Resurgence in Integrable Field Theories

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January 2023

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Introduction with a brief historical excursus

The two main pillars of resurgence are:

1) Observables can be written as an infinite number of ``sufficiently simple" series. One of such series corresponds to the perturbative expansion, the others to series which include non-perturbative corrections.

2) Most (or all) of the information encoded in the trans-series can be reconstructed using only the perturbative series.

While we have several beautiful applications of resurgence in different contexts, understanding if and how resurgence works in Quantum Field Theory (QFT) (no TQFT, no SUSY) is to a large extent an open question.

One obvious reason: computing in QFT the many perturbative orders necessary to "activate" the resurgence program is hard!

Before the advent of resurgence in QFT

Early works were focused on understanding the large order behaviour of perturbative series and ways to possibly resum it.

Borel resummation of the perturbative series was thus considered essential.

In QM and QFT series are asymptotics because of the factorial growth of their large order coefficients

$$Z(\lambda) \sim \sum_{n=0}^{\infty} Z_n \lambda^n \qquad Z_n \approx c\, n! a^n n^b \Big(1 + \mathcal{O}(1/n) \Big) + c.c.$$

Factorial growth gives rise to a pole or branch-cut singularity of the Borel function $\mathcal{B}Z(t)$ at $t = \frac{1}{a}$

A general source for the factorial growth:

O(1) contributions of O(n!) Feynman diagrams

These are governed by **instantons** whose action determines the growing factor. Relatively well-understood.

[Vainhstein 1964; Lam 1968; Bender, Wu 1969; Lipatov 1976; …]

In super-renormalizable QFTs this is typically the only source of factorial growth.

In special cases, like the anharmonic oscillator in QM, and 2d or 3d ϕ^4 theories at weak coupling, perturbation theory was proved to be Borel resummable

[Loeffel et al, 1969; Simon and Dicke, 1970; Eckmann, Magnen and Seneor, 1975; Magnen and Seneor, 1977; MS, Spada, Villadoro, 1805.05882; Sberveglieri, MS, Spada, 1905.02122]

Borel resummation of quartic scalar models and the conjectured Borel resummation of the epsilon expansion in quartic scalar theories led to an active research program over the years in the statistical mechanics community to compute critical exponents using numerical resummation methods [Guida, Zinn-Justin, 1998 + …]

Very successful program. With a few (5-6) perturbative coefficients precision of per cent or per mille can be achieved. Research program still active thanks to the improvements in Feynman diagrams computations. Current status is 7-loops analytic for the epsilon expansion [Schnetz, 2212.03663] and 7-loops numerics for the fixed dimension expansion [Sberveglieri, Spada, in progress].

According to resurgence, when a theory is non-Borel resummable we expect a trans-series. In QFT a trans-series is generally expected and given semiclassically by a sum over saddle points (instantons).

This is e.g. the case for quantum mechanical systems, where resurgence has been shown to work in various instances.

[Voros, 1983; Delabaere, Dillinger, Pham, 1997; Zinn-Justin, Jentschura, 2005; …]

In certain cases in quantum mechanics, the resurgence program can be successfully (and implicitly) completed by repackaging the trans-series to an appropriate Borel resummable series.

[MS, Spada, Villadoro, 2016, 2017]

However, in QFT the story is more complicated. Most interesting theories, such as 4d gauge theories, are just renormalizable.

In such theories the coupling constant turns into a mass scale quantum mechanically and it is no longer a well-defined expansion parameter. Couplings run!

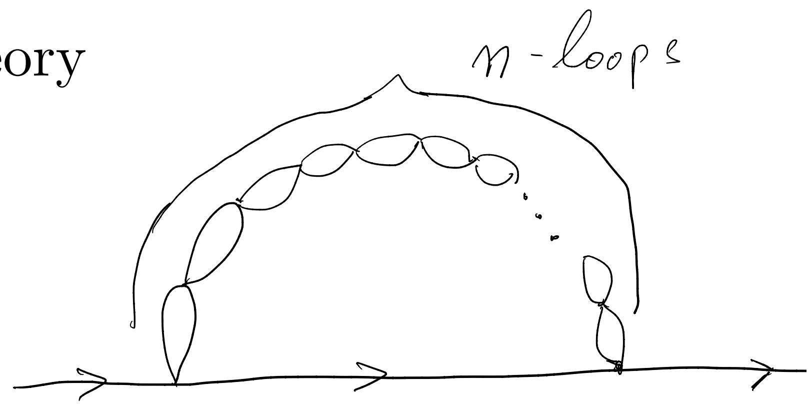
As a consequence, specific Feynman diagrams, related to those which are resummed by the renormalization group (bubble-like diagrams) gives rise to a factorial growth by themselves.

