Non-perturbative Complex Chern-Simons

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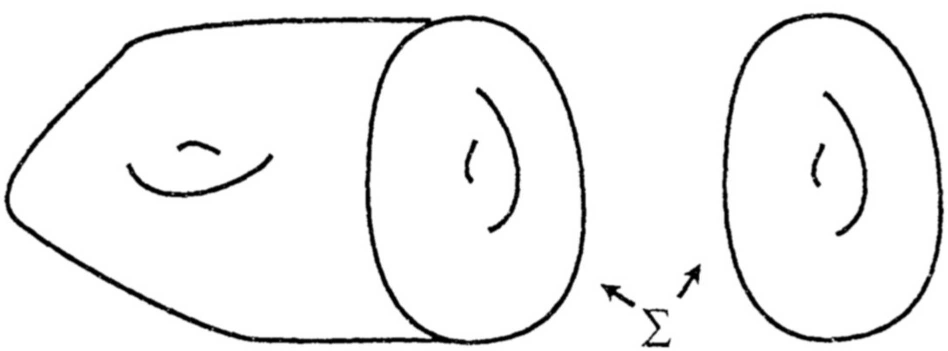


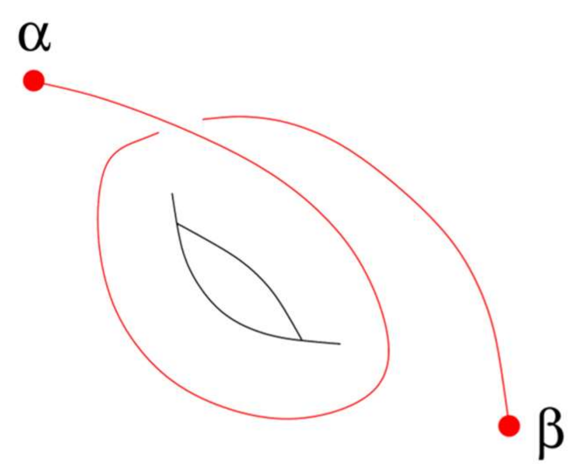
cf. E.Witten (1989)

Non-perturbative Complex Chern-Simons

$$\int \frac{DA}{\text{gauge}} \, e^{-\frac{1}{\hbar} S[A]}$$

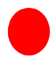
Contains contributions of all complex flat connections $\alpha \in \pi_0(\mathcal{M}_{\text{flat}}(M_3, G))$:

$$\mathcal{Z}_{\alpha}^{\text{pert}}(\hbar) \; = \; \sum_{n=0}^{\infty} a_n^{\alpha} \hbar^{n+c_{\alpha}}$$



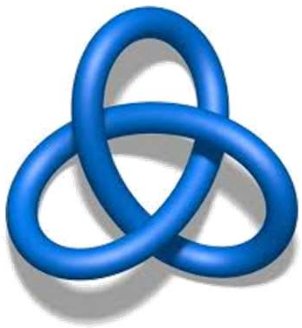
Lift $\alpha \in \pi_0(\mathcal{M}_{\text{flat}}(M_3, G)) \times \mathbb{Z}$

Modern low-dimensional topology:



