with *good* asymptotics as $\hbar \to 0$?
2. What is the WKB method for P and ∇?

§1.1. Formal WKB Method (Quick Reminder)

- Plug the WKB ansatz into ($\bigstar$) to get a nonlinear ODE of order $n-1$:

$$(\hbar \partial_x)^{n-1} s + s^n + \ldots = 0\ ; \qquad \text{explicitly:} \quad \sum_{k=1}^{n} p_k (\hbar \partial_x + s)^{k-1} s = 0 \qquad (\blacklozenge)$$

§1.1. Formal WKB Method (Quick Reminder)

- Plug the WKB ansatz into $(\bigstar)$ to get a nonlinear ODE of order $n-1$:

$$(\hbar\partial_x)^{n-1}s + s^n + \ldots = 0\ ; \qquad \text{explicitly:} \quad \sum_{k=1}^{n} p_k(\hbar\partial_x + s)^{k-1}s = 0 \qquad (\blacklozenge)$$

- **Examples**: **(1)** $\hbar\partial_x s + s^2 + q = 0$ // $(\hbar\partial_x + s)s + q = 0$
  (2) $\hbar^2\partial_x^2 s + 3s\hbar\partial_x s + s^3 + 3s + 2ix = 0$ // $(\hbar\partial_x + s)^2 s + 3s + 2ix = 0$

§1.1. Formal WKB Method (Quick Reminder)

- Plug the WKB ansatz into ($\bigstar$) to get a nonlinear ODE of order $n-1$:

$$(\hbar\partial_x)^{n-1}s + s^n + \ldots = 0 \,; \qquad \text{explicitly:} \quad \sum_{k=1}^{n} p_k(\hbar\partial_x + s)^{k-1}s = 0 \qquad (\blacklozenge)$$

- **Examples:** **(1)** $\hbar\partial_x s + s^2 + q = 0$ // $(\hbar\partial_x + s)s + q = 0$
  (2) $\hbar^2\partial_x^2 s + 3s\hbar\partial_x s + s^3 + 3s + 2ix = 0$ // $(\hbar\partial_x + s)^2 s + 3s + 2ix = 0$

Formal Existence and Uniqueness Theorem [classical]

If the basepoint x_0 is chosen *generically*, there are n formal solutions

$$\widehat{s}_i(x,\hbar) = \sum_{k=0}^{\infty} s_i^{(k)}(x)\hbar^k \in \mathcal{O}_{\mathsf{X},x_0}[\![\hbar]\!] \qquad i = 1,\ldots,n$$

uniquely and recursively determined by leading-orders $s_i^{(0)} = \lambda_i(x)$ that are roots of

$$\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0 \qquad \text{where} \qquad a_k(x) := p_k(x,0) \qquad (\spadesuit)$$

§1.1. Formal WKB Method (Quick Reminder)

- Plug the WKB ansatz into (★) to get a nonlinear ODE of order $n-1$:

$$(\hbar\partial_x)^{n-1}s + s^n + \ldots = 0\,; \qquad \text{explicitly:} \quad \sum_{k=1}^{n} p_k(\hbar\partial_x + s)^{k-1}s = 0 \qquad (\blacklozenge)$$

- **Examples:** **(1)** $\hbar\partial_x s + s^2 + q = 0$ // $(\hbar\partial_x + s)s + q = 0$
  (2) $\hbar^2\partial_x^2 s + 3s\hbar\partial_x s + s^3 + 3s + 2ix = 0$ // $(\hbar\partial_x + s)^2 s + 3s + 2ix = 0$

Formal Existence and Uniqueness Theorem [classical]

If the basepoint x_0 is chosen *generically*, there are n formal solutions

$$\widehat{s}_i(x,\hbar) = \sum_{k=0}^{\infty} s_i^{(k)}(x)\hbar^k \in \mathcal{O}_{\mathsf{X},x_0}[\![\hbar]\!] \qquad i = 1,\ldots,n$$

uniquely and recursively determined by leading-orders $s_i^{(0)} = \lambda_i(x)$ that are roots of

$$\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0 \qquad \text{where} \qquad a_k(x) := p_k(x,0) \qquad (\spadesuit)$$

and therefore n unique ***formal WKB solutions*** normalised at x_0:

$$\widehat{\psi}_i(x,\hbar) = \exp\left(\frac{1}{\hbar}\int_{x_0}^{x}\widehat{s}_i(x,\hbar)\,\mathrm{d}x\right) = e^{\int_{x_0}^{x}\lambda_i/\hbar}\sum_{k=0}^{\infty}\psi_i^{(k)}\hbar^k$$

§1.1. Formal WKB Method (Quick Reminder)

- Plug the WKB ansatz into (★) to get a nonlinear ODE of order $n-1$:

$$(\hbar\partial_x)^{n-1}s + s^n + \ldots = 0\,; \qquad \text{explicitly:} \quad \sum_{k=1}^{n} p_k(\hbar\partial_x + s)^{k-1}s = 0 \qquad (\blacklozenge)$$

- **Examples:** **(1)** $\hbar\partial_x s + s^2 + q = 0$ // $(\hbar\partial_x + s)s + q = 0$
  (2) $\hbar^2\partial_x^2 s + 3s\hbar\partial_x s + s^3 + 3s + 2ix = 0$ // $(\hbar\partial_x + s)^2 s + 3s + 2ix = 0$

Formal Existence and Uniqueness Theorem [classical]

If the basepoint x_0 is chosen *generically*, there are n formal solutions

$$\widehat{s}_i(x,\hbar) = \sum_{k=0}^{\infty} s_i^{(k)}(x)\hbar^k \in \mathcal{O}_{\mathsf{X},x_0}[\![\hbar]\!] \qquad i = 1,\ldots,n$$

uniquely and recursively determined by leading-orders $s_i^{(0)} = \lambda_i(x)$ that are roots of

$$\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0 \qquad \text{where} \qquad a_k(x) := p_k(x,0) \qquad (\spadesuit)$$

and therefore n unique ***formal WKB solutions*** normalised at x_0:

$$\widehat{\psi}_i(x,\hbar) = \exp\left(\frac{1}{\hbar}\int_{x_0}^{x} \widehat{s}_i(x,\hbar)\,\mathrm{d}x\right) = e^{\int_{x_0}^{x}\lambda_i/\hbar}\sum_{k=0}^{\infty}\psi_i^{(k)}\hbar^k$$

- "Generically" := away from ***turning points*** := zeros of the discriminant of (♠)

§1.1. Formal WKB Method (Quick Reminder)

- Plug the WKB ansatz into ($\bigstar$) to get a nonlinear ODE of order $n-1$:

$$(\hbar\partial_x)^{n-1}s + s^n + \ldots = 0\,; \qquad \text{explicitly:} \quad \sum_{k=1}^{n} p_k(\hbar\partial_x + s)^{k-1}s = 0 \qquad (\blacklozenge)$$

- **Examples:** **(1)** $\hbar\partial_x s + s^2 + q = 0$ // $(\hbar\partial_x + s)s + q = 0$
  (2) $\hbar^2\partial_x^2 s + 3s\hbar\partial_x s + s^3 + 3s + 2ix = 0$ // $(\hbar\partial_x + s)^2 s + 3s + 2ix = 0$

Formal Existence and Uniqueness Theorem [classical]

If the basepoint x_0 is chosen *generically*, there are n formal solutions

$$\widehat{s}_i(x,\hbar) = \sum_{k=0}^{\infty} s_i^{(k)}(x)\hbar^k \in \mathcal{O}_{\mathsf{X},x_0}[\![\hbar]\!] \qquad i = 1,\ldots,n$$

uniquely and recursively determined by leading-orders $s_i^{(0)} = \lambda_i(x)$ that are roots of

$$\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0 \qquad \text{where} \qquad a_k(x) := p_k(x,0) \qquad (\spadesuit)$$

and therefore n unique ***formal WKB solutions*** normalised at x_0:

$$\widehat{\psi}_i(x,\hbar) = \exp\left(\frac{1}{\hbar}\int_{x_0}^{x} \widehat{s}_i(x,\hbar)\,\mathrm{d}x\right) = e^{\int_{x_0}^{x}\lambda_i/\hbar}\sum_{k=0}^{\infty}\psi_i^{(k)}\hbar^k$$

- "Generically" := away from ***turning points*** := zeros of the discriminant of ($\spadesuit$)
- $\widehat{\psi}_k$ is very computable but almost always **divergent!**

§1.2. Exact WKB Method (Quick Reminder and Main Results)

Q: Can $\widehat{\psi}_i$ be upgraded to a holomorphic solution ψ_i?

i.e.: is $\widehat{\psi}_i$ the asymptotic/perturbative expansion as $\hbar \to 0$ of a holomorphic ψ_i?

§1.2. Exact WKB Method (Quick Reminder and Main Results)

Q: Can $\widehat{\psi}_i$ be upgraded to a holomorphic solution ψ_i?
i.e.: is $\widehat{\psi}_i$ the asymptotic/perturbative expansion as $\hbar \to 0$ of a holomorphic ψ_i?

A: Yes! [Asymptotic Existence Theorem]

§1.2. Exact WKB Method (Quick Reminder and Main Results)

Q: Can $\widehat{\psi}_i$ be upgraded to a holomorphic solution ψ_i?
i.e.: is $\widehat{\psi}_i$ the asymptotic/perturbative expansion as $\hbar \to 0$ of a holomorphic ψ_i?

A: Yes! [Asymptotic Existence Theorem]

BUT: such ψ_i is highly non-unique and not constructive

§1.2. Exact WKB Method (Quick Reminder and Main Results)

Q: Can $\widehat{\psi}_i$ be upgraded to a holomorphic solution ψ_i?

i.e.: is $\widehat{\psi}_i$ the asymptotic/perturbative expansion as $\hbar \to 0$ of a holomorphic ψ_i?

A: Yes! [Asymptotic Existence Theorem]

BUT: such ψ_i is highly non-unique and not constructive

BETTER Q: Can $\widehat{\psi}_i$ be upgraded to a holomorphic solution ψ_i in a canonical way?

i.e.: is $\widehat{\psi}_i$ Borel-summable to a holomorphic solution ψ_i?

§1.2. Exact WKB Method (Quick Reminder and Main Results)

Q: Can $\widehat{\psi}_i$ be upgraded to a holomorphic solution ψ_i?
i.e.: is $\widehat{\psi}_i$ the asymptotic/perturbative expansion as $\hbar \to 0$ of a holomorphic ψ_i?

A: Yes! [Asymptotic Existence Theorem]

BUT: such ψ_i is highly non-unique and not constructive

BETTER Q: Can $\widehat{\psi}_i$ be upgraded to a holomorphic solution ψ_i in a canonical way?
i.e.: is $\widehat{\psi}_i$ Borel-summable to a holomorphic solution ψ_i?