$\boxed{\text{O(n!) contributions of O(1) Feynman diagrams}}$

Singularities in the Borel function are called **renormalons**.

[Gross, Neveu 1974; Lautrup 1977;'t Hooft 1977; …]

E.g. in 4d ϕ^4 theory



$\propto n!$

Renormalons arise in two classes, so called UV and IR renormalons.

In all known examples renormalon singularities make the perturbative series non-Borel resummable.

UV renormalons for $\beta_0 > 0$

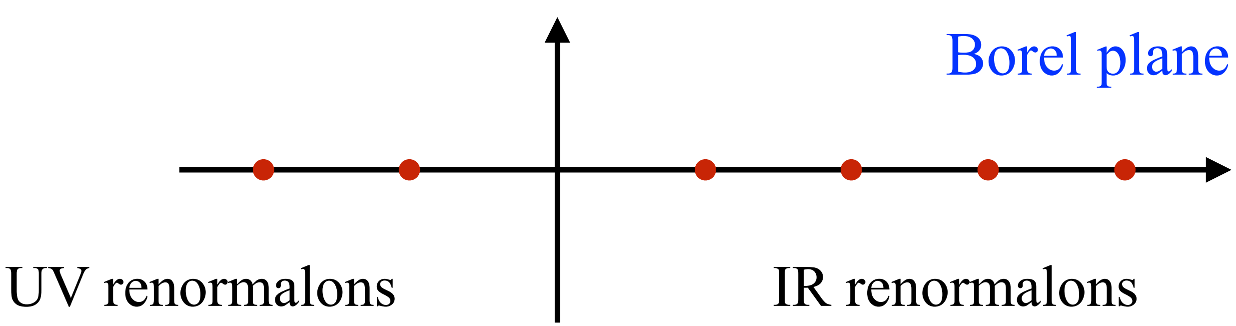
IR renormalons for $\beta_0 < 0$

Physical observables are plagued with renormalon singularities in a renormalization-scheme independent way.

Renormalon singularities are expected to occur at specific locations in the coupling constant Borel t-plane, determined by β_0

They typically dominate over instanton-anti instanton singularities.

E.g., in non-abelian gauge theories



$$t = \frac{n}{\beta_0}$$

$$n \in \mathbb{Z}$$

This pattern has soon be related to the OPE expansion [Parisi, 1979]. In a perturbative computation in a UV-free theory all condensates vanish. On the other hand we expect

$$\langle O(x)O(0)\rangle \propto C_I(\mu x) + \sum_{\mathcal{O}} C_{\mathcal{O}}(\mu x) x^{\Delta_{\mathcal{O}}} \langle \mathcal{O}(0)\rangle$$

Perturbative series

Non-perturbative contributions

By dimensional analysis in the free UV theory

$$\langle \mathcal{O}(0)\rangle \propto \Lambda^{\Delta_{\mathcal{O}}} \sim e^{\frac{\Delta_{\mathcal{O}}}{\beta_0}}$$

Scaling dimensions in free theories are integers $\Longrightarrow$ $t = \frac{n}{\beta_0}$

Not much is known about renormalon singularities, in particular for the most interesting ones, IR renormalons in 4d non-abelian gauge theories.

Although we have little doubt about their presence in 4d gauge theories, as of today we do not even have an analytic proof of their existence.

For this reason, since the early days some attention has been paid to the 2d non-linear sigma model at large N.

These models are well-known to be UV free and gapped in the IR, so they are good laboratories for 4d non-abelian gauge theories.

Early works [David, 1982, 1984] were mostly focused in analytically proving the existence of renormalons and their properties, and understanding and checking the relation between them and the OPE.