"local"
operators

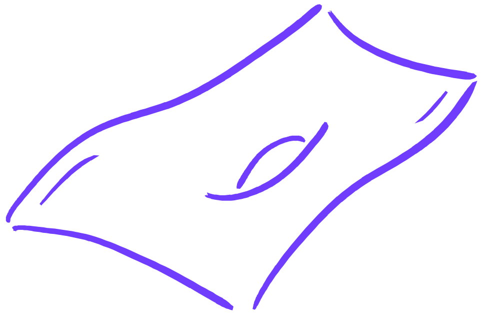
supported
at points



line
operators

webs

supported on
1-manifolds

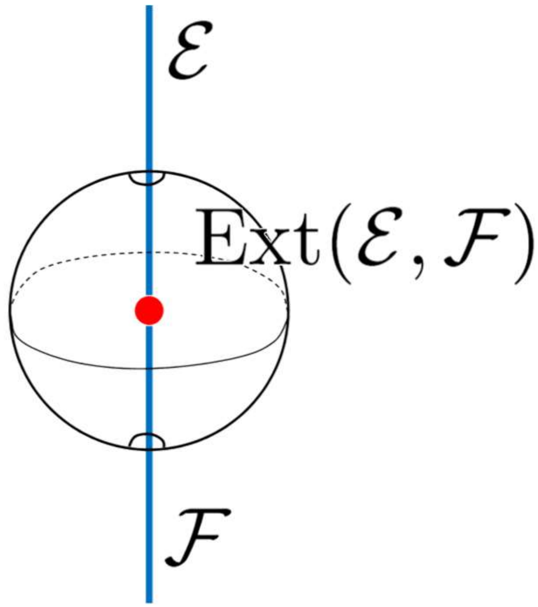


surface
operators

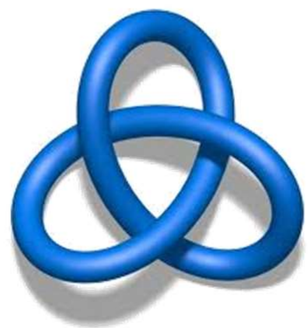
foams

supported on
2-manifolds

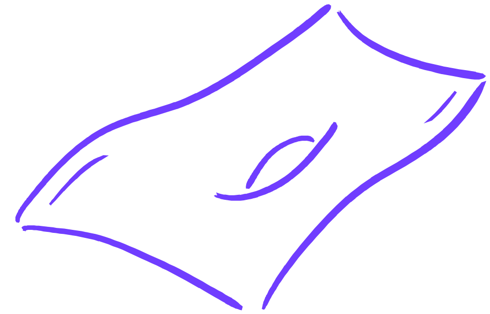
Modern low-dimensional topology:



"local"
operators



line
operators



surface
operators

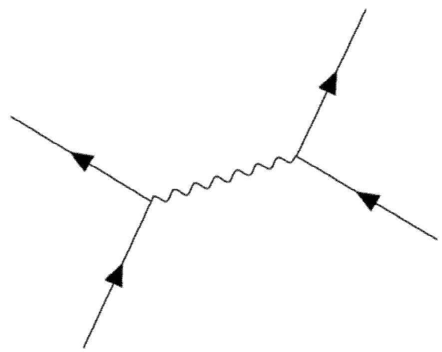
Logarithmic
(non-semisimple)

Symmetries
decorated TQFT

cf. M.Jagadale

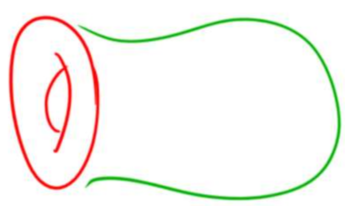
- Perturbative complex Chern-Simons well understood in early 2000s (conceptually and computationally)

- even the direct calculation leads to finite integrals under good analytical control

