Instantons and Renormalons in the O(3) Sigma Model: The Full Resurgent Transseries

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- Introduction: TBA integral equations
- Wiener-Hopf solution of the TBA integral equations
- Building blocks: TBA asymptotic series and transseries solution
- The main conjecture and median resummation of the perturbative series

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Quantization and Resurgence, Les Diablerets, 31 January 2023

TBA(like) integral equation

TBA(-like) integral equation:

$$\chi_n(\theta) - \int_{-B}^{B} \mathrm{d}\theta' K(\theta-\theta')\chi_n(\theta') = \cosh n\theta \qquad |\theta| \le B \qquad n \ge 0$$

TBA context: 2-dim integrable QFT;
$K(\theta)$: logarithmic derivative of S-matrix
$(\mu\cosh\theta, \mu\sinh\theta)$ relativistic 2-momentum
B: (bosonic) fermi-rapidity

densities:

$$O_{n,m} = \frac{1}{2\pi}\int_{-B}^{B} \mathrm{d}\theta \chi_n(\theta)\cosh m\theta \qquad m \ge 0$$

explicit B-dependence dropped:

$$\chi_n(\theta) \sim \chi_n(\theta, B) \qquad O_{n,m} \sim O_{n,m}(B)$$

Relativistic QFT:

$$\varepsilon = \mu^2 O_{1,1} \quad \rho = \mu O_{1,0} \qquad \{\varepsilon(B); \rho(B)\} \Longrightarrow \varepsilon = \varepsilon(\rho)$$

Free energy (Legendre transform)

$$\varepsilon(\rho) \longrightarrow \mathcal{F}(h) = \varepsilon - \rho h \Big|_{h=\frac{\partial \varepsilon}{\partial \rho}}$$

O(N) nonlinear sigma model:

$$\mathcal{F}(h) = -\frac{h^2}{4\pi\Delta}\left(\frac{1}{\gamma} + c - \frac{1}{2} - \frac{\Delta\gamma}{2} + \mathrm{O}(\gamma^2)\right) \qquad \Delta = \frac{1}{N-2}$$

Wiener-Hopf method; NP running coupling γ ($h \to \infty$, $\gamma \to 0$):

$$\frac{1}{\gamma} + \Delta \ln \gamma = \ln \frac{h}{\mu}$$

Ordinary PT:

$$\mathcal{F}(h) = -\frac{h^2}{4\pi\Delta}\left(\frac{1}{\alpha} - \frac{1}{2} - \frac{\Delta\alpha}{2} + \mathrm{O}(\alpha^2)\right)$$

(only these 3 terms are known from explicit perturbative calculations)

$$\frac{1}{\alpha} + \Delta \ln \alpha = \ln \frac{h}{\Lambda} \qquad \Lambda = \Lambda_{\overline{\mathrm{MS}}}$$

1^{st} application of $\mathcal{F}(h)$ as a tool: exact μ/Λ ratio:

$$\frac{\mu}{\Lambda} = \mathrm{e}^{c} = \left(\frac{8}{\mathrm{e}}\right)^{\Delta} \frac{1}{\Gamma(1+\Delta)}$$

Polyakov, Wiegmann '83   Hasenfratz, Maggiore, Niedermayer '90

Lieb-Liniger physics:

(same kernel as for O(3) NLS model; θ: velocity)

$$\psi_k = \left(\frac{\mathrm{d}}{\mathrm{d}m}\right)^k O_{0,m}\Big|_{m=0} = \frac{1}{2\pi}\int_{-B}^{B} \mathrm{d}\theta \chi_0(\theta)\theta^k$$

$\psi_0 \sim$ particle density $\qquad$ $\psi_2 \sim$ NR energy density

$$\frac{\gamma}{\pi} = \frac{1}{O_{0,0}} \qquad e_2 = \left(\frac{\gamma}{\pi}\right)^3 \psi_2 \qquad \left\{e_2(B), \gamma(B)\right\} \longrightarrow e_2(\gamma)$$

Disk capacitor:

(O(3) NLS model kernel)

$O_{0,0}$: ES capacity of two conducting parallel disks

Volin's method

combination of WH method and resolvent expansion

Volin '09

allows calculation of very long PT series and study of Borel plane

NR models: Lieb-Liniger, Gaudin-Yang, Hubbard

Marino, Reis '19

Disk capacitor

Reichert, Ristivojevic '20

Relativistic integrable QFT

Marino, Miravitllas, Reis '19-'22   Abbott, Bajnok, B., Hegedus, Vona '20-'22

Large N

Marino, Miravitllas, Reis '21   DiPietro, Marino, Sberveglieri, Serone '21

Differential relations among densities

$$O_{n,m} = O_{m,n} \qquad (I)\ \dot{O}_{n,m} = \frac{1}{\pi}\rho_n\rho_m \qquad \dot{\phi} = \frac{\mathrm{d}\phi}{\mathrm{d}B}$$

where

$$\rho_n = \chi_n(B) \ \ [\sim \chi_n(B,B)]$$

$$(II)\ \ddot{\rho}_n - n^2\rho_n = \mathcal{F}\rho_n \qquad \mathcal{F} = \mathcal{F}(B) \ \ \{n\ \text{–independent}\}$$