A: Yes! But highly sensitive to the semi-global geometry determined by $\lambda_1, \ldots, \lambda_n$

§1.2. Exact WKB Method (Quick Reminder and Main Results)

Q: Can $\widehat{\psi}_i$ be upgraded to a holomorphic solution ψ_i?
i.e.: is $\widehat{\psi}_i$ the asymptotic/perturbative expansion as $\hbar \to 0$ of a holomorphic ψ_i?

A: Yes! [Asymptotic Existence Theorem]

BUT: such ψ_i is highly non-unique and not constructive

BETTER Q: Can $\widehat{\psi}_i$ be upgraded to a holomorphic solution ψ_i in a canonical way?
i.e.: is $\widehat{\psi}_i$ Borel-summable to a holomorphic solution ψ_i?

A: Yes! But highly sensitive to the semi-global geometry determined by $\lambda_1, \ldots, \lambda_n$

Main Results [N] (rough statement)

1. Formal WKB solutions $\widehat{\psi}_1, \ldots, \widehat{\psi}_n$ are Borel-summable away from relevant Stokes lines.

§1.2. Exact WKB Method (Quick Reminder and Main Results)

Q: Can $\widehat{\psi}_i$ be upgraded to a holomorphic solution ψ_i?
i.e.: is $\widehat{\psi}_i$ the asymptotic/perturbative expansion as $\hbar \to 0$ of a holomorphic ψ_i?

A: Yes! [Asymptotic Existence Theorem]

BUT: such ψ_i is highly non-unique and not constructive

BETTER Q: Can $\widehat{\psi}_i$ be upgraded to a holomorphic solution ψ_i <u>in a canonical way</u>?
i.e.: is $\widehat{\psi}_i$ Borel-summable to a holomorphic solution ψ_i?

A: Yes! But highly sensitive to the semi-global geometry determined by $\lambda_1, \ldots, \lambda_n$

Main Results [N] (rough statement)

1. Formal WKB solutions $\widehat{\psi}_1, \ldots, \widehat{\psi}_n$ are Borel-summable away from relevant Stokes lines.
2. Their Borel resummations $\psi_1, \ldots, \psi_n$ are uniquely determined by an asymptotic condition, and therefore have an invariant geometric meaning for a differential operator P on a line bundle $\mathcal{L}$ over (X, D).

§1.2. Exact WKB Method (Quick Reminder and Main Results)

Q: Can $\widehat{\psi}_i$ be upgraded to a holomorphic solution ψ_i?
i.e.: is $\widehat{\psi}_i$ the asymptotic/perturbative expansion as $\hbar \to 0$ of a holomorphic ψ_i?

A: Yes! [Asymptotic Existence Theorem]

BUT: such ψ_i is highly non-unique and not constructive

BETTER Q: Can $\widehat{\psi}_i$ be upgraded to a holomorphic solution ψ_i in a canonical way?
i.e.: is $\widehat{\psi}_i$ Borel-summable to a holomorphic solution ψ_i?

A: Yes! But highly sensitive to the semi-global geometry determined by $\lambda_1, \ldots, \lambda_n$

Main Results [N] (rough statement)

1. Formal WKB solutions $\widehat{\psi}_1, \ldots, \widehat{\psi}_n$ are Borel-summable away from relevant Stokes lines.
2. Their Borel resummations $\psi_1, \ldots, \psi_n$ are uniquely determined by an asymptotic condition, and therefore have an invariant geometric meaning for a differential operator P on a line bundle $\mathcal{L}$ over (X, D).
3. Geometrically, the WKB method is a method to search for an invariant splitting of an oper structure on $(\mathcal{E}, \nabla)$, so exact WKB solutions make sense for connections.

§2.1. WKB Trajectories and Stokes Lines

- ***WKB trajectory of type ij*** emanating from x_0 is locally given by

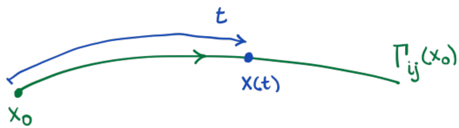
$$\Gamma_{ij}(x_0) \quad : \quad \mathrm{Im}\left(\int_{x_0}^{x} (\lambda_i - \lambda_j)\,\mathrm{d}x\right) = 0 \quad \text{and} \quad \mathrm{Re}\left(\int_{x_0}^{x} (\lambda_i - \lambda_j)\,\mathrm{d}x\right) \geqslant 0$$

§2.1. WKB Trajectories and Stokes Lines

- ***WKB trajectory of type ij*** emanating from x_0 is locally given by

$$\Gamma_{ij}(x_0) \quad : \quad \mathrm{Im}\left(\int_{x_0}^{x} (\lambda_i - \lambda_j)\,\mathrm{d}x\right) = 0 \quad \text{and} \quad \mathrm{Re}\left(\int_{x_0}^{x} (\lambda_i - \lambda_j)\,\mathrm{d}x\right) \geqslant 0$$

- Natural flow time parameter: $t(x) := \displaystyle\int_{x_0}^{x(t)} (\lambda_i - \lambda_j)\,\mathrm{d}x$

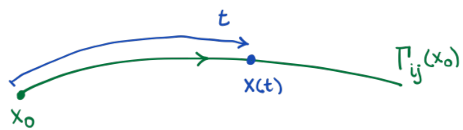


§2.1. WKB Trajectories and Stokes Lines

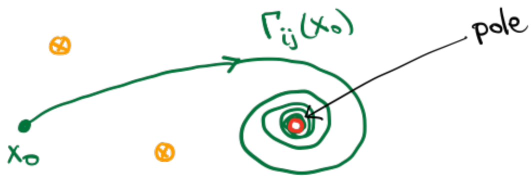
- ***WKB trajectory of type ij*** emanating from x_0 is locally given by

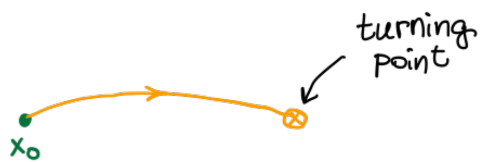
$$\Gamma_{ij}(x_0) \quad : \quad \mathrm{Im}\left(\int_{x_0}^{x} (\lambda_i - \lambda_j)\,\mathrm{d}x\right) = 0 \quad \text{and} \quad \mathrm{Re}\left(\int_{x_0}^{x} (\lambda_i - \lambda_j)\,\mathrm{d}x\right) \geqslant 0$$

- Natural flow time parameter: $t(x) := \displaystyle\int_{x_0}^{x(t)} (\lambda_i - \lambda_j)\,\mathrm{d}x$



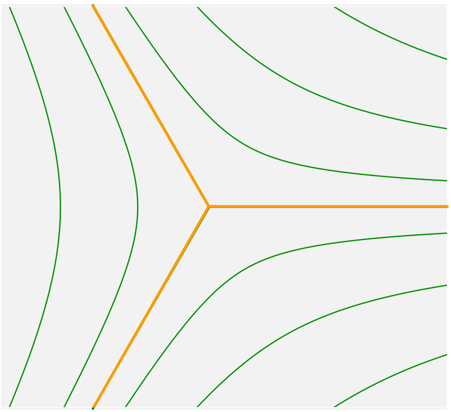
- $\Gamma_{ij}(x_0)$ is ***nonsingular*** if it is infinitely long and encounters no turning points
- $\Gamma_{ij}(x_0)$ is ***singular*** if it flows into a turning point





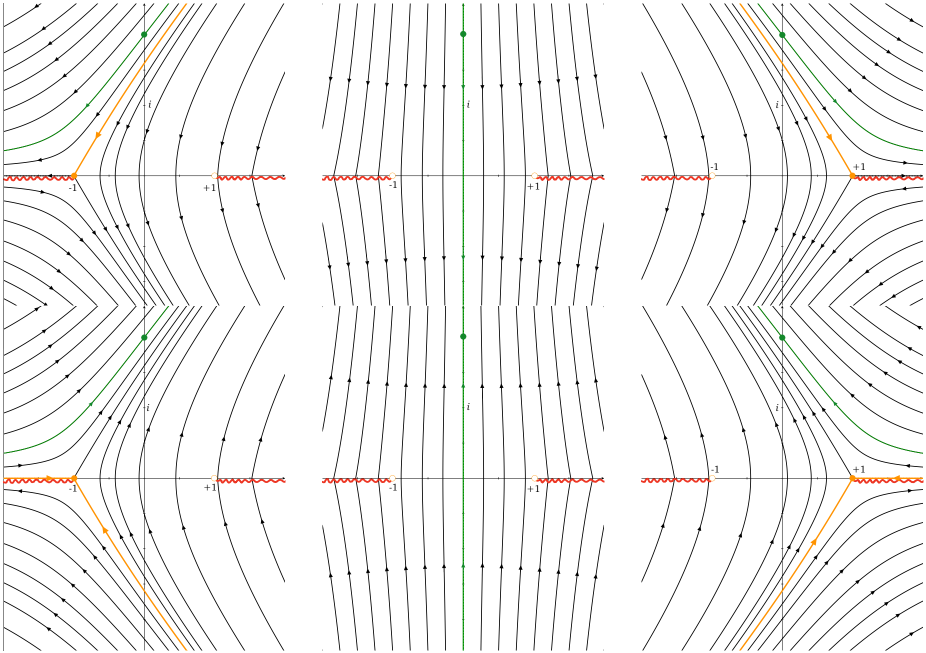
§2.1. WKB Trajectories and Stokes Lines

- A ***Stokes line of type*** ij on X is a maximal singular WKB trajectory of type ij
- ***Stokes 'graph'*** or ***network*** := collection of all Stokes lines on X



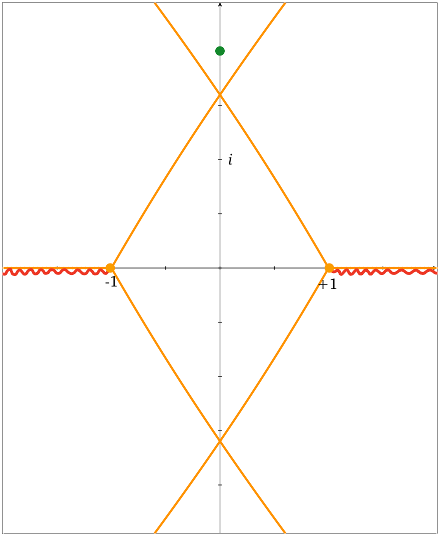
§2.1. WKB Trajectories and Stokes Lines

- A ***Stokes line of type ij*** on X is a maximal singular WKB trajectory of type ij
- ***Stokes 'graph'*** or ***network*** := collection of all Stokes lines on X