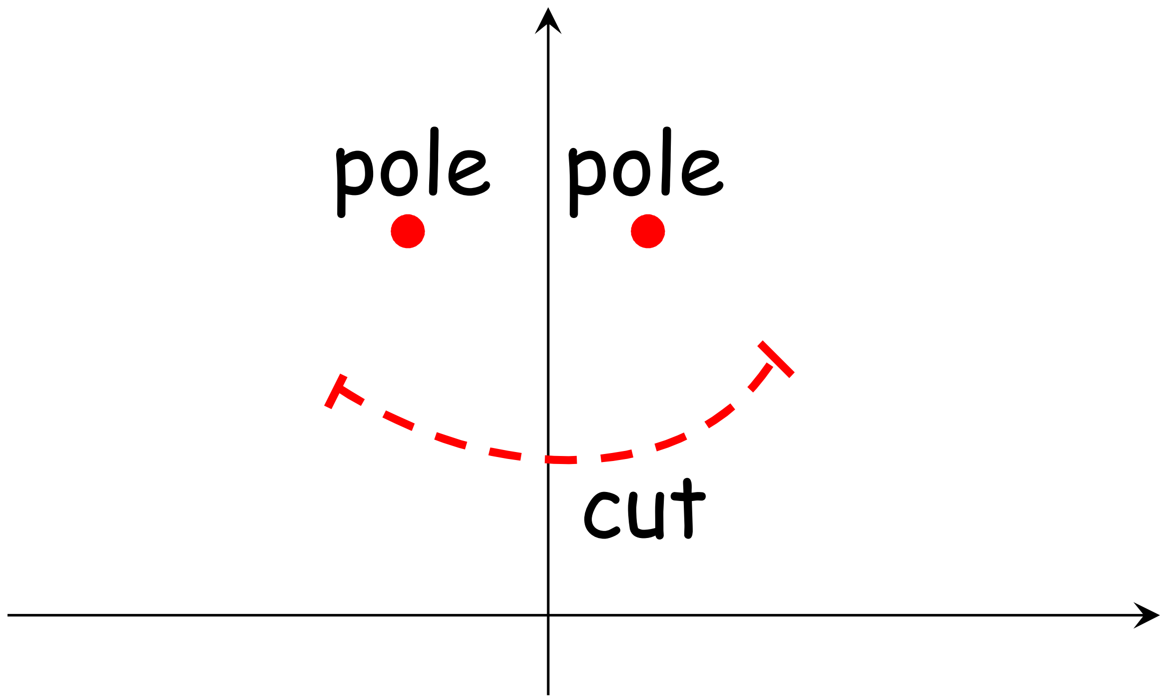


cf. S.Axelrod, I.Singer

- Many partition functions from 3d-3d correspondence ... can not completely close the boundary until 2013



Non-perturbative Complex Chern-Simons



perturbative ⇨ Borel plane ⇨ non-perturbative

$$\sum a_n \hbar^n \qquad B(S) = \sum \frac{a_n}{n!} S^n \qquad \frac{1}{\hbar} \int_0^\infty B(S) e^{-S/\hbar} dS$$