Analysis based on large N QFT techniques.

The non-linear sigma model, as well as other similar 2d models, are actually **integrable** for finite N.

In an independent line of research, integrability has been used to compute the free energy F(h) of the QFT as a function of a chemical potential h for a U(1) subgroup of the global symmetry group.

$$F(h) = -\lim_{V,\beta\to\infty} \frac{1}{V\beta} \log \,\mathrm{tr}\, e^{-\beta(H-hQ)},$$

More precisely the relative free energy

$$\mathcal{F}(h) \equiv F(h) - F(0)$$

In integrable QFTs this quantity can be computed exactly for any N using the known S-matrix and a thermodynamic Bethe ansatz (TBA), in terms of a linear integral equation.

[Polyakov, Wiegmann, 1983; Hasenfratz, Maggiore, Niedermayer, 1990; Hasenfratz, Niedermayer, 1990; Balog et al, 1992; Forgacs, Niedermayer, Weisz, 1991]

Notable integrable 2d models we will focus on: non linear sigma model (NLSM), Gross-Neveu (GN) model and Principal Chiral Field (PCF) model, but other models have been studied as well

The NLSM is described by $$\mathcal{L} = \frac{1}{2}\partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi} \,, \qquad \boldsymbol{\phi} \cdot \boldsymbol{\phi} = \frac{N}{g^2}$$

$$G = SO(N)$$

The PCF model is described by $$\mathcal{L} = \frac{1}{g^2} \mathrm{tr}\left(\partial_\mu \Sigma \, \partial^\mu \Sigma^\dagger\right) , \qquad \Sigma^\dagger \Sigma = I$$

$\Sigma \in SU(N)$ $$G = SU(N)_L \times SU(N)_R$$

The GN model is described by $$\mathcal{L} = \frac{i}{2} \overline{\boldsymbol{\chi}} \cdot \partial\!\!\!/ \, \boldsymbol{\chi} + \frac{g^2}{8} \left(\overline{\boldsymbol{\chi}} \cdot \boldsymbol{\chi}\right)^2 ,$$

N Majorana fermions $$G = SO(N)$$

Let m be the mass of the lightest particle in the theory (mass gap).

Main motivation at that time was to find an exact relation between m and $\Lambda_{\overline{\mathrm{MS}}}$

This is obtained by comparing TBA (in terms of m) and QFT (in terms of $\Lambda_{\overline{\mathrm{MS}}}$) computations at large h.

When $h > m$, the vacuum is populated with such particles. Assume these are the only ones excited by h. Starting from their known **exact** 2-2 S-matrix amplitude $S(\theta)$, we can determine the particle and energy densities ρ and e as a function of the Bethe roots $\chi(\theta)$.

The assumption that the vacuum is populated with only one kind of particles has been confirmed by 1/N QFT computations, where this assumption is not needed. At finite N this is yet to be proved.

$$\chi(\theta) - \int_{-B}^{B} d\theta' \, K(\theta - \theta')\chi(\theta') = m \cosh\theta \qquad K(\theta) = \frac{1}{2\pi i}\frac{d}{d\theta}\log S(\theta)$$

$$e = \frac{m}{2\pi}\int_{-B}^{B} d\theta \, \chi(\theta)\cosh\theta, \qquad \rho = \frac{1}{2\pi}\int_{-B}^{B} d\theta \, \chi(\theta).$$

The free energy $\mathcal{F}(h)$ is the Legendre transform of e :

$$h \equiv \partial_\rho e(\rho)\,, \qquad \mathcal{F}(h) \equiv e(\rho) - \rho h\,.$$

Alternative (more convenient) formulation: introduce function $\epsilon(\theta)$ such that

$$\epsilon(\theta) - \int_{-B}^{B} d\theta' \, K(\theta - \theta')\epsilon(\theta') = h - m \cosh\theta\,,$$

$$\mathcal{F}(h) = -\frac{m}{2\pi}\int_{-B}^{B} d\theta \, \cosh\theta \epsilon(\theta)\,, \qquad \epsilon(\pm B) = 0$$

Sometimes it is useful to study expansion of $e(\rho)$ rather than $\mathcal{F}(h)$.