§2.1. WKB Trajectories and Stokes Lines

- A ***Stokes line of type ij*** on X is a maximal singular WKB trajectory of type ij
- ***Stokes 'graph'*** or ***network*** := collection of all Stokes lines on X

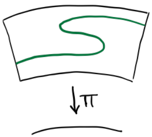


§2.2. WKB Trajectories and Stokes Lines: Invariant Description

- WKB trajectories of type ij are leaves of $\mathbb{R}_+$-foliation of the differential $(\lambda_i - \lambda_j)\,\mathrm{d}x$

§2.2. WKB Trajectories and Stokes Lines: Invariant Description

- WKB trajectories of type ij are leaves of $\mathbb{R}_+$-foliation of the differential $(\lambda_i - \lambda_j)\,\mathrm{d}x$
- The characteristic equation $\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$ ($\spadesuit$) is a ***spectral curve***:

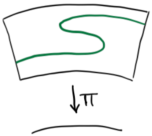


$$\begin{tikzcd} \Sigma \arrow[r, hook] \arrow[d, "\pi"'] & \mathsf{T}^*_{\mathsf{X},\mathsf{D}} = \mathsf{tot}\left(\Omega^1_{\mathsf{X},\mathsf{D}}\right) \arrow[dl] \\ \mathsf{X} & \end{tikzcd}$$

equipped with canonical differential
$\lambda \in \mathsf{H}^0_\Sigma(\Omega^1_{\Sigma,\tilde{\mathsf{D}}})$

§2.2. WKB Trajectories and Stokes Lines: Invariant Description

- WKB trajectories of type ij are leaves of $\mathbb{R}_+$-foliation of the differential $(\lambda_i - \lambda_j)\,\mathrm{d}x$
- The characteristic equation $\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$ (♠) is a ***spectral curve***:



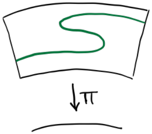
$$\begin{tikzcd} \Sigma \arrow[r, hook] \arrow[d, "\pi"'] & \mathsf{T}^*_{\mathsf{X},\mathsf{D}} = \mathsf{tot}\left(\Omega^1_{\mathsf{X},\mathsf{D}}\right) \arrow[dl] \\ \mathsf{X} & \end{tikzcd}$$

equipped with canonical differential
$\lambda \in \mathsf{H}^0_{\Sigma}(\Omega^1_{\Sigma,\tilde{\mathsf{D}}})$

- $\lambda_i\,\mathrm{d}x$ is the local expression for λ on sheet i of Σ

§2.2. WKB Trajectories and Stokes Lines: Invariant Description

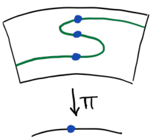
- WKB trajectories of type ij are leaves of $\mathbb{R}_+$-foliation of the differential $(\lambda_i - \lambda_j)\,\mathrm{d}x$
- The characteristic equation $\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$ (♠) is a ***spectral curve***:



$$\begin{tikzcd} \Sigma \arrow[r, hook] \arrow[d, "\pi"'] & \mathsf{T}^*_{\mathsf{X},\mathsf{D}} = \mathsf{tot}\left(\Omega^1_{\mathsf{X},\mathsf{D}}\right) \arrow[dl] \\ \mathsf{X} & \end{tikzcd}$$

equipped with canonical differential
$\lambda \in \mathsf{H}^0_\Sigma(\Omega^1_{\Sigma,\tilde{\mathsf{D}}})$

- $\lambda_i\,\mathrm{d}x$ is the local expression for λ on sheet i of Σ
- **Lemma:** $(\lambda_i - \lambda_j)\,\mathrm{d}x$ are local expressions for ***adjoint canonical differential*** $\mathrm{ad}\,\lambda$ on



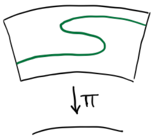
$$\begin{tikzcd} \mathrm{ad}\,\Sigma := \Sigma \underset{\mathsf{X}}{\times} \Sigma \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\ulcorner"] & \Sigma \arrow[d, "\pi"] \\ \Sigma \arrow[r, "\pi"'] & \mathsf{X} \end{tikzcd}$$

$\mathrm{ad}\,\lambda := \pi_1^*\lambda - \pi_2^*\lambda$

adjoint spectral curve

§2.2. WKB Trajectories and Stokes Lines: Invariant Description

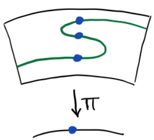
- WKB trajectories of type ij are leaves of $\mathbb{R}_+$-foliation of the differential $(\lambda_i - \lambda_j)\,\mathrm{d}x$
- The characteristic equation $\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$ ($\spadesuit$) is a ***spectral curve***:



$$\begin{tikzcd} \Sigma \arrow[r, hook] \arrow[d, "\pi"'] & \mathsf{T}^*_{\mathsf{X},\mathsf{D}} = \mathrm{tot}\left(\Omega^1_{\mathsf{X},\mathsf{D}}\right) \arrow[dl] \\ \mathsf{X} & \end{tikzcd}$$

equipped with canonical differential
$\lambda \in \mathsf{H}^0_{\Sigma}(\Omega^1_{\Sigma,\tilde{\mathsf{D}}})$

- $\lambda_i\,\mathrm{d}x$ is the local expression for λ on sheet i of Σ
- **Lemma:** $(\lambda_i - \lambda_j)\,\mathrm{d}x$ are local expressions for ***adjoint canonical differential*** $\mathrm{ad}\,\lambda$ on



$$\begin{tikzcd} \mathrm{ad}\,\Sigma := \Sigma \underset{\mathsf{X}}{\times} \Sigma \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\ulcorner"] & \Sigma \arrow[d, "\pi"] \\ \Sigma \arrow[r, "\pi"'] & \mathsf{X} \end{tikzcd}$$

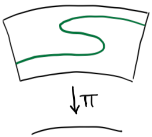
$\mathrm{ad}\,\lambda := \pi_1^*\lambda - \pi_2^*\lambda$

adjoint spectral curve

- ***turning points*** := ramification locus of $\mathrm{ad}\,\pi : \mathrm{ad}\,\Sigma \longrightarrow \mathsf{X}$
- ***WKB trajectories*** := leaves of $\mathbb{R}_+$-foliation of $\mathrm{ad}\,\lambda$ on $\mathrm{ad}\,\Sigma$
- ***Stokes lines*** := maximal singular WKB trajectories on $\mathrm{ad}\,\Sigma$
- ***Stokes graph*** := collection of all Stokes lines on $\mathrm{ad}\,\Sigma$

§2.2. WKB Trajectories and Stokes Lines: Invariant Description

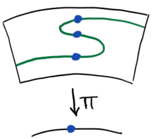
- WKB trajectories of type ij are leaves of $\mathbb{R}_+$-foliation of the differential $(\lambda_i - \lambda_j)\,\mathrm{d}x$
- The characteristic equation $\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$ ($\spadesuit$) is a ***spectral curve***:



$$\begin{tikzcd} \Sigma \arrow[r, hook] \arrow[d, "\pi"'] & \mathsf{T}^*_{\mathsf{X},\mathsf{D}} = \mathsf{tot}\left(\Omega^1_{\mathsf{X},\mathsf{D}}\right) \arrow[dl] \\ \mathsf{X} & \end{tikzcd}$$

equipped with canonical differential
$\lambda \in \mathsf{H}^0_\Sigma(\Omega^1_{\Sigma,\tilde{\mathsf{D}}})$

- $\lambda_i\,\mathrm{d}x$ is the local expression for λ on sheet i of Σ
- **Lemma:** $(\lambda_i - \lambda_j)\,\mathrm{d}x$ are local expressions for ***adjoint canonical differential*** $\mathrm{ad}\,\lambda$ on



$$\begin{tikzcd} \mathrm{ad}\,\Sigma := \Sigma \underset{\mathsf{X}}{\times} \Sigma \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\ulcorner"] & \Sigma \arrow[d, "\pi"] \\ \Sigma \arrow[r, "\pi"'] & \mathsf{X} \end{tikzcd}$$

$\mathrm{ad}\,\lambda := \pi_1^*\lambda - \pi_2^*\lambda$

adjoint spectral curve

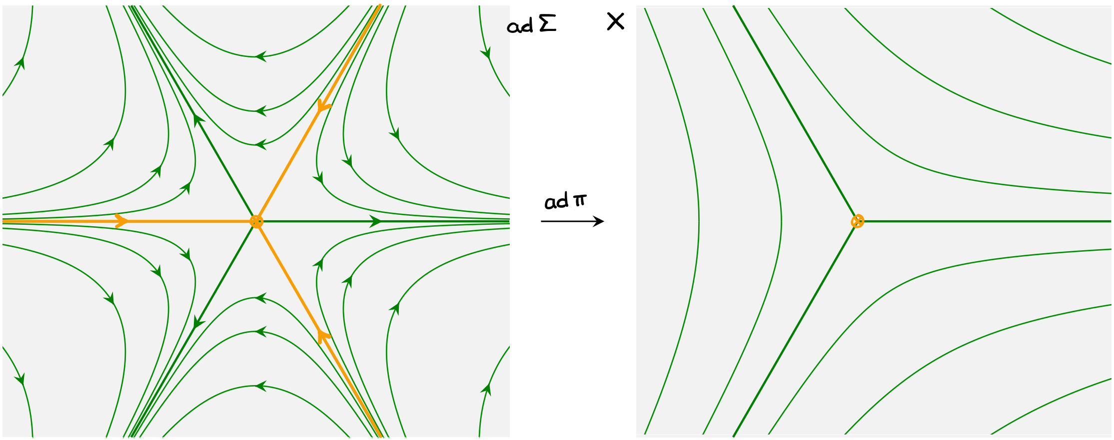
- ***turning points*** := ramification locus of $\mathrm{ad}\,\pi : \mathrm{ad}\,\Sigma \longrightarrow \mathsf{X}$
- ***WKB trajectories*** := leaves of $\mathbb{R}_+$-foliation of $\mathrm{ad}\,\lambda$ on $\mathrm{ad}\,\Sigma$
- ***Stokes lines*** := maximal singular WKB trajectories on $\mathrm{ad}\,\Sigma$
- ***Stokes graph*** := collection of all Stokes lines on $\mathrm{ad}\,\Sigma$
- Stokes network on X is the projection of the Stokes graph under $\mathrm{ad}\,\pi : \mathrm{ad}\,\Sigma \longrightarrow \mathsf{X}$

§2.3. WKB Trajectories and Stokes Lines: Nonsingular WKB Flow

Fix $x_0 \in \mathsf{X}$ ***ordinary point*** := neither a turning point nor a pole

Definition ($n = 2$)

The ***WKB flow of x_0 of type i is nonsingular*** if the WKB trajectory $\Gamma_{ij}(x_0)$ is nonsingular.