Ristivojevic '22

1 density determines all:

$$O_{1,1} \longrightarrow \rho_1 \longrightarrow \mathcal{F} \longrightarrow \rho_n \longrightarrow O_{n,m}$$

Wiener-Hopf solution of TBA integral equations

Riemann-Hilbert problem

$$1-\tilde{K}(\omega)=\frac{1}{G_+(\omega)G_-(\omega)} \qquad G_-(\omega)=G_+(-\omega)$$

O(N) NLS model

$$\tilde{K}(\omega)=\mathrm{e}^{-\pi\Delta|\omega|}\frac{\cosh\frac{\pi}{2}(1-2\Delta)\omega}{\cosh\frac{\pi}{2}\omega} \qquad \left\{\tilde{K}(\omega)=\mathrm{e}^{-\pi|\omega|} \quad \mathrm{O}(3)\right\}$$

$$\sigma(\omega)=\frac{G_-(\omega)}{G_+(\omega)}: \quad \text{cut and poles along positive imaginary axis}$$

poles and residues:

$$\sigma(iz\pm\epsilon)\approx\mp i\frac{S_\ell}{z-2\ell\xi_o} \qquad \ell=1,2,\dots$$

$$\xi_o=\begin{cases}\frac{N-2}{2} & N\ \text{even}\\ N-2 & N\ \text{odd}\end{cases} \qquad [\xi_o=1\ \text{for O(3), O(4)}]$$

discontinuity:

$$\sigma(iz+\epsilon)-\sigma(iz-\epsilon)=-2i\beta(z)$$

$\beta(z)$: meromorphic with poles at $z=2\ell\xi_o$

expansion in terms of running coupling:

$$2B=\tfrac{1}{v}+\gamma\ln v+L \qquad \gamma=2\Delta-1$$

$$\mathrm{e}^{-2Bvx}\beta(vx)=\mathrm{e}^{-x}\mathcal{A}(x) \qquad \{\mathcal{A}(x)\sim\mathcal{A}(x,v)\}$$

$$\mathcal{A}(x)=1+\sum_{k=1}^{\infty}(vx)^k\,L_k(\ln x)$$

(L_k polynomial of degree k)

$\nu = \mathrm{e}^{-2B}$: NP expansion parameter v: PT expansion parameter

$$c_n = \nu^n \sigma(in + \epsilon)$$

reduced densities:

$$O_{n,m} = \tfrac{1}{4\pi} G_+(in) G_+(im) \mathrm{e}^{(n+m)B} W_{n,m}$$
$$\rho_n = \tfrac{1}{2} G_+(in) \mathrm{e}^{nB} w_n$$

WH integral equation:

$$Q_n(x) + \tfrac{c_n}{n+vx} + i \sum_{\ell=1}^{\infty} \frac{\nu^{2\ell\xi_o} S_\ell q_{n,\ell}}{2\ell\xi_o + vx} + \tfrac{1}{\pi} \int_{\mathcal{C}_+} \frac{\mathrm{e}^{-y} \mathcal{A}(y) Q_n(y)}{x+y} \mathrm{d}y = \tfrac{1}{n - vx}$$

$$q_{n,\ell} = Q_n\left(\tfrac{2\ell\xi_o}{v}\right)$$

$\mathcal{C}_+$: (just) above the real line

$Q_n(x)$: meromorphic upper half-plane, only pole at $x = n/v$

$$q_{n,s} + \frac{c_n}{n+2s\xi_o} + \frac{i}{2\xi_o}\sum_{\ell=1}^{\infty}\frac{\nu^{2\ell\xi_o}S_\ell q_{n,\ell}}{\ell+s} + \frac{v}{\pi}\int_{\mathcal{C}_+}\frac{\mathrm{e}^{-x}\mathcal{A}(x)Q_n(x)}{2s\xi_o+vx}\,\mathrm{d}x = \frac{1}{n-2s\xi_o}$$

densities:

$$w_n = 1 + c_n + i\sum_{\ell=1}^{\infty}\nu^{2\ell\xi_o}S_\ell q_{n,\ell} + \frac{v}{\pi}\int_{\mathcal{C}_+}\mathrm{e}^{-x}\mathcal{A}(x)Q_n(x)\mathrm{d}x$$