Non-perturbative Complex Chern-Simons

$U_q(\mathfrak{g})$
at generic q

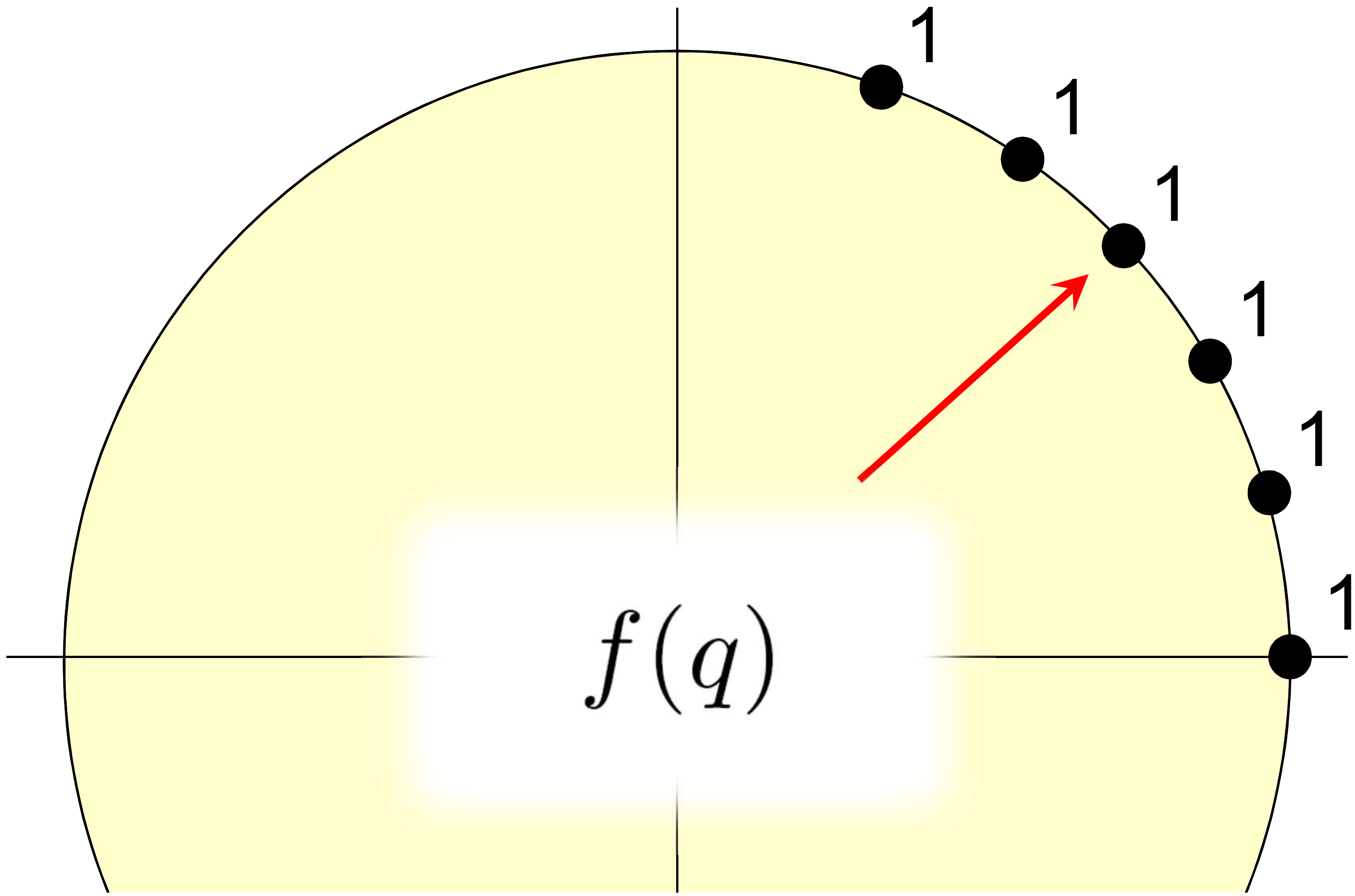
$q = e^{\hbar} \to 1$

$$\mathcal{Z}_{\alpha}^{\text{pert}}(\hbar) \;=\; \sum_{n=0}^{\infty} a_n^{\alpha} \hbar^{n+c_{\alpha}}$$

function of q

Borel resum

$q = e^{\hbar}$ complex, continuous



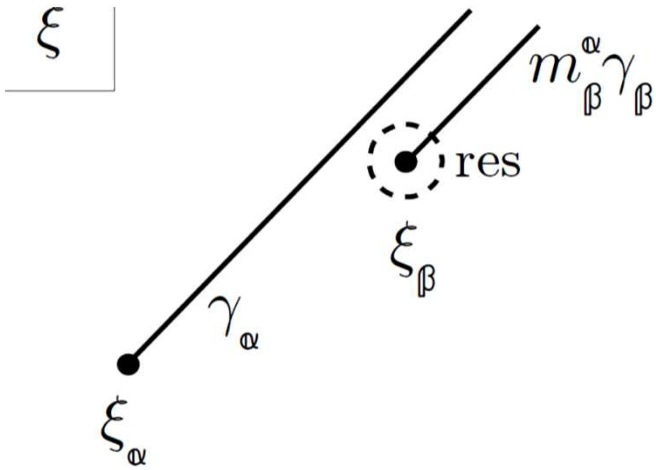
$\mathfrak{g} = \mathfrak{sl}_2$:

$$KEK^{-1} = q^2 E\,, \;\; KFK^{-1} = q^{-2} F$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

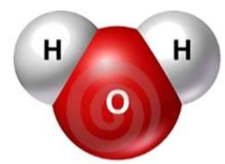
$$Z_\alpha(q = e^\hbar) \Rightarrow Z_a(q) \Rightarrow \widehat{Z}_b(M_3; q)$$

thimble integral



labeled by complex flat connection

Borel resummation, function of q



labeled by $a \in H_1(M_3; \mathbb{Z})$

integral q-series, functoriality, cutting-and-gluing

labeled by $b \in \mathrm{Spin}^c(M_3)$

$$\widehat{Z}_b = \sum_{\substack{a \\ \text{abelian}}} S^{ab} \left(\mathcal{S} Z_a^{\text{pert}} + \sum_{\substack{\beta \\ \text{non-abelian}}} n_a^\beta \, \mathcal{S} Z_\beta^{\text{pert}} \right)$$

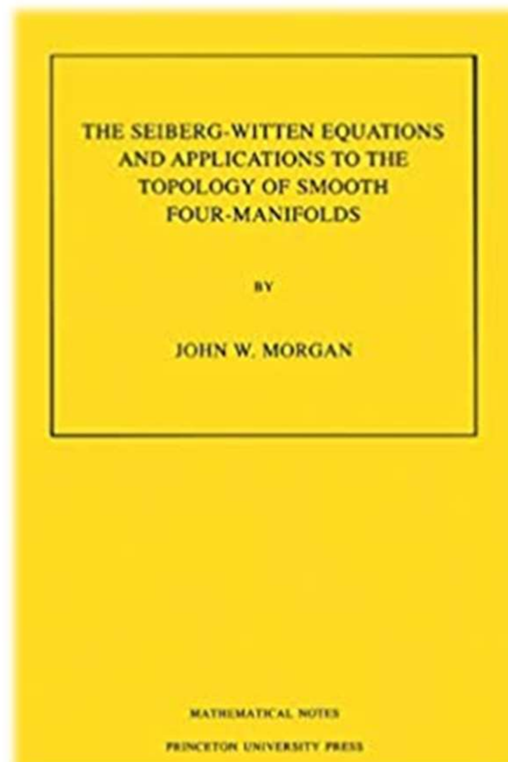
There is a canonical map:

$$\sigma : \mathrm{Spin}(M_3) \times H_1(M_3, \mathbb{Z}) \longrightarrow \mathrm{Spin}^c(M_3)$$

induced by

$$B\mathrm{Spin} \times BU(1) \to B\mathrm{Spin}^c$$

which, in turn, is part of the fiber sequence for the classifying spaces.



$$b \in \mathrm{Spin}^c(M_3) \cong H_1(M_3; \mathbb{Z})$$

cf. M.Jagadale

Example: Lens spaces & some mapping tori

$$\widehat{Z}(M_3) = q = e^{\hbar}$$

S.Chun, S.G., S.Park, N.Sopenko
J.Andersen, W.Mistegaard

→ trivial Borel plane,
"almost abelian" flat connections

Theorem:

$$n_{\beta}^{\alpha} = 0 \qquad \begin{array}{rcl} \alpha & = & \text{any} \\ \beta & = & \text{abelian} \end{array}$$

S.G., M.Marino, P.Putrov

Stokes / trans-series coefficients are *not* symmetric!

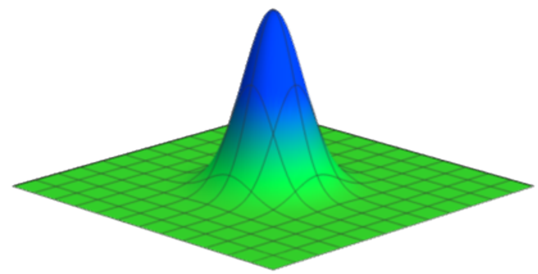

α
β

2d-3d (half)-index / elliptic genus of 3d $\mathcal{N} = 2$ theory with 2d (0,2) boundary condition $\mathcal{B}_b$:

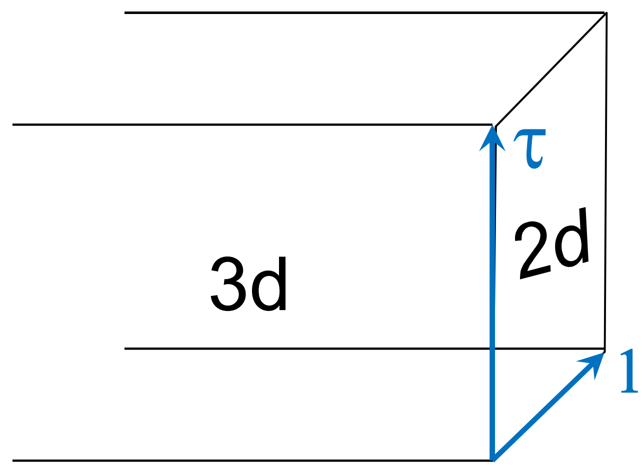
A.Gadde, S.G., P.Putrov (2013)

$$\widehat{Z}(q) \;=\; \text{partition function on } S^1 \times_q D^2$$

Counting BPS states



$$S^1 \times_q D^2 \quad =$$



3d “distorts” modular symmetry

Confluence of many perspectives

- 6d perspective
- Integer coefficients (categorification)

$$\widehat{Z}_{b}(M_3;q) = \sum_{i,j} (-1)^i q^j \dim H^{i,j}(M_3;b)$$

- Quantum groups at generic q
- Resurgent analysis
- Rozansky-Witten theory based on $T^*\mathrm{Gr}_G$
- Moduli spaces in gauge theory and in enumerative geometry