Recent results

After the advent of resurgence in QFT

Change of perspective: use integrability directly to get many perturbative orders, which is the first necessary step for a resurgence study [Mariño,Reis, 1909.12134; …].

The observable of interest is still the relative free energy.

When computed using TBA (or large N QFT techniques) it is a function of the chemical potential h and of the non-perturbative mass gap m.

In order to recast the results in terms of asymptotic expansions of ordinary perturbation theory, we have to define a (running) coupling constant of some kind.

Define 't Hooft coupling $\alpha \equiv 2\beta_0 g^2$

$$\beta(\alpha) = -\alpha^2 - \xi\alpha^3 + \mathcal{O}(\alpha^4)$$

$$\beta_0 \propto N \qquad \xi = \frac{\beta_1}{2\beta_0^2}$$

and the running coupling $\frac{1}{\alpha(\mu)} + \xi \log \alpha(\mu) \equiv \log\left(\frac{\mu}{m}\right)$

Then $\alpha \equiv \alpha(\mu \sim h)$

Note that the scheme so defined, which we might call "TBA" renormalization scheme, is **not** related in a simple way to ordinary renormalization schemes we use in QFT, such as minimal subtraction. This is a scheme where the beta function is exactly determined as

$$\beta_\alpha^{\mathrm{TBA}} = -\frac{\alpha^2}{1-\xi\alpha} = -\alpha^2 - \xi\alpha^3 + \dots$$

An important technical advance was provided in 2009, who introduced a method based on resolvents to get many perturbative terms in TBA-like schemes [Volin, 2009].

In this way, for the first time the occurrence of singularities in the Borel plane of UV free QFTs at finite N has been firmly shown.

It makes sense to call such singularities renormalons:

- no instantons for generic N
- location in agreement with beta function at leading order in N
- reproduced from chain Feynman diagrams in QFT computations

[Mariño, Miravitllas, Reis, 2102.03078]

For specific cases, i.e. N=4 NLSM, up to 2000 terms have been computed!
[Abbott et al, 2011.09897, 2011.12254, 2111.15390]

More recently another technical advance, based on the Wiener-Hopf method, has allowed to get the first perturbative terms of the trans-series expansion
[Mariño, Miravitllas, Reis, 2111.11951].

This has been further developed and by now the full trans-series can be accessed
[Bajnok, Balog, Hegedus, Vona, 2204.13365; Bajnok, Balog, Vona, 2212.09416]

See Janos talk

Key questions we would like to address:

- Can we decode the exact results in terms of a trans-series?
- Can we recover the trans-series terms from the perturbative series, in the spirit of resurgence?

The answer is yes to both questions, in an interesting way, as we will see.

$$\mathcal{F}(h) \sim h^2 \Phi(\alpha, N, C)$$

$$\Phi(\alpha, N, C_\pm) = \phi_0(\alpha, N) + \sum_{\ell=1}^{\infty} e^{-\frac{2\ell A(N)}{\alpha}} \phi_\ell(\alpha, N, C_\pm)$$

Ordinary QFT IR renormalons corresponds to $A(N) = 1$

$C_\pm$ Stokes constants

Trans-series studied at large and at finite N.

Large N [Di Pietro et al., 2108.02647]

$$\mathcal{F}(h) = \sum_{k=0}^{\infty} \mathcal{F}_k(h) \Delta^{k-1} \qquad \Delta = \frac{1}{N-2}$$

$$\Phi(\alpha, N, C_\pm) = \sum_{k,l=0}^{\infty} e^{-\frac{2\ell}{\alpha}} \varphi_k^{(\ell)}(\alpha, C_\pm) \Delta^{k-1}$$

Knowing the perturbative series is **not** generally enough to reproduce the non-perturbative pieces with suitable Stokes constants.

This happens when large N ``trivializes" the perturbative series.

Example: Gross-Neveu

Considered leading and next-to-leading order terms $\mathcal{F}_0$ and $\mathcal{F}_1$

[Forgacs,Niedermayer,Weisz,1991]

1/N series is **convergent** in the GN model.