conditions: $n, m > 0 \quad n \neq m \quad n, m \neq 2\ell\xi_o$

$$W_{n,m} = \frac{1}{n+m} + \frac{c_m - c_n}{n-m} + c_m k_{n,m} + i\sum_{\ell=1}^{\infty}\frac{\nu^{2\ell\xi_o}S_\ell q_{n,\ell}}{m-2\ell\xi_o} + \frac{v}{\pi}\int_{\mathcal{C}_+}\frac{\mathrm{e}^{-x}\mathcal{A}(x)Q_n(x)}{m-vx}\mathrm{d}x$$

$$k_{n,m} = -\frac{c_n}{n+m} - i\sum_{\ell=1}^{\infty}\frac{\nu^{2\ell\xi_o}S_\ell q_{n,\ell}}{m+2\ell\xi_o} - \frac{v}{\pi}\int_{\mathcal{C}_+}\frac{\mathrm{e}^{-x}\mathcal{A}(x)Q_n(x)}{m+vx}\mathrm{d}x$$

Building blocks: TBA asymptotic series and transseries

Purely perturbative:

$$P_\alpha(x) + \frac{1}{\pi}\int_{\mathcal{C}_+} \frac{\mathrm{e}^{-y}\mathcal{A}(y)P_\alpha(y)}{x+y}\,\mathrm{d}y = \frac{1}{\alpha - vx}$$

well-defined (as asymptotic series) for all $\alpha \neq 0$

$$P_{\alpha,\beta} = \frac{v}{\pi}\int_{\mathcal{C}_+} \frac{\mathrm{e}^{-x}\mathcal{A}(x)P_\alpha(x)}{\beta - vx}\,\mathrm{d}x$$

well-defined (as asymptotic series) for all $\alpha, \beta \neq 0$
perturbatively:

$$W_{\alpha,\beta} \longrightarrow A_{\alpha,\beta} = \frac{1}{\alpha+\beta} + P_{\alpha,\beta}$$

well-defined for all $\alpha, \beta, \alpha + \beta \neq 0$

$$w_\alpha \longrightarrow a_\alpha = 1 + \frac{v}{\pi}\int_{\mathcal{C}_+} \mathrm{e}^{-x}\mathcal{A}(x)P_\alpha(x)\,\mathrm{d}x = \lim_{\beta\to\infty} \beta A_{\alpha,\beta}$$

differential relations for the reduced densities:

$$(n+m)W_{n,m} + \dot{W}_{n,m} = w_n w_m$$
$$\ddot{w}_n + 2n\dot{w}_n = \mathcal{F} w_n$$

perturbative limit:

$$(n+m)A_{n,m} + \dot{A}_{n,m} = a_n a_m$$
$$\ddot{a}_n + 2n\dot{a}_n = \mathcal{F}_o a_n$$

Volin's result determines all:

$$A_{1,1} \longrightarrow a_1 \longrightarrow \mathcal{F}_o \longrightarrow a_n \longrightarrow A_{n,m}$$

Volin's result for $A_{1,1}$:

$$2A_{1,1} = 1 + \frac{v}{2} + \left(\frac{5\gamma}{4} + \frac{9}{8}\right) v^2 + \left(\frac{10\gamma^2}{3} + \frac{53\gamma}{8} + \frac{57}{16}\right) v^3 + \frac{v^4}{384}\left(-36\gamma^3(21\zeta_3 - 94) + 10924\gamma^2 + 13344\gamma + 9(144z_3 + 625)\right) + \cdots$$

where

$$z_{2k+1} = 2\frac{\zeta_{2k+1}}{2k+1}(\Delta^{2k+1} - 1 + 2^{-2k-1}) \quad .$$

next step

$$a_1 = 1 + \frac{v}{4} + \left(\frac{5\gamma}{8} + \frac{9}{32}\right) v^2 + \left(\frac{5\gamma^2}{3} + \frac{53\gamma}{32} + \frac{75}{128}\right) v^3 + \frac{v^4\left(-288\gamma^3(21\zeta_3 - 94) + 43696\gamma^2 + 35160\gamma + 9(1152z_3 + 1225)\right)}{6144} + \cdots$$

next step

$$\mathcal{F}_o = -v^2 - 6\gamma v^3 - 26\gamma^2 v^4 + v^5\left(\frac{1}{4}\gamma^3(63\zeta_3 - 386) - 27z_3\right) + \cdots$$

next step

$$a_n = 1 + \frac{v}{4n} + \frac{v^2(20\gamma n+9)}{32n^2} + \frac{v^3\left(640\gamma^2 n^2+636\gamma n+225\right)}{384n^3}$$
$$+ \frac{v^4\left(288n^3\left(\gamma^3(94-21\zeta_3)+36z_3\right)+43696\gamma^2n^2+35160\gamma n+11025\right)}{6144n^4} + \cdots$$

finally

$$A_{n,m} = \frac{1}{m+n} + \frac{v}{4mn} + \frac{v^2(20\gamma mn+9m+9n)}{32m^2n^2}$$
$$+ \frac{v^3\left(m^2\left(640\gamma^2n^2+636\gamma n+225\right)+6mn(106\gamma n+39)+225n^2\right)}{384m^3n^3} + \cdots$$

transseries solution of the linear problem:

$$\tilde{Q}_n(x) = P_n(x) + c_n P_{-n}(x) + i \sum_{\ell=1}^{\infty} \nu^{2\ell\xi_o} S_\ell \tilde{q}_{n,\ell} P_{-2\ell\xi_o}(x)$$

