Tau functions, exact WKB and BPS-states

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Based on joint work with

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Holomorphic connections on Riemann surfaces

We consider connections ∇ on holomorphic vector bundles E over Riemann surfaces C, locally of the form

$$\nabla_\hbar = dx\left(\hbar\partial_z + \begin{pmatrix} A_0(z) & A_+(z) \\ A_-(z) & -A_0(z) \end{pmatrix}\right),$$

modulo gauge transformations. There exist finite-dimensional moduli spaces $\mathcal{M}_{\rm dR}(C)$ of such connections, $\dim\mathcal{M}_{\rm dR}(C) = 2d$, with $d := 3g-3+n$ if $C = C_{g,n}$.

Gauge equivalence classes of connections are characterised by their **monodromy**, representations of $\pi_1(C) \to \mathrm{SL}(2,\mathbb{C})$ modulo overall conjugation. The moduli spaces of monodromy data are denoted $\mathcal{M}_{\rm B}(C)$.

Both $\mathcal{M}_{\rm dR}(C)$ and $\mathcal{M}_{\rm B}(C)$ have natural complex structures and Poisson structures. The holonomy map $\mathbf{Hol}$ defines a local biholomorphism between $\mathcal{M}_{\rm dR}(C)$ and $\mathcal{M}_{\rm B}(C)$, the inverse being the Riemann-Hilbert correspondence. $\mathbf{Hol}$ preserves the Poisson structures.[1]

[1] Hitchin; Alekseev, Malkin; Korotkin, Samtleben; Bertola, Korotkin

From connections to opers

There exists a matrix function g such that

$$g^{-1}\cdot\nabla_\hbar\cdot g = dx\left(\hbar\partial_x + \begin{pmatrix} 0 & q_\hbar(x) \\ 1 & 0 \end{pmatrix}\right) \;,$$

$$q_\hbar = A_0^2 + A_+A_- + \hbar\left(A_0' - \frac{A_0A_-'}{A_-}\right) + \hbar^2\left(\frac{3}{4}\left(\frac{A_-'}{A_-}\right)^2 - \frac{A_-''}{2A_-}\right)\;.$$

Connections gauge equivalent to the form $dx\left(\hbar\partial_x + \left(\begin{smallmatrix} 0 & q(x) \\ 1 & 0 \end{smallmatrix}\right)\right)$ are called **opers**.

Flat sections or oper connections are of the form $\left(\begin{smallmatrix} \chi' \\ \chi \end{smallmatrix}\right)$, with χ solving

$$\left(\frac{\partial^2}{\partial x^2} - q_\hbar(x)\right)\chi(x) = 0\,.$$

Note that $q_\hbar(x)$ defined above has poles at any zero u_k of A_-, **apparent singularities**,

$$q_\hbar(x) = \frac{3}{4}\frac{\hbar^2}{(x-u_k)^2} + \hbar\frac{v_k}{x-u_k} + \dots\,.$$

Variables $(u_k, v_k)_{k=1,\dots,d} \equiv (\boldsymbol{u},\boldsymbol{v})$: Useful **Darboux** coordinates for $\mathcal{M}_{\mathrm{dR}}(C)$!

Isomonodromic Tau-functions

Connections ∇ are holomorphic w.r.t. an underlying complex structure on C. Let $\mathcal{M}(C)$: moduli space of complex structures on C, with coordinates $\boldsymbol{z} = (z_1, \ldots, z_d)$.

Considering families of connections with fixed holonomy defines flows on the total space of the fibration $\mathcal{M}_{\mathrm{dR}}(C)$ over $\mathcal{M}(C)$, the **isomonodromic deformation flows**.

These flows are Hamiltonian, there locally exist functions $H_r(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{z})$ such that[2]

$$\frac{\partial u^r}{\partial z_k} = \frac{\partial H_r}{\partial v^r}, \qquad \frac{\partial v_r}{\partial z_k} = -\frac{\partial H_r}{\partial u_r}.$$

The Riemann-Hilbert correspondence defines H_r as functions $H_r(\boldsymbol{\mu}; \boldsymbol{z})$ of the monodromy data $\boldsymbol{\mu}$.