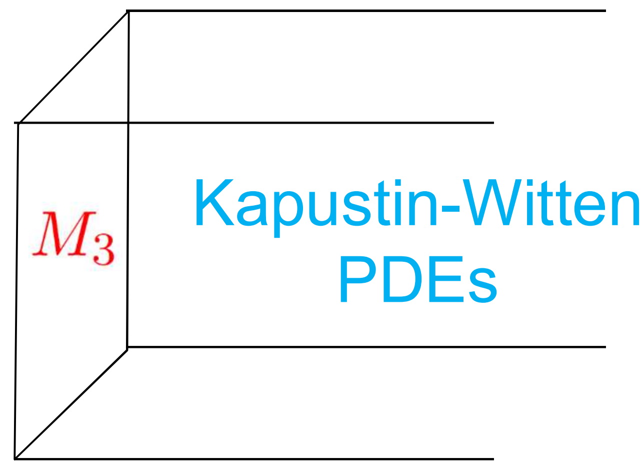
$$M_3 \times S^1 \times_q D^2$$

<u>Conjecture:</u> $\widehat{Z}_b(M_3;q)$ should admit a definition via moduli spaces in gauge theory *

S.G., D.Pei, P.Putrov, C.Vafa

$\mathcal{B}_{\text{Nahm}}$

$$\frac{\sum_{i=1}^3 e_\mu^i t_i}{y} + \ldots$$



$\mathcal{B}_b$

---

* Symmetries: require a new skyrmion / caloron / K-theory / multiplicative version

Early hints: S.Chun, S.G., S.Park, N.Sopenko
More detailed analysis: S.G., P.-S.Hsin, D.Pei (to appear)

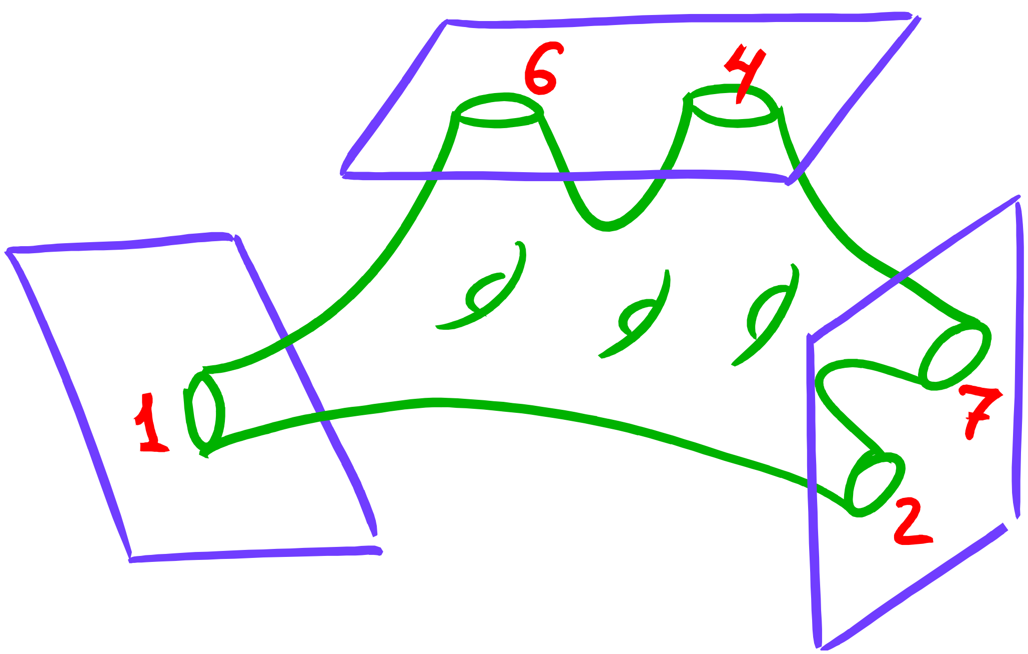
$$M_3 \times S^1 \times_q D^2$$

<u>Conjecture:</u> $\widehat{Z}_b(M_3; q)$ should admit a definition via moduli spaces in curve counting

S.G., D.Pei, P.Putrov, C.Vafa

$$\phi : (\Sigma, \partial\Sigma) \longrightarrow (X, L)$$

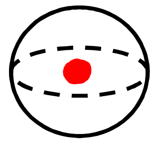
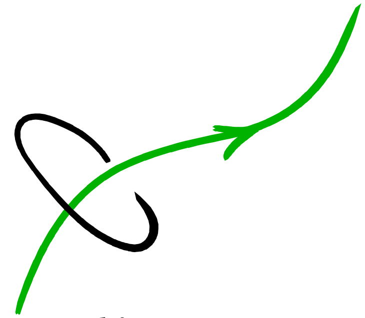
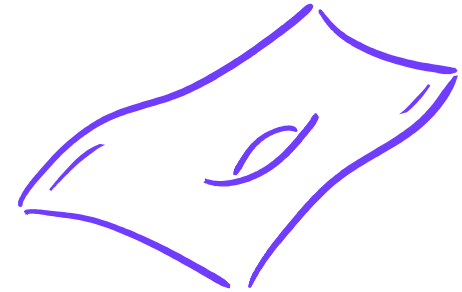
Σ genus g, with n boundary components