Complete (transseries) solution:

$$\tilde{q}_{n,s} - i \sum_{\ell=1}^{\infty} \nu^{2\ell\xi_o} S_\ell \tilde{q}_{n,\ell} A_{-2\ell\xi_o,-2s\xi_o} = A_{n,-2s\xi_o} + c_n A_{-n,-2s\xi_o}$$

$$\tilde{W}_{n,m} = A_{n,m} + i \sum_{\ell=1}^{\infty} \nu^{2\ell\xi_o} S_\ell \tilde{q}_{n,\ell} A_{-2\ell\xi_o,m} + c_n A_{-n,m} + c_m A_{n,-m}$$

$$+ c_n c_m A_{-n,-m} + i c_m \sum_{\ell=1}^{\infty} \nu^{2\ell\xi_o} S_\ell \tilde{q}_{n,\ell} A_{-2\ell\xi_o,-m}$$

$A_{\alpha,\beta}$ (PT) and Stokes constants S_ℓ (NP) universal building blocks!

Energy density $\tilde{W}_{1,1}$

$$n \approx 1 \qquad c_n = \nu\Big\{h_o + \tfrac{M}{2}(1-n) + \mathrm{O}\big((1-n)^2\big)\Big\} \qquad h_0 = \frac{\delta_{N,3}}{\mathrm{e}}$$

$N \geq 4$ and $N = 3$ cases drastically different!

$$M = \begin{cases} -2\mathrm{e}\left(\frac{\Delta}{\mathrm{e}}\right)^{2\Delta}\frac{\Gamma(1-\Delta)}{\Gamma(1+\Delta)}\mathrm{e}^{i\pi\Delta} & N \geq 4 \\ -\frac{i\pi}{\mathrm{e}} + \frac{2}{\mathrm{e}}(2B + 1 - \gamma_E - \ln 2) & N = 3 \end{cases}$$

$n = 1 \quad m = 1 + \delta \to 1$

$N \geq 4$

$$\tilde{q}_{1,s} - i \sum_{\ell=1}^{\infty} \nu^{2\ell\xi_o} S_\ell \tilde{q}_{1,\ell} A_{-2\ell\xi_o,-2s\xi_o} = A_{1,-2s\xi_o}$$

$$\tilde{W}_{1,1} = A_{1,1} + \tfrac{M\nu}{2} + i \sum_{\ell=1}^{\infty} \nu^{2\ell\xi_o} S_\ell \tilde{q}_{n,\ell} A_{1,-2\ell\xi_o}$$

$N = 3$

$$\tilde{q}_{1,s} - i \sum_{\ell=1}^{\infty} \nu^{2\ell} S_\ell \tilde{q}_{1,\ell} A_{-2\ell,-2s} = A_{1,-2s} + \tfrac{\nu}{\mathrm{e}} A_{-1,-2s}$$

$$\tilde{W}_{1,1} = A_{1,1} + \tfrac{M\nu}{2} + \tfrac{2\nu}{\mathrm{e}} P_{1,-1} + \tfrac{\nu^2}{\mathrm{e}^2} A_{-1,-1}$$

$$+ i \sum_{\ell=1}^{\infty} \nu^{2\ell} S_\ell \tilde{q}_{n,\ell} \left[A_{1,-2\ell} + \tfrac{\nu}{\mathrm{e}} A_{-1,-2\ell} \right]$$

Compact form $N \geq 4$

$$\mathcal{N} \sim \mathcal{N}_{ab} = A_{-2a\xi_o,-2b\xi_o}$$

$$Y \sim Y_\ell = A_{1,-2\ell\xi_o}$$

$$q \sim \tilde{q}_{1,\ell}$$

$$\mathcal{D} \sim \mathcal{D}_{ab} = i\delta_{ab}\nu^{2a\xi_o}S_a$$

$$\mathcal{M} = \mathcal{N}\mathcal{D}$$

Neumann series solution

$$(1-\mathcal{M})q = Y \qquad q = \bar{\mathcal{M}}Y \qquad \bar{\mathcal{M}} = (1-\mathcal{M})^{-1} = \sum_{k=0}^{\infty} \mathcal{M}^k$$

energy density in compact form

$$\tilde{W} = 2\tilde{W}_{1,1} = (M_0 + iM_1)\nu + f + 2Y^{\mathrm{T}}\mathcal{D}\bar{\mathcal{M}}Y$$