There exists generating functions for the functions H_r,

$$\frac{\partial}{\partial z_k} \log \mathcal{T}(\boldsymbol{\mu}; \boldsymbol{z}) = H_r(\boldsymbol{\mu}; \boldsymbol{z}), \qquad r = 1, \ldots, d.$$

The functions $\mathcal{T}(\boldsymbol{\mu}; \boldsymbol{z})$ are called **isomonodromic tau-functions**.

[2] Schlesinger; Garnier; Okamoto; Iwasaki; ... ; Dinh, J.T. in preparation.

Coordinates from exact WKB I

Expansion in $\hbar$ - exact WKB: Solutions to $\big(\hbar^2\frac{\partial^2}{\partial x^2} - q_\hbar(x)\big)\chi(x) = 0$,

$$\chi^{(b)}_{\pm}(x) = \frac{1}{\sqrt{S_{\rm odd}(x)}}\exp\bigg[\pm\int^x dx'\, S_{\rm odd}(x')\bigg],$$

with $S_{\rm odd} = \frac{1}{2}(S^{(+)} - S^{(-)})$, $S^{(\pm)}(x)$ being formal series solutions to

$$q_\hbar = \hbar^2(S^2 + S'), \qquad S(x) = \sum_{k=-1}^{\infty}\hbar^k S_k(x), \qquad S^{(\pm)}_{-1} = \pm\sqrt{q_0}. \tag{1}$$

It is believed[3] that series (1) is **Borel-summable** away from the **Stokes-lines**, $\mathrm{Im}(w(x)) = \mathrm{const.}$, $w(x) = e^{-\mathrm{i}\arg(\hbar)}\int^x dx'\,\sqrt{q_0(x')}$. The **Voros symbols**

$$V_\beta := \int_\beta dx\; S_{\rm odd}(x), \qquad \beta \in H_1^{\rm odd}(\Sigma, \mathbb{Z}),$$

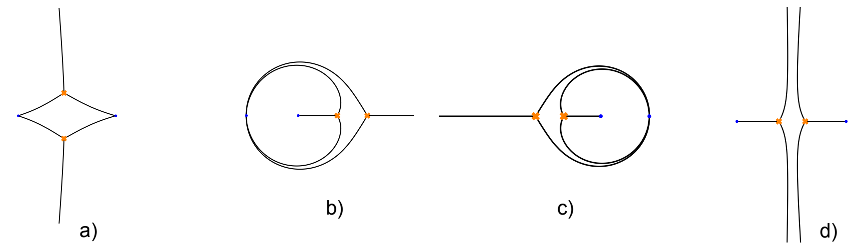
can be Borel-summable, then representing **coordinates on** $\mathcal{M}_{\rm B}(C)$.

[3] Probably proven by Koike-Schäfke (unpublished), and by Nikolaev

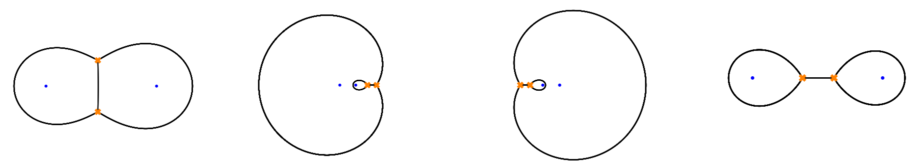
Coordinates from exact WKB II

Borel summability depends on the topology of Stokes graph formed by Stokes lines (determined by $q_0 \sim$ point on $\mathcal{B} = H^0(C, K^2)$). Two "extreme" cases:

FG Stokes graph $\leftrightsquigarrow$ triangulation of C



FN Stokes graph $\leftrightsquigarrow$ pants decomposition



In between there exist several hybrid types of graphs.