Leading order: all trans-series terms appear, though with simple coefficients

$$\varphi_0^{(0)}(\alpha) = 1\,, \quad \varphi_0^{(1)}(\alpha) = -2 - \frac{4}{\alpha}\,, \quad \varphi_0^{(2)}(\alpha) = 2\,, \quad \varphi_0^{(3)}(\alpha) = 2\,, \ldots$$

At **next-to-leading order** asymptotic series appears

$$\varphi_1^{(0)}(\alpha) = -\Big(\alpha + \alpha^2 + \frac{3}{2}\alpha^3 + \mathcal{O}(\alpha^4)\Big) = -2\sum_{n=1}^{\infty}\Gamma(n+1)\Big(\frac{\alpha}{2}\Big)^n$$

Resurgence analysis would predict

$$\varphi_1^{(1)}(\alpha, C_\pm) = C_\pm\frac{4\pi}{\alpha}\,, \quad \varphi_1^{(\ell>1)} = 0\,, \quad C_\pm = \pm\mathrm{i}$$

In contrast, we get

$$\varphi_1^{(1)}(\alpha, C_\pm) = C_\pm\frac{4\pi}{\alpha} + \frac{8}{\alpha^2} - \frac{4}{\alpha}\log(\alpha^2) - 4 + \alpha\Big(2 + \alpha + \alpha^2 + \mathcal{O}(\alpha^3)\Big)$$

$$\varphi_1^{(\ell>1)}(\alpha, C_\pm) \neq 0$$

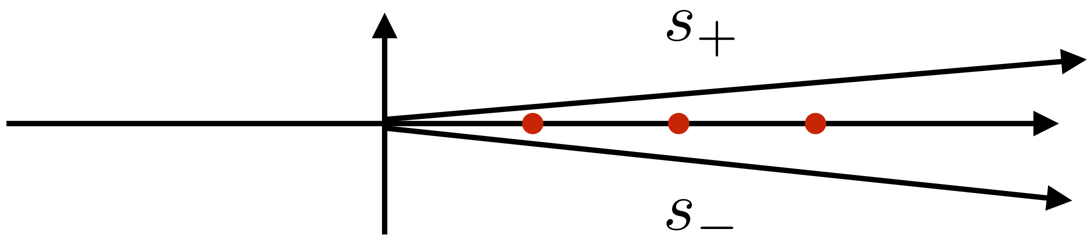
When large N does not ``trivializes'' the series, resurgence works (median resummation).

Example: Principal Chiral Field

Computed leading order term $\mathcal{F}_0$ by using TBA

$$\mathcal{F}_0(h) \sim -\frac{h^2}{8\pi}\Phi(\alpha, C_\pm) = -\frac{h^2}{8\pi}\sum_{\ell\geq 0} C_\pm^\ell e^{-2\ell/\alpha}\varphi^{(\ell)}(\alpha)\,, \qquad C_\pm = \mp \mathrm{i}$$

Resurgence structure particularly easy



$$\Delta_k = s_+^{-1} e^{\frac{2k}{\alpha}}(s_+ - s_-) \qquad \sigma \equiv \exp\left(\sum_k e^{-\frac{2k}{\alpha}}\Delta_k\right)$$

$$\Delta_1\varphi^{(\ell)} = 2i(\ell+1)\varphi^{(\ell+1)} \qquad \Delta_k\varphi^{(\ell)} = 0\,, \qquad k > 1$$

$$\boxed{\mathcal{F}_0(h) = -\frac{h^2}{8\pi} s_+(\sigma^{1/2})\varphi^{(0)}}$$

Finite N

Resurgence always works!

Detailed resurgent analysis up to $\ell = 8$ for the O(4) NLSM
[Abbott et al, 2011.09897, 2011.12254]

Agreement checked up to $\ell = 4$ for general O(N) NLSM, PCF and GN models.