where

$$M = M_0 + iM_1 \qquad f = 2A_{1,1}$$

Compact form $N = 3$

$$Z \sim Z_\ell = \tfrac{1}{\mathrm{e}} A_{-1,-2\ell}$$

Neumann series solution

$$(1 - \mathcal{M})q = Y + \nu Z \qquad q = \bar{\mathcal{M}} Y + \nu \bar{\mathcal{M}} Z$$

energy density in compact form

$$\tilde{W} = 2\tilde{W}_{1,1} = f^{(0)} + \nu f^{(1)} + \nu^2 f^{(2)}$$
$$iM_1\nu + 2Y^{\mathrm{T}}\mathcal{D}\bar{\mathcal{M}}Y + 4\nu Z^{\mathrm{T}}\mathcal{D}\bar{\mathcal{M}}Y + +2\nu^2 Z^{\mathrm{T}}\mathcal{D}\bar{\mathcal{M}}Z$$

where

$$f^{(0)} = 2A_{1,1} \qquad f^{(1)} = M_0 + \tfrac{4}{\mathrm{e}} P_{1,-1} \qquad f^{(2)} = \tfrac{2}{\mathrm{e}^2} A_{-1,-1}$$

The main conjecture and median resummation of the asymptotic series

Main conjecture

$$\boxed{W = W_{\rm TBA} = S_+(\tilde{W})}$$

ambiguity: $S_-(\tilde{W})$ if we use $\mathcal{C}_+ \to \mathcal{C}_-$ and $c_n \to \nu^n \sigma(in - \epsilon)$

Stokes automorphism:

$$\mathfrak{S} = \exp\{-\tilde{D}\} \qquad \tilde{D} = \nu \Delta_1 + D \quad D = \sum_{k=1}^{\infty} \nu^{2k\xi_o} \Delta_{2k\xi_o}$$

modified alien derivatives:

$$\Delta_\omega = \left(v^\gamma {\rm e}^L\right)^\omega \Delta_\omega^{\rm st} \qquad N = 4 \;\; \Delta_\omega = \Delta_\omega^{\rm st} \qquad N = 3 \;\; \Delta_\omega = \left(\frac{{\rm e}v}{8}\right)^\omega \Delta_\omega^{\rm st}$$

for $\mathcal{F}$ `real` asymptotic series:

$$2i{\rm Im}\, S_+(\mathcal{F}) = S_+\left[(1 - \mathfrak{S}^{-1})\mathcal{F}\right]$$

$N \geq 4$

$$\tilde{W} = (M_0 + iM_1)\nu + f + 2iS_1 Y_1^2 \nu^{2\xi_o} + (2iS_2 Y_2^2 - 2S_1^2 Y_1^2 \mathcal{N}_{11})\nu^{4\xi_o} + \mathrm{O}(\nu^{6\xi_o})$$

O(ν):

$$W = \mathrm{Re}\, S_+(f) + S_+\big\{M_0\nu + iM_1\nu - \tfrac{\nu}{2}\Delta_1 f\big\} + \mathrm{O}(\nu^{2\xi_o})$$

reality condition:

$$\Delta_1 A_{1,1} = iM_1 \text{ (const.)} \Longrightarrow \Delta_1 A_{n,m} = 0$$

Stokes automorphism simplifies:

$$D^k \Delta_1 f = \Delta_1 D^k f = 0 \qquad \mathfrak{S}^{-1} f = iM_1\nu + \exp\{-D\} f$$

O$(\nu^{2\xi_o})$:

$$W = \mathrm{Re}\, S_+(f) + S_+\left\{M_0\nu - \tfrac{1}{2}\nu^{2\xi_o}\Delta_{2\xi_o}f + 2iS_1\nu^{2\xi_o}Y_1^2\right\} + \mathrm{O}(\nu^{4\xi_o})$$

reality condition:

$$\Delta_{2\xi_o}A_{1,1} = 2iS_1A^2_{1,-2\xi_o}$$

Lemma:

$$\Delta_\omega A_{1,1} = \xi_\omega A^2_{1,-\omega} \Longrightarrow \Delta_\omega A_{n,m} = \xi_\omega A_{n,-\omega}A_{m,-\omega}$$

all $\Delta_{2\xi_o}$ alien derivatives known

O($\nu^{4\xi_o}$):

$$W = \mathrm{Re}\, S_+(f) + S_+\big\{M_0\nu - \tfrac{1}{2}\nu^{4\xi_o}\Delta_{4\xi_o} f - \tfrac{1}{4}\Delta^2_{2\xi_o} f \nu^{4\xi_o} + (2iS_2Y_2^2 - 2S_1^2Y_1^2\mathcal{N}_{11})\nu^{4\xi_o}\big\} + \mathrm{O}(\nu^{6\xi_o})$$

reality condition:

$$\Delta_{4\xi_o}A_{1,1} = 2iS_2A^2_{1,-4\xi_o} \Longrightarrow \Delta_{4\xi_o}A_{n,m} = 2iS_2A_{n,-4\xi_o}A_{m,-4\xi_o}$$

next step:

$$W = \mathrm{Re}\, S_+\big\{f + M_0\nu + 2S_1^2A^2_{1,-2\xi_o}A_{-2\xi_o,-2\xi_o}\nu^{4\xi_o}\big\} + \mathrm{O}(\nu^{6\xi_o})$$

$$\vdots$$

step by step full resurgence structure (closed $A_{n,m}$):

$$\Delta_{2k\xi_o}A_{1,1} = 2iS_kA^2_{1,-2k\xi_o} \Longrightarrow \Delta_{2k\xi_o}A_{n,m} = 2iS_kA_{n,-2k\xi_o}A_{m,-2k\xi_o}$$

Median resummation

$$D = \sum_{k=1}^{\infty} \nu^{2k\xi_o} \Delta_{2k\xi_o}$$

$$DY = 2\mathcal{M}Y \qquad DY^{\mathrm{T}}\mathcal{D} = 2Y^{\mathrm{T}}\mathcal{D}\mathcal{M} \qquad D\mathcal{M} = 2\mathcal{M}^2$$

$$\begin{aligned}
Df &= 4Y^{\mathrm{T}}\mathcal{D}Y \\
D^2 f &= 16Y^{\mathrm{T}}\mathcal{D}\mathcal{M}Y \\
D^3 f &= 96Y^{\mathrm{T}}\mathcal{D}\mathcal{M}^2 Y \\
&\vdots \\
D^k f &= 2^{k+1}k!Y^{\mathrm{T}}\mathcal{D}\mathcal{M}^{k-1}Y
\end{aligned}$$

Ambiguity free `median` resummation:

$$S_{\text{med}}(f) = S_-(\mathfrak{S}^{1/2} f) = S_+(\mathfrak{S}^{-1/2} f)$$

I.Aniceto, R.Schiappa '13

In our case:

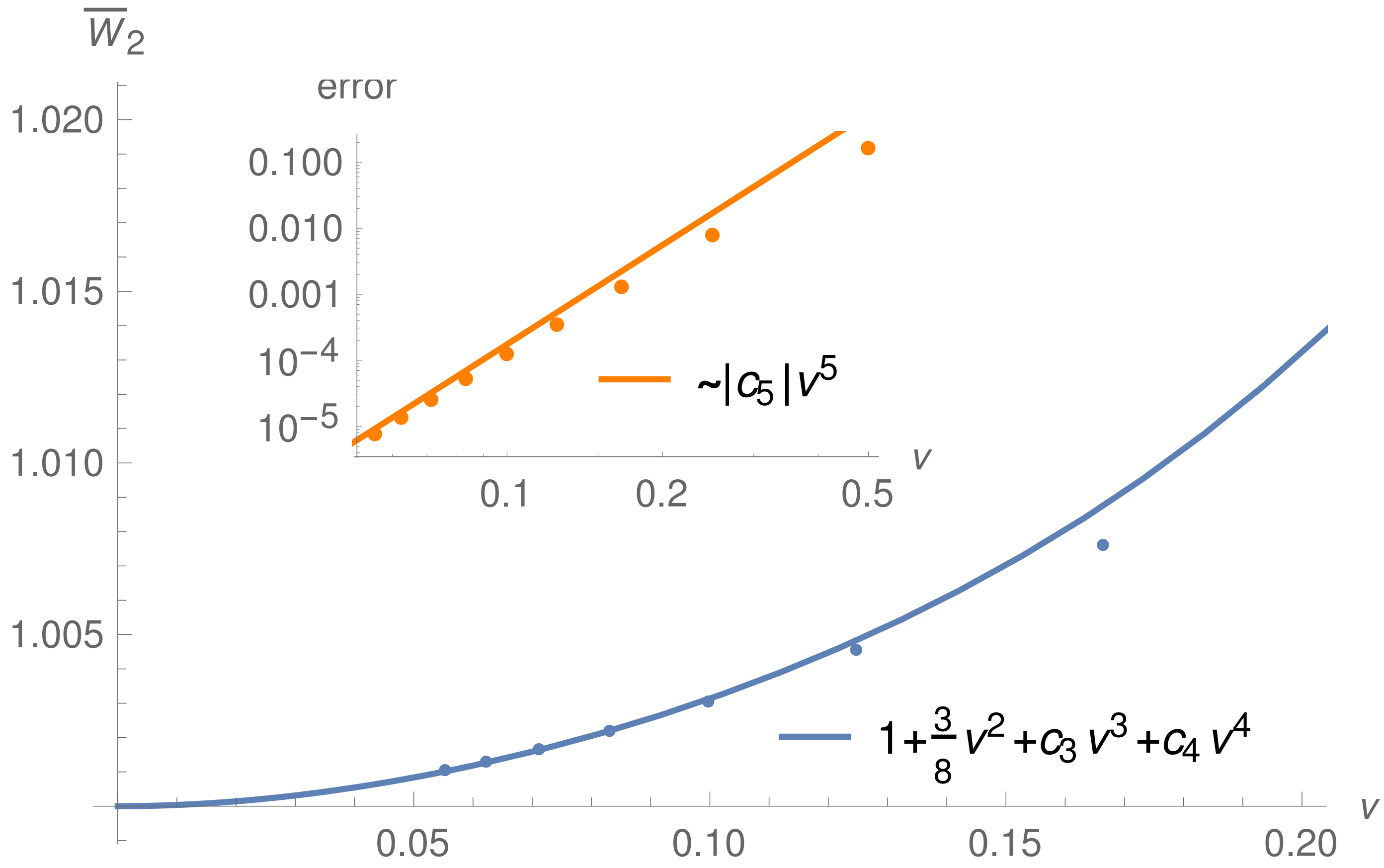
$$\sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{D}{2}\right)^k f = 2 \sum_{k=1}^{\infty} Y^{\mathrm{T}} \mathcal{D} \mathcal{M}^{k-1} Y = 2 Y^{\mathrm{T}} \mathcal{D} \bar{\mathcal{M}} Y$$