Case FG: For $q_\hbar(x)$ **without** apparent singularities D. Allegretti has proven conjecture of T. Bridgeland:

$$\text{Voros symbols} \quad \sim \quad \text{Fock-Goncharov (FG) type coordinates}$$

Conjectures:

- **Case FG:** This also holds if there are apparent singularities.
- **Case FN:** Coordinates are of Fenchel-Nielsen type (pants decompositions).

Coordinates from exact WKB III

Consider spectral curve Σ:

$$\Sigma = \big\{(x,y) \in T^*C\,;\, y^2 - q_0(x) = 0\big\} \subset T^*C,$$

with $q_0(x)(dx)^2$: quadratic differential on surface C.

Special geometry: Periods $\sim$ coordinates for $\mathcal{B} \simeq H^0(C, K^2)$ (choices of q_0),

$$a^r = \int_{\alpha^r} \sqrt{q_0}, \qquad \check{a}_r = \int_{\check{\alpha}^r} \sqrt{q_0} = \frac{\partial}{\partial a^r}\mathcal{F}(a),$$

where $(\alpha_r, \check{\alpha}^r)$, $r = 1, \dots, d$, is a canonical basis for $H_1^{\rm odd}(\Sigma, \mathbb{Z})$.

Further coordinates from NLO in $\hbar$: (with convenient normalisation)

$$\theta_r = \frac{1}{2}\int_{\check{\alpha}^r} \frac{p}{\sqrt{q_0}}, \qquad q_\hbar = q_0 + \hbar p + \mathcal{O}(\hbar^2),$$

coordinates for the Prym of Σ.

coordinates $(\boldsymbol{u}, \boldsymbol{v}) \quad \leftrightarrow \quad$ coordinates $(\boldsymbol{a}, \boldsymbol{\theta})$

Coordinates from exact WKB II

- Choose canonical basis $(\alpha^r, \check{\alpha}_r)_{r=1,\dots,d}$, for $H_1^{\text{odd}}(\Sigma, \mathbb{Z})$,
- denote corresponding Voros symbols $(\sigma^r, \eta_r)_{r=1,\dots,d} \equiv (\boldsymbol{\sigma}, \boldsymbol{\eta})$
  (coordinates on total space $\mathcal{M}_{\mathrm{H}}(C)$ of Prym fibration over $\mathcal{B}$)
- asymptotic behaviour for $\hbar \to 0$ is of the form

$$\sigma_r(\boldsymbol{a}, \boldsymbol{\theta}; \boldsymbol{z}; \hbar) \sim \frac{1}{\hbar} a^r + \mathcal{O}(\hbar), \qquad \eta_r(\boldsymbol{a}, \boldsymbol{\theta}; \boldsymbol{z}; \hbar) \sim \frac{1}{\hbar} \check{a}_r + \theta_r + \mathcal{O}(\hbar),$$

with $(a^r, \check{a}_r)$ period coordinates on $\mathcal{B}$, and θ_r linear coordinates on the Prym.

coordinates $(\boldsymbol{u}, \boldsymbol{v})$ $\leftrightarrow$ coordinates $(\boldsymbol{a}, \boldsymbol{\theta})$ $\leftrightarrow$ coordinates $(\boldsymbol{\sigma}, \boldsymbol{\eta})$

Relation to BPS states[4]

Changes of coordinates from WKB happen when Stokes graph changes topology.

This happens across the rays $\{\hbar \in \mathbb{C}^{\times}; a_\gamma/\hbar \in \mathrm{i}\mathbb{R}_-\}$.

The change of coordinates can be represented in the form[5]

$$\tilde{X}_{\gamma'} = X_{\gamma'}(1 - X_\gamma)^{\langle\gamma',\gamma\rangle\Omega(\gamma)}, \qquad X_\gamma^{\jmath} = e^{2\pi\mathrm{i}(q_r\sigma^r - p^r\eta_r)}, \;\; a_\gamma = q_r a^r - p^r \check{a}_r, \qquad \gamma = \sum_r (\alpha^r q_r - \check{\alpha}_r p^r) \in H_1^{\mathrm{odd}}(\Sigma, \mathbb{Z}),$$

determining integers $\Omega(\gamma)$ satisfying Kontsevich-Soibelman-WCF.