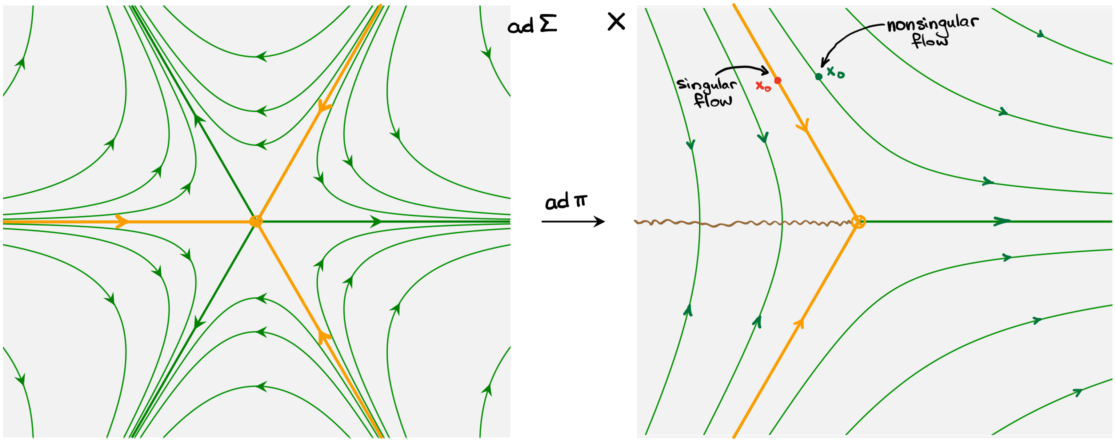


§2.3. WKB Trajectories and Stokes Lines: Nonsingular WKB Flow

Fix $x_0 \in \mathsf{X}$ ***ordinary point*** := neither a turning point nor a pole

Definition ($n = 2$)

The ***WKB flow of x_0 of type i is nonsingular*** if the WKB trajectory $\Gamma_{ij}(x_0)$ is nonsingular.



§2.3. WKB Trajectories and Stokes Lines: Nonsingular WKB Flow

Definition ($n \geqslant 3$)

The ***WKB flow of x_0 of type i is nonsingular*** if

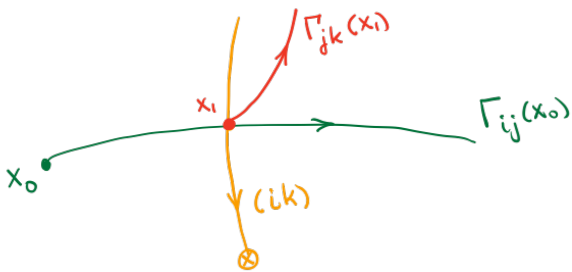
- each WKB trajectory $\Gamma_{i1}(x_0), \Gamma_{i2}(x_0), \ldots, \Gamma_{in}(x_0)$ is nonsingular

§2.3. WKB Trajectories and Stokes Lines: Nonsingular WKB Flow

Definition ($n \geqslant 3$)

The ***WKB flow of x_0 of type i is nonsingular*** if

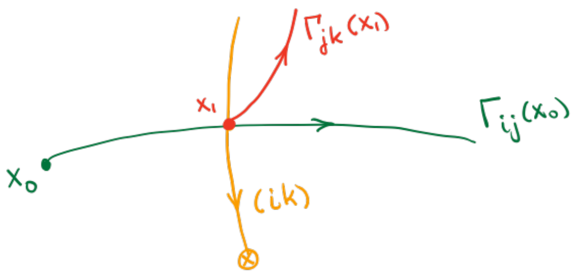
- each WKB trajectory $\Gamma_{i1}(x_0), \Gamma_{i2}(x_0), \ldots, \Gamma_{in}(x_0)$ is nonsingular
- Whenever $\Gamma_{ij}(x_0)$ intersects a singular trajectory of type ik, let $x_1 \in \mathrm{X}$ be an intersection point, and assume $\Gamma_{jk}(x_1)$ encounters no turning points



Definition ($n \geqslant 3$)

The ***WKB flow of x_0 of type i is nonsingular*** if

- each WKB trajectory $\Gamma_{i1}(x_0), \Gamma_{i2}(x_0), \ldots, \Gamma_{in}(x_0)$ is nonsingular
- Whenever $\Gamma_{ij}(x_0)$ intersects a singular trajectory of type ik, let $x_1 \in \mathrm{X}$ be an intersection point, and assume $\Gamma_{jk}(x_1)$ encounters no turning points
- Repeat for $\Gamma_{kj}(x_1)$

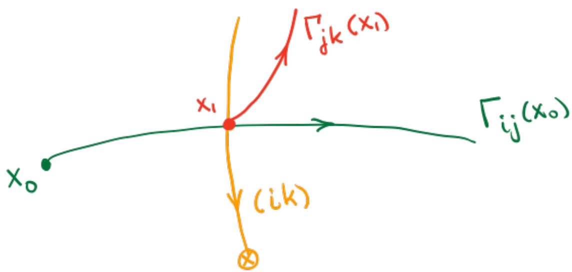


§2.3. WKB Trajectories and Stokes Lines: Nonsingular WKB Flow

Definition ($n \geqslant 3$)

The ***WKB flow of x_0 of type i is nonsingular*** if

- each WKB trajectory $\Gamma_{i1}(x_0), \Gamma_{i2}(x_0), \ldots, \Gamma_{in}(x_0)$ is nonsingular
- Whenever $\Gamma_{ij}(x_0)$ intersects a singular trajectory of type ik, let $x_1 \in \mathsf{X}$ be an intersection point, and assume $\Gamma_{jk}(x_1)$ encounters no turning points
- Repeat for $\Gamma_{kj}(x_1)$
- This process terminates at a finite number of iterations

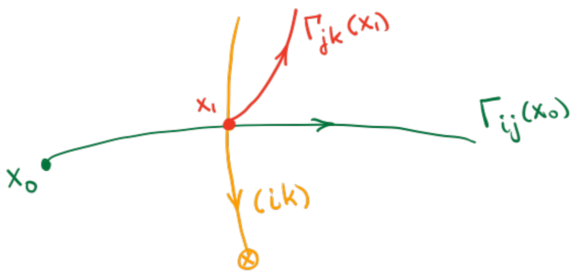


§2.3. WKB Trajectories and Stokes Lines: Nonsingular WKB Flow

Definition ($n \geqslant 3$)

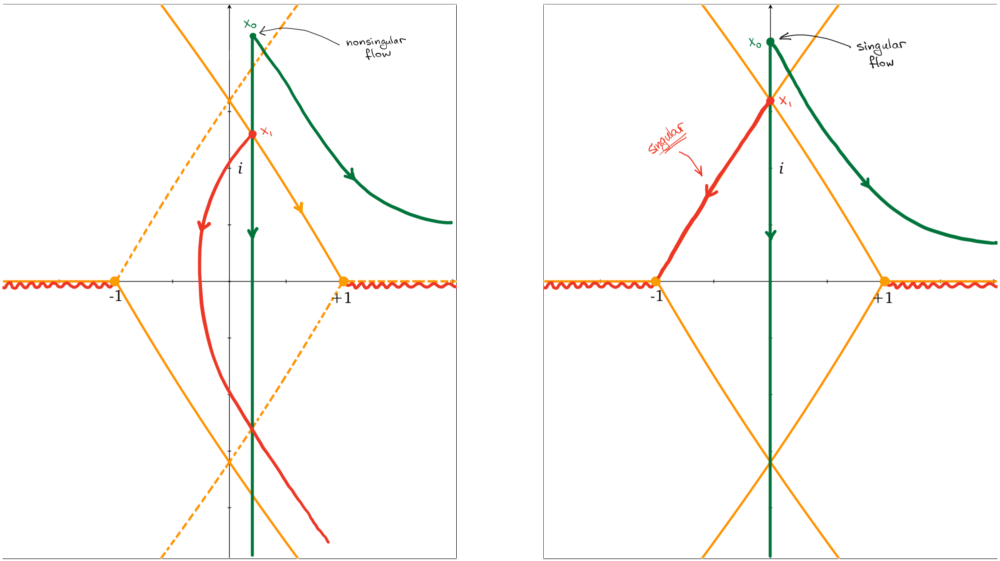
The ***WKB flow of x_0 of type i is nonsingular*** if

- each WKB trajectory $\Gamma_{i1}(x_0), \Gamma_{i2}(x_0), \ldots, \Gamma_{in}(x_0)$ is nonsingular
- Whenever $\Gamma_{ij}(x_0)$ intersects a singular trajectory of type ik, let $x_1 \in \mathrm{X}$ be an intersection point, and assume $\Gamma_{jk}(x_1)$ encounters no turning points
- Repeat for $\Gamma_{kj}(x_1)$
- This process terminates at a finite number of iterations



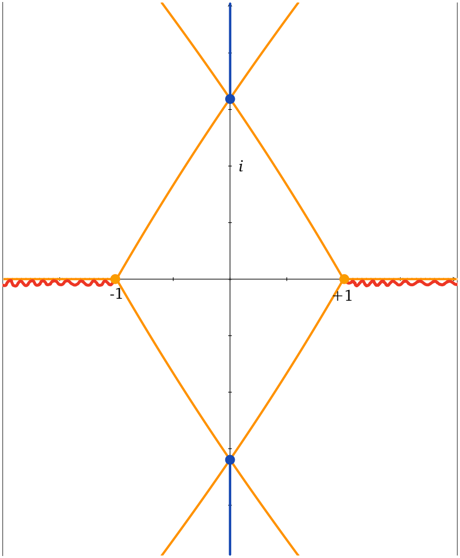
- ***Complete Stokes network*** := locus of all points on X with singular WKB flow

Example (BNR): $\left(\hbar^3\partial_x^3 + 3\hbar\partial_x + 2ix\right)\psi = 0$



§2.3. WKB Trajectories and Stokes Lines: Nonsingular WKB Flow

Example (BNR): $\left(\hbar^3\partial_x^3 + 3\hbar\partial_x + 2ix\right)\psi = 0$



§3. Borel Summability of WKB Solutions

Theorem [N] (Existence and Uniqueness of Exact WKB Solutions)

Fix $x_0 \in \mathsf{X}$ ordinary point and λ_i leading-order characteristic root near x_0.

§3. Borel Summability of WKB Solutions

Theorem [N] (Existence and Uniqueness of Exact WKB Solutions)

Fix $x_0 \in \mathsf{X}$ ordinary point and λ_i leading-order characteristic root near x_0.
Assume that the WKB flow of x_0 of type i is nonsingular.