$$\mathfrak{S}^{-1/2} f = f + \tfrac{1}{2} \nu \Delta_1 f + 2 Y^{\mathrm{T}} \mathcal{D} \bar{\mathcal{M}} Y$$

$$\tilde{W} = M_0 \nu + \mathfrak{S}^{-1/2} f$$

Stronger conjecture:

$$\boxed{W = W_{\text{TBA}} = S_+(\tilde{W}) = M_0 \nu + S_{\text{med}}(f)}$$



Plot of $-8\mathrm{e}^{8B}\left[W_{\mathrm{TBA}} - \mathrm{Re}S_+(f)\right]$ for $\mathrm{O}(4)$

Difference absolute magnitude for smallest v: $\sim 10^{-34}$

conclusion: $W_{\mathrm{exact}} = W_{\mathrm{TBA}} = S_{\mathrm{med}}(f)$

Strongest resurgence!

The exceptional $O(3)$ case

Main conjecture ($N = 3$)

$$\boxed{W = W_{\mathrm{TBA}} = S_+(\tilde{W})}$$

reality of W (recursively):

$$\Delta_1 f^{(a)} = 2iM_1\delta_{a,0} \qquad a = 0, 1, 2$$

$$\Delta_{2k}A_{1,1} = 2iS_k A_{1,-2k}^2 \Longrightarrow \Delta_{2k}A_{n,m} = 2iS_k A_{n,-2k}A_{m,-2k}$$

calculation of Stokes automorphism:

$$\tilde{D} = \nu\Delta_1 + D \qquad D = \sum_{k=1}^{\infty} \nu^{2k}\Delta_{2k}$$

$$\mathrm{e}^{\alpha\tilde{D}} f^{(a)} = 2i\alpha M_1 \nu\delta_{a,0} + \mathrm{e}^{\alpha D} f^{(a)} \qquad a = 0, 1, 2$$

extra building blocks Z:

$$DZ = 2\mathcal{M}Z$$

$$Df^{(0)} = 4Y^{\mathrm{T}}\mathcal{D}Y \qquad Df^{(1)} = 8Z^{\mathrm{T}}\mathcal{D}Y \qquad Df^{(2)} = 4Z^{\mathrm{T}}\mathcal{D}Z$$

Stokes automorphism:

$$\mathrm{e}^{\frac{1}{2}\tilde{D}}f^{(0)} = iM_1\nu + f^{(0)} + 2Y^{\mathrm{T}}\mathcal{D}\bar{\mathcal{M}}Y$$