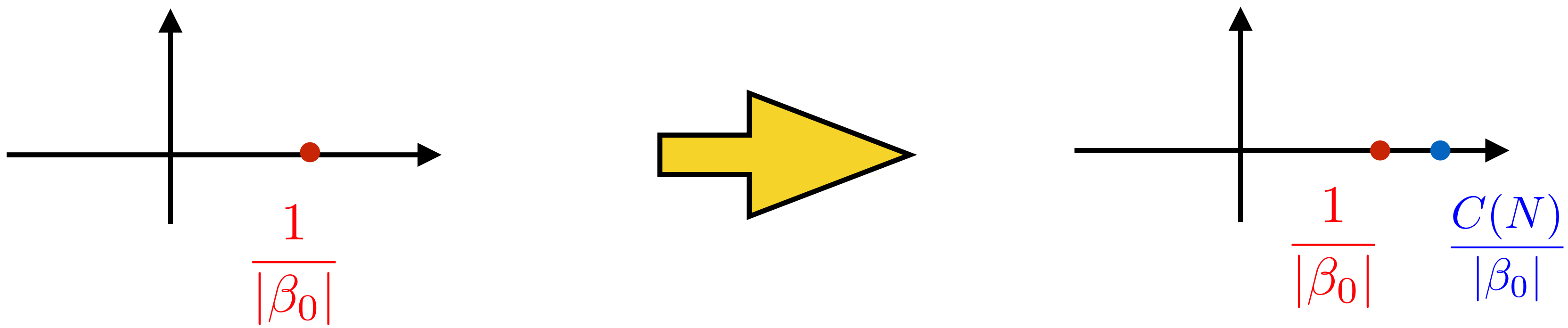
See Janos talk for a detailed study of the O(3) model, where we have both stable instantons as well as renormalons.

However, in the GN and PCF model, a surprise occurs

[Mariño, Miravitllas, Reis, 2111.11951].

$$\Phi(\alpha, N, C_\pm) = \phi_0(\alpha, N) + \sum_{\ell=1}^{\infty} e^{-\frac{2\ell A(N)}{\alpha}} \phi_\ell(\alpha, N, C_\pm)$$

The first IR renormalon splits into two



$$C^{\mathrm{GN}}(N) = \frac{N-2}{N-4} \qquad C^{\mathrm{PCF}}(N) = \frac{N}{N-2}$$

Other IR renormalons are shifted by the same factor. UV renormalons as expected.

New IR renormalons are puzzling from an OPE point of view.

Conclusions

The study of large orders behaviours of the perturbative series, its Borel resummation, resurgence properties, etc. in QFT has been limited for several decades to theories with no renormalons.

Understanding resurgence in theories with renormalons is crucial, since most interesting QFTs (including 4d non-abelian gauge theories) have renormalons.

Renormalons are difficult to study, mostly because we still do not know if and how they are not associated to semi-classical configurations in the path integral, despite various attempts [Cherman et al., 2014; Dunne, Unsal, 2015; …]

Significant progress has been achieved in the last few years within 2d integrable models. These are UV free theories which develop a non-perturbative mass gap in the IR.

Integrable $\neq$ trivial!

In fact, significant technical developments were at the base of this progress.

- Established analytically the existence of renormalons in a QFT at finite N.

- Established in QFTs with renormalons the trans-series expression in terms of ``sufficiently simple'' series.

- Established in QFTs with renormalons that resurgence works beautifully at finite N.

- Resurgence does not commute with the large N limit in general because large N can trivialize the series. On the other hand, in the PCF model at leading order in N a complete resurgence analysis has been achieved.

- Unexpected renormalons have been discovered.

All such results are based on the study of a single observable, the free energy of the theory in presence of a chemical potential.

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They do **not** depend on a would-be semiclassical interpretation of renormalons.

From a QFT point of view they hold in a specific class of renormalization schemes, TBA schemes.

Assumption of specific particles present in the ensemble at finite density (proved at large N).

Future perspectives

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- Extend the analysis to **non-integrable** theories, where large N techniques still can give us access to non-perturbative results.

More generally, it is important to establish when we expect a trans-series and when perturbation theory is Borel resummable.

For instance, is QED_3 (no renormalons or instantons) Borel resummable?

In quantum mechanics, a trans-series can be repackaged to an appropriate Borel resummable series. Is it possible in QFT as well?

So far, one application only in QFT.
More work should be done in this direction.

Obviously, finding out new ways to compute perturbative terms in non-integrable QFTs would help

Thank You