Theorem [N] (Existence and Uniqueness of Exact WKB Solutions)

Fix $x_0 \in \mathsf{X}$ ordinary point and λ_i leading-order characteristic root near x_0.
Assume that the WKB flow of x_0 of type i is nonsingular.
Then the formal WKB solution

$$\widehat{\psi}_i(x,\hbar) = \exp\left(\frac{1}{\hbar}\int_{x_0}^{x} \widehat{s}_i(x,\hbar)\,\mathrm{d}x\right) = e^{\int_{x_0}^{x}\lambda_i/\hbar}\sum_{k=0}^{\infty}\psi_i^{(k)}(x)\hbar^k$$

is uniformly Borel summable near x_0:

$$\psi_i(x,\hbar) := \mathcal{S}\big[\widehat{\psi}_i\big](x,\hbar) = e^{\int_{x_0}^{x}\lambda_i/\hbar}\mathcal{S}\left(\sum_{k=0}^{\infty}\psi_i^{(k)}(x)\hbar^k\right)$$

§3. Borel Summability of WKB Solutions

Theorem [N] (Existence and Uniqueness of Exact WKB Solutions)

Fix $x_0 \in \mathsf{X}$ ordinary point and λ_i leading-order characteristic root near x_0.
Assume that the WKB flow of x_0 of type i is nonsingular.
Then the formal WKB solution

$$\widehat{\psi}_i(x,\hbar) = \exp\left(\frac{1}{\hbar}\int_{x_0}^{x} \widehat{s}_i(x,\hbar)\,\mathrm{d}x\right) = e^{\int_{x_0}^{x} \lambda_i/\hbar} \sum_{k=0}^{\infty} \psi_i^{(k)}(x)\hbar^k$$

is uniformly Borel summable near x_0:

$$\psi_i(x,\hbar) := \mathcal{S}\big[\widehat{\psi}_i\big](x,\hbar) = e^{\int_{x_0}^{x} \lambda_i/\hbar}\mathcal{S}\left(\sum_{k=0}^{\infty} \psi_i^{(k)}(x)\hbar^k\right)$$

In fact, ψ_i is the unique solution for x near x_0 which satisfies

$$\psi_i(x_0,\hbar) = 1 \qquad \text{and} \qquad \psi_i(x,\hbar) \simeq \widehat{\psi}_i(x,\hbar) \qquad \text{as } \hbar \to 0 \text{ with } \mathrm{Re}(\hbar) > 0$$

uniformly in x and $\arg(\hbar)$.

§3. Borel Summability of WKB Solutions

Theorem [N] (Existence and Uniqueness of Exact WKB Solutions)

Fix $x_0 \in \mathsf{X}$ ordinary point and λ_i leading-order characteristic root near x_0.
Assume that the WKB flow of x_0 of type i is nonsingular.
Then the formal WKB solution

$$\widehat{\psi}_i(x,\hbar) = \exp\left(\frac{1}{\hbar}\int_{x_0}^{x} \widehat{s}_i(x,\hbar)\,\mathrm{d}x\right) = e^{\int_{x_0}^{x} \lambda_i/\hbar} \sum_{k=0}^{\infty} \psi_i^{(k)}(x)\hbar^k$$

is uniformly Borel summable near x_0:

$$\psi_i(x,\hbar) := \mathcal{S}\big[\widehat{\psi}_i\big](x,\hbar) = e^{\int_{x_0}^{x} \lambda_i/\hbar}\mathcal{S}\left(\sum_{k=0}^{\infty} \psi_i^{(k)}(x)\hbar^k\right)$$

In fact, ψ_i is the unique solution for x near x_0 which satisfies

$$\psi_i(x_0,\hbar) = 1 \qquad \text{and} \qquad \psi_i(x,\hbar) \simeq \widehat{\psi}_i(x,\hbar) \qquad \text{as } \hbar \to 0 \text{ with } \mathrm{Re}(\hbar) > 0$$

uniformly in x and $\arg(\hbar)$.

Corollary

Uniqueness yields a notion of ***exact WKB flat sections*** of $\mathcal{L}$ for P on (X, D).

§3.1. Proof Outline ($n = 2$)

Focus on the Riccati equation $\hbar \partial_x s + s^2 + p_1 s + p_2 = 0$

§3.1. Proof Outline ($n = 2$)

Focus on the Riccati equation $\hbar\partial_x s + s^2 + p_1 s + p_2 = 0$

Lemma

The Borel transform of $\widehat{\boldsymbol{s}}_{\boldsymbol{i}}$ *is uniformly convergent near* x_0:

$$\widehat{\boldsymbol{\sigma}}_{\boldsymbol{i}}(\boldsymbol{x},\boldsymbol{\xi}) := \mathfrak{B}\big[\widehat{s}_i\big] = \mathfrak{B}\left[\lambda_i + \sum_{k=1}^{\infty} s_i^{(k)}(x)\hbar^k\right] = \sum_{k=0}^{\infty} \tfrac{1}{k!} s_i^{(k+1)}(x)\xi^k \in \mathcal{O}_{\mathsf{X},x_0}\{\xi\}$$

§3.1. Proof Outline ($n = 2$)

Focus on the Riccati equation $\hbar\partial_x s + s^2 + p_1 s + p_2 = 0$

Lemma

The Borel transform of $\widehat{s}_i$ is uniformly convergent near x_0:

$$\widehat{\sigma}_i(x,\xi) := \mathfrak{B}\big[\widehat{s}_i\big] = \mathfrak{B}\left[\lambda_i + \sum_{k=1}^{\infty} s_i^{(k)}(x)\hbar^k\right] = \sum_{k=0}^{\infty} \tfrac{1}{k!} s_i^{(k+1)}(x)\xi^k \in \mathcal{O}_{\mathsf{X},x_0}\{\xi\}$$

Goal

Construct the analytic continuation σ_i of $\widehat{\sigma}_i$ for all $\xi \in \mathbb{R}_+$ and define

$$s_i(x,\hbar) := \lambda_i + \mathfrak{L}\big[\sigma_i\big] = \lambda_i(x) + \int_0^{+\infty} e^{-\xi/\hbar}\sigma_i(x,\xi)\,\mathrm{d}\xi$$

$$\psi_i(x,\hbar) := \exp\left(\frac{1}{\hbar}\int_{x_0}^{x} s_i(x',\hbar)\,\mathrm{d}x'\right)$$

§3.1. Proof Outline ($n = 2$)

Focus on the Riccati equation $\hbar \partial_x s + s^2 + p_1 s + p_2 = 0$

Lemma

The Borel transform of $\widehat{\boldsymbol{s}}_{\boldsymbol{i}}$ is uniformly convergent near x_0:

$$\widehat{\boldsymbol{\sigma}}_{\boldsymbol{i}}(\boldsymbol{x}, \boldsymbol{\xi}) := \mathfrak{B}\big[\widehat{s}_i\big] = \mathfrak{B}\left[\lambda_i + \sum_{k=1}^{\infty} s_i^{(k)}(x)\hbar^k\right] = \sum_{k=0}^{\infty} \tfrac{1}{k!} s_i^{(k+1)}(x)\xi^k \in \mathcal{O}_{\mathsf{X},x_0}\{\xi\}$$

Goal

Construct the analytic continuation $\boldsymbol{\sigma}_{\boldsymbol{i}}$ of $\widehat{\boldsymbol{\sigma}}_{\boldsymbol{i}}$ for all $\xi \in \mathbb{R}_+$ and define

$$\boldsymbol{s}_{\boldsymbol{i}}(\boldsymbol{x}, \boldsymbol{\hbar}) := \lambda_i + \mathfrak{L}\big[\sigma_i\big] = \lambda_i(x) + \int_0^{+\infty} e^{-\xi/\hbar} \sigma_i(x, \xi)\, \mathrm{d}\xi$$

$$\boldsymbol{\psi}_{\boldsymbol{i}}(\boldsymbol{x}, \boldsymbol{\hbar}) := \exp\left(\tfrac{1}{\hbar} \int_{x_0}^{x} s_i(x', \hbar)\, \mathrm{d}x'\right)$$

Recall: uniform summability $\implies$ $\mathcal{S}\left[\exp\left(\tfrac{1}{\hbar}\int_{x_0}^{x} \widehat{s}\, \mathrm{d}x \,/\hbar\right)\right] = \exp\left(\tfrac{1}{\hbar}\int_{x_0}^{x} \mathcal{S}\big[\widehat{s}\big]\, \mathrm{d}x\right)$

§3.1. Proof Outline ($n = 2$)

To construct the analytic continuation $\boldsymbol{\sigma_i}$, argue as follows.

§3.1. Proof Outline ($n = 2$)

To construct the analytic continuation $\boldsymbol{\sigma_i}$, argue as follows.

1. Simplify by linearising the Riccati equation around λ_i:

$$\text{Let } s = \lambda_i + S \qquad \Longrightarrow \qquad \hbar \partial_x S + (\lambda_i - \lambda_j) S = \hbar A_0 + \hbar A_1 S - S^2$$

§3.1. Proof Outline ($n = 2$)

To construct the analytic continuation $\boldsymbol{\sigma_i}$, argue as follows.

1. Simplify by linearising the Riccati equation around λ_i:

$$\text{Let } s = \lambda_i + S \qquad \Longrightarrow \qquad \hbar \partial_x S + (\lambda_i - \lambda_j) S = \hbar A_0 + \hbar A_1 S - S^2$$

2. Apply the Borel transform:

$$\text{Let } \sigma = \mathfrak{B}[S] \qquad \Longrightarrow \qquad \partial_x \sigma + (\lambda_i - \lambda_j) \partial_\xi \sigma = \alpha_0 + a_1 \sigma + \alpha_1 * \sigma - \partial_\xi \sigma^{*2}$$

§3.1. Proof Outline ($n = 2$)

To construct the analytic continuation $\boldsymbol{\sigma_i}$, argue as follows.

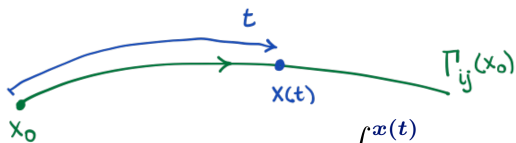
1. Simplify by linearising the Riccati equation around λ_i:

$$\text{Let } s = \lambda_i + S \qquad \Longrightarrow \qquad \hbar \partial_x S + (\lambda_i - \lambda_j) S = \hbar A_0 + \hbar A_1 S - S^2$$

2. Apply the Borel transform:

$$\text{Let } \sigma = \mathfrak{B}[S] \qquad \Longrightarrow \qquad \partial_x \sigma + (\lambda_i - \lambda_j) \partial_\xi \sigma = \alpha_0 + a_1 \sigma + \alpha_1 * \sigma - \partial_\xi \sigma^{*2}$$

3. Rewrite as an integral equation:



$$\sigma(x, \xi) = a_0 - \int_0^\xi (\text{righthand side}) \Big|_{(\boldsymbol{x(t)}, \xi - t)} \mathrm{d}t \qquad \text{where} \qquad t = \int_{x_0}^{\boldsymbol{x(t)}} \lambda_{ij} \, \mathrm{d}x$$

§3.1. Proof Outline ($n = 2$)

To construct the analytic continuation $\boldsymbol{\sigma_i}$, argue as follows.