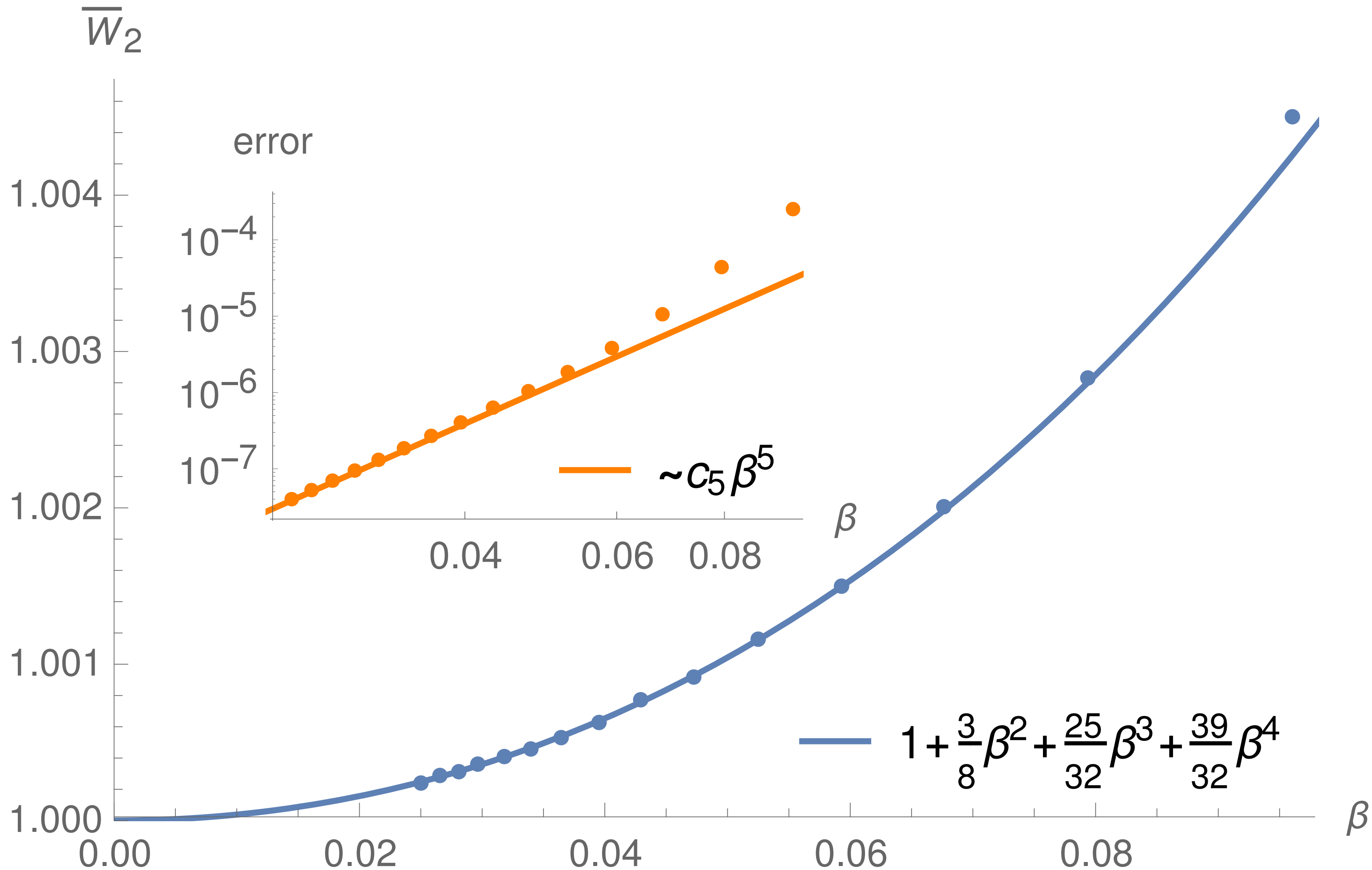
$$\mathrm{e}^{\frac{1}{2}\tilde{D}}f^{(1)} = f^{(1)} + 4Z^{\mathrm{T}}\mathcal{D}\bar{\mathcal{M}}Y$$

$$\mathrm{e}^{\frac{1}{2}\tilde{D}}f^{(2)} = f^{(2)} + 2Z^{\mathrm{T}}\mathcal{D}\bar{\mathcal{M}}Z$$

summing up:

$$\tilde{W} = \mathfrak{S}^{-1/2}\left[f^{(0)} + \nu f^{(1)} + \nu^2 f^{(2)}\right]$$

$$\boxed{W = W_{\mathrm{TBA}} = S_{+}(\tilde{W}) = S_{\mathrm{med}}\left[f^{(0)} + \nu f^{(1)} + \nu^2 f^{(2)}\right]}$$



Plot of $-\frac{e^4}{2}e^{8B}\left\{W_{\mathrm{TBA}} - \mathrm{Re}S_+\left[f^{(0)} + \nu f^{(1)} + \nu^2 f^{(2)}\right]\right\}$ for O(3)

Difference absolute magnitude for smallest β: $\sim 10^{-63}$

conclusion: resurgence for $0+1+2$ instanton sectors!

Instantons and/or renormalons?

- found (by WH) $\tilde{W}$ as transseries
- is $S_+(\tilde{W})$ convergent?
- found (by WH) $f^{(a)}$ $a = 0, 1, 2$ sectors for O(3)
- $f^{(a)}$ a-instanton sector?
- adding θ-term at $\theta = \pi$ gives $f^{(a)} \Longrightarrow (-1)^a f^{(a)}$

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TO DO

- construct by direct field theory calculation instanton sectors
- numerical study of the convergence of the transseries

Thank you!