The integers $\Omega(\gamma)$ have an interpretation as **BPS-indices**:[6]

Changes of Stokes graph
$\Leftrightarrow$ Existence of saddle or ring trajectories
$\Leftrightarrow$ Stable objects in Fukaya category of Calabi-Yau mfd Y_Σ

$$uv - f_\Sigma(x,y) = 0, \qquad f_\Sigma(x,y) = y^2 - q_0(x).$$

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[4] Gaiotto-Moore-Neitzke

[5] Dillinger-Delabaere-Pham, Gaiotto-Moore-Neitzke

[6] Klemm-Lerche-Mayr-Vafa-Warner, Gaiotto-Moore-Neitzke, Bridgeland-Smith, Smith

Generating functions

for change of coordinates $(\boldsymbol{u}, \boldsymbol{v}) \quad \leftrightarrow \quad (\boldsymbol{\sigma}, \boldsymbol{\eta})$

Key result: **Holonomy map is symplectic**, $\mathbf{Hol}^* \Omega_{\rm B} = \Omega_{\rm dR}$

(Hitchin; Alekseev, Malkin; Korotkin, Samtleben; Bertola, Korotkin)

Let us introduce Darboux coordinates:

- u_r, v_r, $r = 1, \ldots, d$, coordinates for $\mathcal{M}_{\rm dR}(C)$ s.t. $\Omega_{\rm dR} = \sum_r du^r \wedge dv_r$.

- σ_r, η_r, $r = 1, \ldots, d$, coordinates for $\mathcal{M}_{\rm B}(C)$ s.t. $\Omega_{\rm B} = 2\pi \mathrm{i} \sum_r d\eta^r \wedge d\sigma_r$.

We then have

$$\Rightarrow \quad d(2\pi \mathrm{i}\, \eta^r d\sigma_r + v_r du^r) = 0 \quad \Rightarrow \quad 2\pi \mathrm{i}\, \eta^r d\sigma_r + v_r du^r = dS(\boldsymbol{\sigma}, \boldsymbol{u}; \boldsymbol{z}),$$
$$\frac{\partial S}{\partial \sigma_r} = 2\pi \mathrm{i}\, \eta^r(\boldsymbol{\sigma}, \boldsymbol{u}; \boldsymbol{z}), \qquad \frac{\partial S}{\partial u^r} = -v_r(\boldsymbol{\sigma}, \boldsymbol{u}; \boldsymbol{z}).$$

$S = S(\boldsymbol{\sigma}, \boldsymbol{u}; \boldsymbol{z})$: **Generating function**, uniquely defined up to constants by the choices of Darboux coordinates.

The canonical line bundle

Let (σ_r, η^r), and $(\tilde{\sigma}_r, \tilde{\eta}^r)$, $r = 1, \ldots, d$, be coordinates on open sets $U, V \subset \mathcal{M}_{\mathrm{B}}(C)$ such that

$$\sum_r d\eta^r \wedge d\sigma_r = \sum_r d\tilde{\eta}^r \wedge d\tilde{\sigma}_r.$$

There will then exist a function $G(\boldsymbol{\sigma}, \tilde{\boldsymbol{\sigma}})$ on $U \cap V$ such that

$$\frac{\partial G}{\partial \sigma_r} = \eta^r(\boldsymbol{\sigma}, \tilde{\boldsymbol{\sigma}}), \qquad \frac{\partial G}{\partial \tilde{\sigma}_r} = -\tilde{\eta}_r(\boldsymbol{\sigma}, \tilde{\boldsymbol{\sigma}}).$$

The collection of generating functions G defined on overlaps of an atlas of Darboux coordinates defines a canonical projective[7] holomorphic line bundle $\mathcal{L}$ on $\mathcal{M}_{\mathrm{B}}(C)$.[8]

[7] Condition on triple overlaps satisfied up to roots of unity.