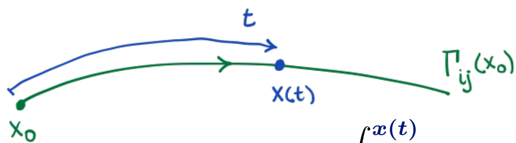
1. Simplify by linearising the Riccati equation around λ_i:

$$\text{Let } s = \lambda_i + S \qquad \Longrightarrow \qquad \hbar\partial_x S + (\lambda_i - \lambda_j)S = \hbar A_0 + \hbar A_1 S - S^2$$

2. Apply the Borel transform:

$$\text{Let } \sigma = \mathfrak{B}[S] \qquad \Longrightarrow \qquad \partial_x \sigma + (\lambda_i - \lambda_j)\partial_\xi \sigma = \alpha_0 + a_1\sigma + \alpha_1 * \sigma - \partial_\xi \sigma^{*2}$$

3. Rewrite as an integral equation:



$$\sigma(x,\xi) = a_0 - \int_0^\xi (\text{righthand side})\Big|_{(\boldsymbol{x(t)},\, \xi - t)} \mathrm{d}t \qquad \text{where} \qquad t = \int_{x_0}^{\boldsymbol{x(t)}} \lambda_{ij}\, \mathrm{d}x$$

4. Construct σ_i using the method of successive approximations: define $\{\tau_k(x,\xi)\}$ by

$$\tau_0 := a_0 \ , \qquad \tau_1 := -\int_0^\xi \big(\alpha_0 + a_1\tau_0\big)\, \mathrm{d}t \ , \qquad \tau_2 := -\int_0^\xi \big(a_1\tau_1 + \alpha_1 * \tau_0\big)\, \mathrm{d}t \ , \qquad \cdots$$

§3.1. Proof Outline ($n = 2$)

To construct the analytic continuation $\boldsymbol{\sigma_i}$, argue as follows.

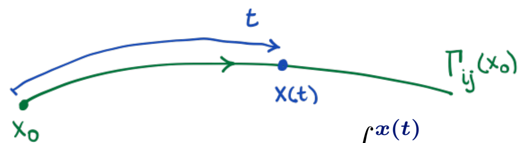
1. Simplify by linearising the Riccati equation around λ_i:

$$\text{Let } s = \lambda_i + S \qquad \Longrightarrow \qquad \hbar\partial_x S + (\lambda_i - \lambda_j)S = \hbar A_0 + \hbar A_1 S - S^2$$

2. Apply the Borel transform:

$$\text{Let } \sigma = \mathfrak{B}[S] \qquad \Longrightarrow \qquad \partial_x\sigma + (\lambda_i - \lambda_j)\partial_\xi\sigma = \alpha_0 + a_1\sigma + \alpha_1 * \sigma - \partial_\xi\sigma^{*2}$$

3. Rewrite as an integral equation:



$$\sigma(x,\xi) = a_0 - \int_0^\xi (\text{righthand side})\Big|_{(\boldsymbol{x(t)},\, \xi - t)} \mathrm{d}t \qquad \text{where} \qquad t = \int_{x_0}^{\boldsymbol{x(t)}} \lambda_{ij}\, \mathrm{d}x$$

4. Construct σ_i using the method of successive approximations: define $\{\tau_k(x,\xi)\}$ by

$$\tau_0 := a_0 \,, \qquad \tau_1 := -\int_0^\xi \big(\alpha_0 + a_1\tau_0\big)\,\mathrm{d}t \,, \qquad \tau_2 := -\int_0^\xi \big(a_1\tau_1 + \alpha_1 * \tau_0\big)\,\mathrm{d}t \,, \qquad \cdots$$

5. **Lemma**: $\boldsymbol{\sigma_i(x,\xi)} := \sum_{k=0}^{\infty} \tau_k(x,\xi)$ is uniformly convergent for all $\xi \in \mathbb{R}_+$, of exponential type, and $\widehat{\sigma}_i$ is its Taylor series at $\xi = 0$ $\square$

§3.2. Proof Outline ($n \geqslant 3$) | skip!

Focus on the equation $(\hbar\partial_x)^{n-1}s + s^n + \ldots = 0 \quad (\blacklozenge)$ and argue as follows.

❶ Rewrite as a nonlinear system: put $y_1 = s, \;\; y_2 = \hbar\partial_x y, \ldots$, and consider

$$\hbar\partial_x y = F(x, \hbar, y\,)$$

Example (BNR): $\qquad (\hbar^3\partial_x^3 + 3\hbar\partial_x + 2ix)\psi = 0$

$$\rightsquigarrow \quad \hbar^2\partial_x^2 s + 3s\hbar\partial_x s + s^3 + 3s + 2ix = 0$$

$$\rightsquigarrow \quad \hbar\partial_x \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = F(x,y) = -\begin{bmatrix} y_1^2 - y_2 \\ y_1 y_2 + 3y_1 + 2ix \end{bmatrix}$$

$\rightsquigarrow$ leading-order solution $y_i^{(0)} = \begin{bmatrix} \lambda_i \\ \lambda_i^2 \end{bmatrix}$

$\rightsquigarrow$ leading-order Jacobian at $y_i^{(0)}$ is $J_i = -\left.\dfrac{\partial F}{\partial y}\right|_{y=y_i^{(0)}} = \begin{bmatrix} 2\lambda_i & -1 \\ \lambda_i^2 + 3 & \lambda_i \end{bmatrix}$

$\rightsquigarrow$ J_i is diagonalisable to $\Lambda_i := \begin{bmatrix} \lambda_i - \lambda_j & \\ & \lambda_i - \lambda_k \end{bmatrix}$

❷ Linearise around the leading-order solution $y_i^{(0)}$ and apply a gauge transformation G to diagonalise the Jacobian J_i:

$$\text{Let } y = y_i^{(0)} + GS \quad \Longrightarrow \quad \hbar\partial_x S + \Lambda_i S = \hbar A_0 + \hbar A_1 S + \underbrace{\cdots}_{\substack{\text{at least quadratic} \\ \text{in } \hbar \text{ or } S}}$$

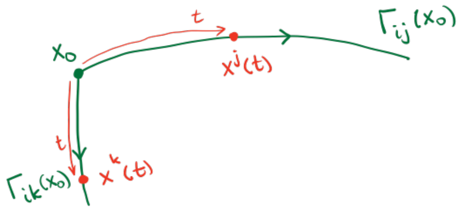
§3.2. Proof Outline ($n \geqslant 3$) | skip!

❸ Apply the Borel transform:

$$\text{Let } \sigma = \mathfrak{B}[S] \quad \Longrightarrow \quad \partial_x \sigma + \Lambda_i \partial_\xi \sigma = \alpha_0 + a_1 \sigma + \alpha_1 * \sigma + \cdots$$

❹ Rewrite as a system of integral equations: $j = 1, \dots, n-1$

$$\sigma^j(x,\xi) = a_0^j - \int_0^\xi (\text{righthand side})\Big|_{(\boldsymbol{x^j(t)},\, \xi - t)} \mathrm{d}t \qquad \text{where} \qquad t = \int_{x_0}^{\boldsymbol{x^j(t)}} \lambda_{ij}\, \mathrm{d}x$$



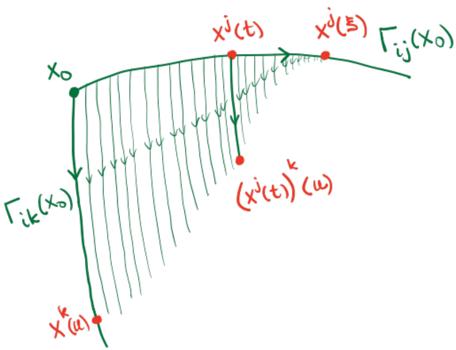
❺ Construct σ_i using the method of successive approximations: define $\{\tau_k(x,\xi)\}$ by

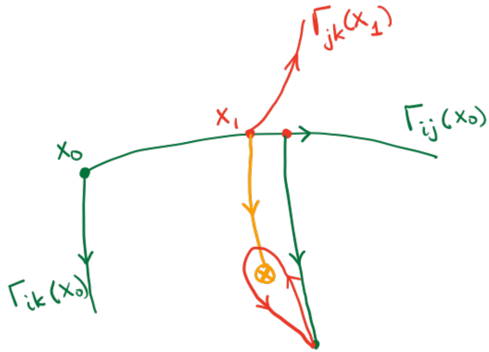
$$\tau_0 := a_0\,, \qquad \tau_1 := -\int_0^\xi \big(\alpha_0 + a_1 \tau_0\big)\,\mathrm{d}t\,, \qquad \tau_2 := -\int_0^\xi \big(a_1\tau_1 + \alpha_1 * \tau_0\big)\,\mathrm{d}t\,, \qquad \cdots$$

❻ **Lemma 1**: $\sigma_i(x,\xi) := \sum_{k=0}^{\infty} \tau_k(x,\xi)$ is uniformly convergent near $\xi = 0$, and $\widehat{\sigma}_i$ is its Taylor series at $\xi = 0$

❻ To analytically continue σ to all $\xi \in \mathbb{R}_+$, carefully examine cross-terms starting in τ_2:

$$\tau_2 := -\int_0^{\xi} \Big(\underbrace{\boldsymbol{a_1 \tau_1}}_{\begin{bmatrix} \vdots \\ \boldsymbol{a^j_{11}\tau^1_1 + \ldots + a^j_{1n}\tau^n_1} \\ \vdots \end{bmatrix}} + \alpha_1 * \tau_0 \Big)\,\mathrm{d}t \quad \rightsquigarrow \int_0^{\xi}\int_0^{\xi - t} \tau\Big(\big(x^j(t)\big)^k(u), \xi - t - u\Big)\,\mathrm{d}u\,\mathrm{d}t$$





❼ **Lemma 2**: thanks to the assumption that the (complete) WKB flow is nonsingular, $\sigma(x, \xi)$ admits analytic continuation to $\xi \in \mathbb{R}_+$ of exponential type □

The Geometric WKB Problem

The Geometric WKB Problem

0 GIVEN: $(\mathcal{E}, \nabla)$ an oper: $$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

§4. The WKB Method: Invariant Formulation

The Geometric WKB Problem

❶ <u>GIVEN</u>: $(\mathcal{E}, \nabla)$ an oper: $$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$
<u>FIND</u>: a ∇-invariant splitting $W : \mathcal{E}'' \to \mathcal{E}$.

§4. The WKB Method: Invariant Formulation

The Geometric WKB Problem

⓪ <u>GIVEN</u>: $(\mathcal{E}, \nabla)$ an oper: $\quad 0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$
<u>FIND</u>: a ∇-invariant splitting $W : \mathcal{E}'' \to \mathcal{E}$.

The Geometric WKB Method

① Fix a reference pair (W_0, ∇_0) where

- $W_0 : \mathcal{E}'' \to \mathcal{E}$ any **reference splitting**, so $\mathcal{E} \xrightarrow{\sim} \mathcal{E}' \oplus \mathcal{E}''$;
- $\nabla_0 = \nabla' \oplus \nabla''$ any block-diagonal connection on $\mathcal{E}' \oplus \mathcal{E}''$.

§4. The WKB Method: Invariant Formulation

The Geometric WKB Problem

0. <u>GIVEN</u>: $(\mathcal{E}, \nabla)$ an oper: $\quad 0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$
   <u>FIND</u>: a ∇-invariant splitting $W : \mathcal{E}'' \to \mathcal{E}$.

The Geometric WKB Method

1. Fix a reference pair (W_0, ∇_0) where
   - $W_0 : \mathcal{E}'' \to \mathcal{E}$ any **reference splitting**, so $\mathcal{E} \xrightarrow{\sim} \mathcal{E}' \oplus \mathcal{E}''$;
   - $\nabla_0 = \nabla' \oplus \nabla''$ any block-diagonal connection on $\mathcal{E}' \oplus \mathcal{E}''$.
2. Write $W : \mathcal{E}'' \to \mathcal{E}' \oplus \mathcal{E}''$ as $S \oplus \mathrm{id}$ and solve for S by searching for a unipotent gauge transformation

$$\begin{bmatrix} \mathrm{id} & \\ 0 & W \end{bmatrix} = \begin{bmatrix} \mathrm{id} & S \\ 0 & \mathrm{id} \end{bmatrix} : \quad \begin{tikzcd} \mathcal{E}' \arrow[r, equal] & \mathcal{E}' \\ \oplus & \oplus \\ \mathcal{E}'' \arrow[r, equal] \arrow[uur] & \mathcal{E}'' \end{tikzcd}$$

§4. The WKB Method: Invariant Formulation

The Geometric WKB Problem

0 GIVEN: $(\mathcal{E}, \nabla)$ an oper: $\quad 0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$
FIND: a ∇-invariant splitting $W : \mathcal{E}'' \to \mathcal{E}$.

The Geometric WKB Method

1 Fix a reference pair (W_0, ∇_0) where

- $W_0 : \mathcal{E}'' \to \mathcal{E}$ any **reference splitting**, so $\mathcal{E} \xrightarrow{\sim} \mathcal{E}' \oplus \mathcal{E}''$;
- $\nabla_0 = \nabla' \oplus \nabla''$ any block-diagonal connection on $\mathcal{E}' \oplus \mathcal{E}''$.

2 Write $W : \mathcal{E}'' \to \mathcal{E}' \oplus \mathcal{E}''$ as $S \oplus \mathrm{id}$ and solve for S by searching for a unipotent gauge transformation

$$\begin{bmatrix} \mathrm{id} & \\ 0 & W \end{bmatrix} = \begin{bmatrix} \mathrm{id} & S \\ 0 & \mathrm{id} \end{bmatrix} : \quad \begin{tikzcd} \mathcal{E}' \arrow[r, equal] & \mathcal{E}' \\ \oplus & \oplus \\ \mathcal{E}'' \arrow[r, equal] \arrow[uur] & \mathcal{E}'' \end{tikzcd}$$

3 Write $\nabla = \nabla_0 - \phi \quad$ where $\quad \phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$

§4. The WKB Method: Invariant Formulation

The Geometric WKB Problem

0. GIVEN: $(\mathcal{E}, \nabla)$ an oper: $\quad 0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$
   FIND: a ∇-invariant splitting $W : \mathcal{E}'' \to \mathcal{E}$.

The Geometric WKB Method

1. Fix a reference pair (W_0, ∇_0) where
   - $W_0 : \mathcal{E}'' \to \mathcal{E}$ any **reference splitting**, so $\mathcal{E} \xrightarrow{\sim} \mathcal{E}' \oplus \mathcal{E}''$;
   - $\nabla_0 = \nabla' \oplus \nabla''$ any block-diagonal connection on $\mathcal{E}' \oplus \mathcal{E}''$.
2. Write $W : \mathcal{E}'' \to \mathcal{E}' \oplus \mathcal{E}''$ as $S \oplus \mathrm{id}$ and solve for S by searching for a unipotent gauge transformation

$$\begin{bmatrix} \mathrm{id} & W \\ 0 & \end{bmatrix} = \begin{bmatrix} \mathrm{id} & S \\ 0 & \mathrm{id} \end{bmatrix} : \quad \begin{tikzcd} \mathcal{E}' \arrow[r, equal] & \mathcal{E}' \\ \oplus & \oplus \\ \mathcal{E}'' \arrow[r, equal] \arrow[uur] & \mathcal{E}'' \end{tikzcd}$$

3. Write $\nabla = \nabla_0 - \phi \quad$ where $\quad \phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$
4. Then W is a ∇-invariant splitting $\Leftrightarrow$ S satisfies the **geometric Riccati equation**:

$$\mathrm{ad}_{\nabla_0} S - \phi_{11} S + S \phi_{21} S - \phi_{12} + S \phi_{22} = 0$$

Its exact solutions yield ***exact WKB flat sections*** for $(\mathcal{E}, \nabla)$

§4. The WKB Method: Invariant Formulation

The Geometric WKB Problem

0 GIVEN: $(\mathcal{E}, \nabla)$ an oper: $\quad 0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$

FIND: a ∇-invariant splitting $W : \mathcal{E}'' \to \mathcal{E}$.

The Geometric WKB Method

1 Fix a reference pair (W_0, ∇_0) where

- $W_0 : \mathcal{E}'' \to \mathcal{E}$ any **reference splitting**, so $\mathcal{E} \xrightarrow{\sim} \mathcal{E}' \oplus \mathcal{E}''$;
- $\nabla_0 = \nabla' \oplus \nabla''$ any block-diagonal connection on $\mathcal{E}' \oplus \mathcal{E}''$.

2 Write $W : \mathcal{E}'' \to \mathcal{E}' \oplus \mathcal{E}''$ as $S \oplus \mathrm{id}$ and solve for S by searching for a unipotent gauge transformation

$$\begin{bmatrix} \mathrm{id} & \\ 0 & W \end{bmatrix} = \begin{bmatrix} \mathrm{id} & S \\ 0 & \mathrm{id} \end{bmatrix} : \quad \begin{tikzcd} \mathcal{E}' \arrow[r, equal] & \mathcal{E}' \\ \oplus & \oplus \\ \mathcal{E}'' \arrow[r, equal] \arrow[uur] & \mathcal{E}'' \end{tikzcd}$$

3 Write $\nabla = \nabla_0 - \phi \quad$ where $\quad \phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$

4 Then W is a ∇-invariant splitting $\Leftrightarrow$ S satisfies the **geometric Riccati equation**:

$$\mathrm{ad}_{\nabla_0} S - \phi_{11} S + S \phi_{21} S - \phi_{12} + S \phi_{22} = 0$$

Its exact solutions yield ***exact WKB flat sections*** for $(\mathcal{E}, \nabla)$

Remark: $\overset{?}{\Longrightarrow} \quad S \in \mathcal{E}xt^1_X(\mathcal{E}'', \mathcal{E}') \quad \overset{?}{\Longrightarrow}$ cohomological WKB method?

§4. The WKB Method: Invariant Formulation

Traditional Point of View:

⓿ $\hbar^2\partial_x^2\psi + q\psi = 0$

❶ $\psi = \exp\left(\int s \, \mathrm{d}x \, / \hbar\right)$

❷ $\hbar\partial_x s + s^2 + q = 0$

Geometric Point of View:

⓿ <u>Given</u>: $(\mathcal{E}, \nabla)$ oper:

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

<u>Find</u>: ∇-invariant splitting $W : \mathcal{E}'' \to \mathcal{E}$

❶ Fix reference pair (W_0, ∇_0)

❷ Search for $\begin{bmatrix} \mathrm{id} & W \\ 0 & \end{bmatrix} = \begin{bmatrix} \mathrm{id} & S \\ 0 & \mathrm{id} \end{bmatrix}$:

$$\begin{tikzcd} \mathcal{E}' \arrow[r, equal] & \mathcal{E}' \\ \oplus & \oplus \\ \mathcal{E}'' \arrow[r, equal] \arrow[uur] & \mathcal{E}'' \end{tikzcd}$$

❸ Write $\nabla = \nabla_0 - \phi$ where $\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$.

❹ $\mathrm{ad}_{\nabla_0} S - \phi_{11} S + S\phi_{21} S - \phi_{12} + S\phi_{22} = 0$

§4. The WKB Method: Invariant Formulation

Traditional Point of View:

⓿ $\hbar^2 \partial_x^2 \psi + q\psi = 0$

❶ $\psi = \exp\left(\int s \, \mathrm{d}x \,/\hbar\right)$

❷ $\hbar \partial_x s + s^2 + q = 0$

Geometric Point of View:

⓿ <u>GIVEN</u>: $(\mathcal{E}, \nabla)$ oper:

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

<u>FIND</u>: ∇-invariant splitting $W : \mathcal{E}'' \to \mathcal{E}$

❶ Fix reference pair (W_0, ∇_0)

❷ Search for $\begin{bmatrix} \mathrm{id} & W \\ 0 & \end{bmatrix} = \begin{bmatrix} \mathrm{id} & S \\ 0 & \mathrm{id} \end{bmatrix}$:

$$\begin{tikzcd} \mathcal{E}' \arrow[r, equal] & \mathcal{E}' \\ \oplus & \oplus \\ \mathcal{E}'' \arrow[r, equal] \arrow[uur] & \mathcal{E}'' \end{tikzcd}$$

❸ Write $\nabla = \nabla_0 - \phi$ where $\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$.