[8] cf. Freed-Neitzke; related line bundles on cluster varieties previously defined by Alexandrov-Persson-Pioline and Neitzke.

Tau-functions from generating functions

Let $\mathrm{U}(\boldsymbol{\sigma},\boldsymbol{\eta};\boldsymbol{z})$ be defined by the equations

$$\frac{\partial}{\partial\sigma_r}S(\boldsymbol{u},\boldsymbol{\sigma};\boldsymbol{z})\Big|_{\boldsymbol{u}=\mathrm{U}(\boldsymbol{\sigma},\boldsymbol{\eta};\boldsymbol{z})}=2\pi\mathrm{i}\,\eta^r.$$

It was recently conjectured[9], and later shown that[10]

$$\mathcal{T}(\boldsymbol{\sigma},\boldsymbol{\eta};\boldsymbol{z}):=S(\boldsymbol{u},\boldsymbol{\sigma};\boldsymbol{z})\Big|_{\boldsymbol{u}=\mathrm{U}(\boldsymbol{\sigma},\boldsymbol{\eta};\boldsymbol{z})}-2\pi\mathrm{i}\sum_r\sigma_r\eta^r, \tag{2}$$

satisfies the defining equations for the tau-function $\mathcal{T}(\boldsymbol{\sigma},\boldsymbol{\eta};\boldsymbol{z})\equiv\mathcal{T}(\boldsymbol{\mu};\boldsymbol{z})$,

$$\frac{\partial}{\partial z_r}\log\mathcal{T}(\boldsymbol{\sigma},\boldsymbol{\eta};\boldsymbol{z})=H_r(\boldsymbol{\sigma},\boldsymbol{\eta};\boldsymbol{z}).$$

This fixes the dependence of the isomonodromic tau-functions on the monodromy data.

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[9] Its, Lisovyy, Tykhyy
[10] Bertola-Korotkin, Nekrasov

Generalised theta series

For $C = C_{0,4}$ and FN-type coordinates $(\boldsymbol{\sigma}, \boldsymbol{\eta})$ it has recently been shown by Nekrasov that the function $\mathcal{T}(\boldsymbol{\sigma}, \boldsymbol{\eta}; \boldsymbol{z})$ defined in (2) has an expansion of the form

$$\mathcal{T}(\boldsymbol{\sigma}, \boldsymbol{\eta}; \boldsymbol{z}) = \sum_{\boldsymbol{n} \in \mathbb{Z}} e^{2\pi \mathrm{i}\, \boldsymbol{n}\boldsymbol{\eta}} \mathcal{Z}(\boldsymbol{\sigma} + \boldsymbol{n}; \boldsymbol{z}),$$

where $\mathcal{Z}(\boldsymbol{\sigma}; \boldsymbol{z})$ are **instanton partition functions**. (Nekrasov, Nekrasov-Okounkov, . . .)

Both $\mathcal{T}(\boldsymbol{\sigma}, \boldsymbol{\eta}; \boldsymbol{z})$ and $\mathcal{Z}(\boldsymbol{\sigma}; \boldsymbol{z})$ admit several alternative representations:

- $\mathcal{Z}(\boldsymbol{\sigma}; \boldsymbol{z})$: Conformal blocks of the Virasoro algebra at $c = 1$,
  (Gamayun-Iorgov-Lisovyy, Iorgov-Lisovyy-J.T., Bershtein-Shchechkin)
- $\mathcal{T}(\boldsymbol{\sigma}, \boldsymbol{\eta}; \boldsymbol{z})$: Conformal blocks of free fermion VOA
  (Gavrylenko-Marshakov; Coman-Pomoni-J.T.; Coman-Longhi-J.T.)
- Fredholm determinants (Lisovyy-Gavrylenko; Cafasso-Lisovyy-Gavrylenko).