❹ $\mathrm{ad}_{\nabla_0} S - \phi_{11} S + S\phi_{21} S - \phi_{12} + S\phi_{22} = 0$

- Schrödinger equation = 2-nd order $\hbar$-differential operator on $\mathcal{L} := \omega_X^{-1/2}$

§4. The WKB Method: Invariant Formulation

Traditional Point of View:

⓿ $\hbar^2 \partial_x^2 \psi + q\psi = 0$

❶ $\psi = \exp\bigl(\int s \, \mathrm{d}x \, / \hbar\bigr)$

❷ $\hbar \partial_x s + s^2 + q = 0$

Geometric Point of View:

⓿ <u>GIVEN</u>: $(\mathcal{E}, \nabla)$ oper:

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

<u>FIND</u>: ∇-invariant splitting $W : \mathcal{E}'' \to \mathcal{E}$

❶ Fix reference pair (W_0, ∇_0)

❷ Search for $\begin{bmatrix} \mathrm{id} & W \\ 0 & \end{bmatrix} = \begin{bmatrix} \mathrm{id} & S \\ 0 & \mathrm{id} \end{bmatrix}$:

$$\begin{tikzcd} \mathcal{E}' \arrow[r, equal] & \mathcal{E}' \\ \oplus & \oplus \\ \mathcal{E}'' \arrow[r, equal] \arrow[uur] & \mathcal{E}'' \end{tikzcd}$$

❸ Write $\nabla = \nabla_0 - \phi$ where $\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$.

❹ $\mathrm{ad}_{\nabla_0} S - \phi_{11} S + S\phi_{21} S - \phi_{12} + S\phi_{22} = 0$

- Schrödinger equation = 2-nd order $\hbar$-differential operator on $\mathcal{L} := \omega_{\mathsf{X}}^{-1/2}$
- Equivalently, $\hbar$-connection ∇ on the 1-jet bundle $\mathcal{E} := \mathcal{J}^1 \mathcal{L}$

Traditional Point of View:

⓿ $\hbar^2\partial_x^2\psi + q\psi = 0$

❶ $\psi = \exp\big(\int s \, \mathrm{d}x \,/\hbar\big)$

❷ $\hbar\partial_x s + s^2 + q = 0$

Geometric Point of View:

⓿ <u>GIVEN</u>: $(\mathcal{E}, \nabla)$ oper:

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

<u>FIND</u>: ∇-invariant splitting $W : \mathcal{E}'' \to \mathcal{E}$

❶ Fix reference pair (W_0, ∇_0)

❷ Search for $\begin{bmatrix} \begin{matrix}\mathrm{id} \\ 0\end{matrix} & W \end{bmatrix} = \begin{bmatrix} \mathrm{id} & S \\ 0 & \mathrm{id} \end{bmatrix}$:
$$\begin{tikzcd}[row sep=small] \mathcal{E}' \arrow[r, equal] & \mathcal{E}' \\ \oplus & \oplus \\ \mathcal{E}'' \arrow[r, equal] \arrow[uur] & \mathcal{E}'' \end{tikzcd}$$

❸ Write $\nabla = \nabla_0 - \phi$ where $\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$.

❹ $\mathrm{ad}_{\nabla_0} S - \phi_{11} S + S\phi_{21} S - \phi_{12} + S\phi_{22} = 0$

- Schrödinger equation = 2-nd order $\hbar$-differential operator on $\mathcal{L} := \omega_{\mathsf{X}}^{-1/2}$
- Equivalently, $\hbar$-connection ∇ on the 1-jet bundle $\mathcal{E} := \mathcal{J}^1\mathcal{L}$
- Oper structure = jet sequence: $0 \longrightarrow \omega_{\mathsf{X}} \otimes \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$

§4. The WKB Method: Invariant Formulation

Traditional Point of View:

0 $\hbar^2 \partial_x^2 \psi + q\psi = 0$

1 $\psi = \exp\left(\int s \, \mathrm{d}x \, / \hbar\right)$

2 $\hbar \partial_x s + s^2 + q = 0$

Geometric Point of View:

0 <u>GIVEN</u>: $(\mathcal{E}, \nabla)$ oper:

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

<u>FIND</u>: ∇-invariant splitting $W : \mathcal{E}'' \to \mathcal{E}$

1 Fix reference pair (W_0, ∇_0)

2 Search for $\begin{bmatrix} \mathrm{id} & W \\ 0 & \end{bmatrix} = \begin{bmatrix} \mathrm{id} & S \\ 0 & \mathrm{id} \end{bmatrix}$:

$$\begin{array}{ccc} \mathcal{E}' & = & \mathcal{E}' \\ \oplus & \nearrow & \oplus \\ \mathcal{E}'' & = & \mathcal{E}'' \end{array}$$

3 Write $\nabla = \nabla_0 - \phi$ where $\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$.

4 $\mathrm{ad}_{\nabla_0} S - \phi_{11} S + S\phi_{21} S - \phi_{12} + S\phi_{22} = 0$

- Schrödinger equation = 2-nd order $\hbar$-differential operator on $\mathcal{L} := \omega_{\mathsf{X}}^{-1/2}$
- Equivalently, $\hbar$-connection ∇ on the 1-jet bundle $\mathcal{E} := \mathcal{J}^1 \mathcal{L}$
- Oper structure = jet sequence: $0 \longrightarrow \omega_{\mathsf{X}} \otimes \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$
- Reference splitting W_0 is given by choice of coordinate x because

$$\mathcal{E} \xrightarrow[x]{\sim} \left\langle \mathrm{d}x \otimes \mathrm{d}x^{-1/2} \right\rangle \oplus \left\langle \mathrm{d}x^{-1/2} \right\rangle = \mathcal{E}' \oplus \mathcal{E}'' \qquad \text{and} \qquad S = s(x, \hbar)\, \mathrm{d}x$$

§4. The WKB Method: Invariant Formulation

Traditional Point of View:

0. $\hbar^2 \partial_x^2 \psi + q\psi = 0$
1. $\psi = \exp\big(\int s \, \mathrm{d}x \, / \hbar\big)$
2. $\hbar \partial_x s + s^2 + q = 0$

Geometric Point of View:

0. GIVEN: $(\mathcal{E}, \nabla)$ oper:
$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$
FIND: ∇-invariant splitting $W : \mathcal{E}'' \to \mathcal{E}$
1. Fix reference pair (W_0, ∇_0)
2. Search for $\begin{bmatrix} \mathrm{id} & W \\ 0 & \end{bmatrix} = \begin{bmatrix} \mathrm{id} & S \\ 0 & \mathrm{id} \end{bmatrix}$:
$$\begin{tikzcd} \mathcal{E}' \arrow[r, equal] & \mathcal{E}' \\ \oplus & \oplus \\ \mathcal{E}'' \arrow[r, equal] \arrow[uur] & \mathcal{E}'' \end{tikzcd}$$
3. Write $\nabla = \nabla_0 - \phi$ where $\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$.
4. $\mathrm{ad}_{\nabla_0} S - \phi_{11} S + S \phi_{21} S - \phi_{12} + S\phi_{22} = 0$

- Schrödinger equation = 2-nd order $\hbar$-differential operator on $\mathcal{L} := \omega_{\mathsf{X}}^{-1/2}$
- Equivalently, $\hbar$-connection ∇ on the 1-jet bundle $\mathcal{E} := \mathcal{J}^1 \mathcal{L}$
- Oper structure = jet sequence: $0 \longrightarrow \omega_{\mathsf{X}} \otimes \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$
- Reference splitting W_0 is given by choice of coordinate x because

$$\mathcal{E} \underset{x}{\xrightarrow{\sim}} \left\langle \mathrm{d}x \otimes \mathrm{d}x^{-1/2} \right\rangle \oplus \left\langle \mathrm{d}x^{-1/2} \right\rangle = \mathcal{E}' \oplus \mathcal{E}'' \qquad \text{and} \qquad S = s(x, \hbar)\, \mathrm{d}x$$

- Reference connection $\nabla_0 = \hbar \mathrm{d}$, then $\nabla \underset{\mathrm{loc}}{=} \hbar \mathrm{d} - \begin{bmatrix} 0 & -q \\ 1 & 0 \end{bmatrix} \mathrm{d}x = \nabla_0 - \phi$

§4. The WKB Method: Invariant Formulation

Traditional Point of View:

⓪ $\hbar^2 \partial_x^2 \psi + q\psi = 0$

❶ $\psi = \exp\left(\int s \,\mathrm{d}x \,/\hbar\right)$

❷ $\hbar \partial_x s + s^2 + q = 0$

Geometric Point of View:

⓪ <u>GIVEN</u>: $(\mathcal{E}, \nabla)$ oper:

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

<u>FIND</u>: ∇-invariant splitting $W : \mathcal{E}'' \to \mathcal{E}$

❶ Fix reference pair (W_0, ∇_0)

❷ Search for $\begin{bmatrix} \mathrm{id} & W \\ 0 & \end{bmatrix} = \begin{bmatrix} \mathrm{id} & S \\ 0 & \mathrm{id} \end{bmatrix}$:

$$\begin{tikzcd} \mathcal{E}' \arrow[r, equal] & \mathcal{E}' \\ \oplus & \oplus \\ \mathcal{E}'' \arrow[r, equal] \arrow[uur] & \mathcal{E}'' \end{tikzcd}$$

❸ Write $\nabla = \nabla_0 - \phi$ where $\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$.

❹ $\mathrm{ad}_{\nabla_0} S - \phi_{11} S + S\phi_{21} S - \phi_{12} + S\phi_{22} = 0$

- Schrödinger equation = 2-nd order $\hbar$-differential operator on $\mathcal{L} := \omega_{\mathsf{X}}^{-1/2}$
- Equivalently, $\hbar$-connection ∇ on the 1-jet bundle $\mathcal{E} := \mathcal{J}^1 \mathcal{L}$
- Oper structure = jet sequence: $0 \longrightarrow \omega_{\mathsf{X}} \otimes \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$
- Reference splitting W_0 is given by choice of coordinate x because

$$\mathcal{E} \xrightarrow[x]{\sim} \left\langle \mathrm{d}x \otimes \mathrm{d}x^{-1/2} \right\rangle \oplus \left\langle \mathrm{d}x^{-1/2} \right\rangle = \mathcal{E}' \oplus \mathcal{E}'' \qquad \text{and} \qquad S = s(x, \hbar)\,\mathrm{d}x$$

- Reference connection $\nabla_0 = \hbar\mathrm{d}$, then $\nabla \underset{\mathrm{loc}}{=} \hbar\mathrm{d} - \begin{bmatrix} 0 & -q \\ 1 & 0 \end{bmatrix} \mathrm{d}x = \nabla_0 - \phi$
- Riccati equation: $\hbar \partial_x s + s^2 + q = 0$

Thank you for your attention!