Both $\mathcal{T}(\boldsymbol{\sigma}, \boldsymbol{\eta}; \boldsymbol{z})$ and $\mathcal{Z}(\boldsymbol{\sigma}; \boldsymbol{z})$ are **topological string partition functions** from counting of **framed** BPS-states (D0-D2-D4-D6 and D0-D2-D6 bound states). (Coman-Pomoni-J.T.; Coman-Longhi-J.T.).

Global geometric picture

Conjecture:

Let $(\boldsymbol{\sigma}, \boldsymbol{\eta})$ be a **good** system of coordinates of FG or FN type for $\mathcal{M}_{\mathrm{B}}(C)$, and let $\mathcal{T}(\boldsymbol{\sigma}, \boldsymbol{\eta}; \boldsymbol{z})$ be the associated isomonodromic tau-function defined in (2).

(i) $\mathcal{T}(\boldsymbol{\sigma}, \boldsymbol{\eta}; \boldsymbol{z})$ has generalised theta series expansions of the form

$$\mathcal{T}(\boldsymbol{\sigma}, \boldsymbol{\eta}; \boldsymbol{z}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^d} e^{2\pi \mathrm{i}\, \boldsymbol{n} \cdot \boldsymbol{\eta}} \mathcal{Z}(\boldsymbol{\sigma} + \boldsymbol{n}; \boldsymbol{z}). \tag{3}$$

(ii) Whenever $(\boldsymbol{\sigma}, \boldsymbol{\eta})$ are coordinates of FN-type, one may identify the expansion coefficients $\mathcal{Z}(\boldsymbol{\sigma}; \boldsymbol{z})$ with conformal blocks of the Virasoro algebra at $c = 1$.

The Tau-functions associated to different sets of coordinates $(\boldsymbol{\sigma}, \boldsymbol{\eta})$ and $(\boldsymbol{\sigma}', \boldsymbol{\eta}')$ are related by multiplication with **difference generating functions** for $(\boldsymbol{\sigma}, \boldsymbol{\eta}) \leftrightarrow (\boldsymbol{\sigma}', \boldsymbol{\eta}')$.

Outline of proof: Coman-Longhi-J.T., combined with the observation that changes of Darboux coordinates of FG, FN, and mixed types preserve form of the expansions (3).
(J.T., in preparation)

Generalised theta functions

The Voros symbol map from $\mathcal{M}_{\mathrm{H}}(C)$ to $\mathcal{M}_{\mathrm{B}}(C)$ allows one to define

$$\Theta_{\hbar}(\boldsymbol{a},\boldsymbol{\theta};\boldsymbol{z}) = \mathcal{T}\big(\boldsymbol{\sigma}(\boldsymbol{a},\boldsymbol{\theta};\boldsymbol{z};\hbar),\boldsymbol{\eta}(\boldsymbol{a},\boldsymbol{\theta};\boldsymbol{z};\hbar);\boldsymbol{z}\big).$$

Claim: $\Theta_{\hbar}$ is an $\hbar$-deformation of the Riemann-theta functions Θ_{Σ} on the Prym:

$$\lim_{\hbar\to 0}\left[\log\Theta_{\hbar}(\boldsymbol{a},\boldsymbol{\theta};\boldsymbol{z}) - \frac{1}{\hbar^2}\log\mathcal{F}(\boldsymbol{a})\right] = \log\Theta_{\Sigma}(\boldsymbol{a},\boldsymbol{\theta};\boldsymbol{z}) + \mathcal{F}_1(\boldsymbol{a}).$$

Tau-functions: Canonical deformations of the theta-functions defined by the integrable structures.

- Expect relation to objects defined by **topological recursion**.

- Probably related to recent work by T. Bridgeland.

BPS-Riemann-Hilbert bootstrap

- Exact WKB provides a beautiful geometric coding of **unframed** BPS indices
- The tau-functions are related to the generating functions of **framed** BPS indices.
- Jumps of tau-functions reflect **framed wall-crossing**, determined by the spectrum of **unframed** BPS states.

Generalisation to other local CY will involve **non-perturbative partition functions** of Marino and collaborators.