$$\partial\Sigma = \gamma_1 \sqcup \ldots \sqcup \gamma_n$$

$$\beta = \phi_*[\Sigma] \in H_2(X, L)$$

$$b_i = \phi_*[\gamma_i] \in H_1(L)$$

quantum groups $U_q(\mathfrak{g})$	 "local" operators	 line operators	 surface operators
WRT : roots of 1	✗	✓	✗
GPPV : generic q	✓	✓	✓

Decorated TQFT ⟷ 2d VOA / CFT

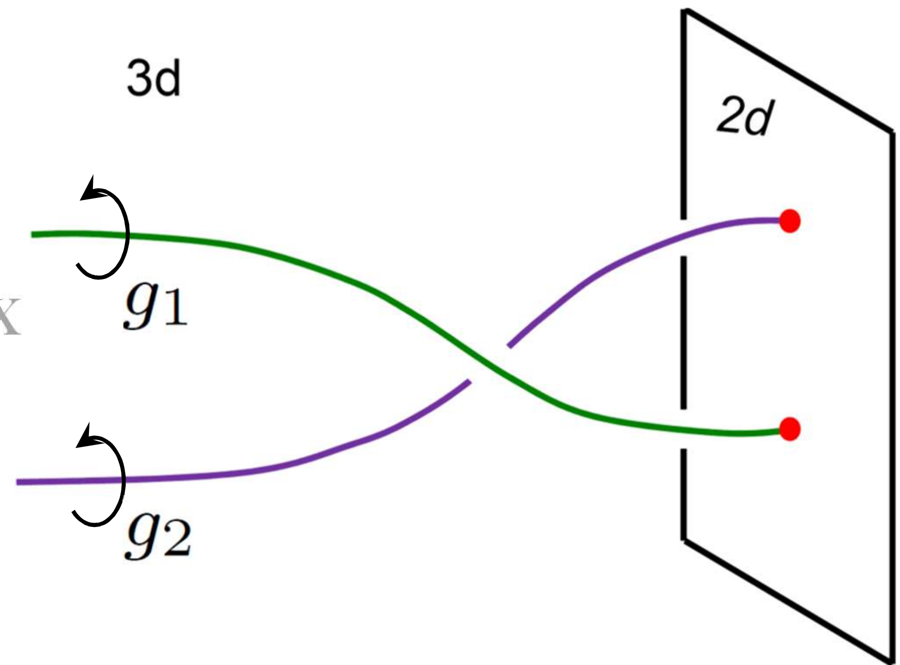
cf. M.Jagadale

G-crossed MTCs, ⟷ twisted sectors
“enriched” SPT phases,
bordered, sutured, ...

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

$$\mathcal{C}_{g_1} \boxtimes \mathcal{C}_{g_2} \to \mathcal{C}_{g_1 g_2}$$

complex
weights



Kazhdan-Lusztig

$$U_q^{\cdots}(\mathfrak{g})\text{-mod}^{\cdots} \simeq \text{VOA-mod}^{\cdots}$$

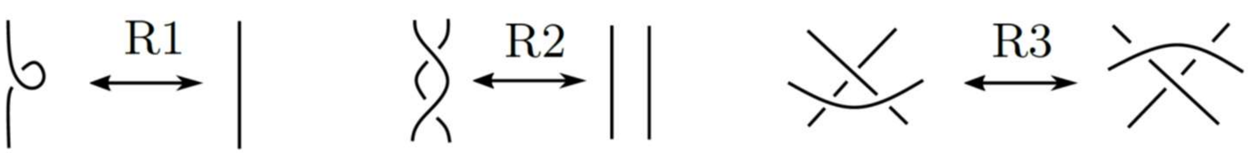
$S^3_{-1}($  $) = S^3_{+1}($  $)$

<u>Theorem:</u> Using the R-matrix for Verma modules, for all links of unknots (plumbings), torus links, positive braid links, fibered knots up to 10 Xs, and homogeneous braid links the two-variable series

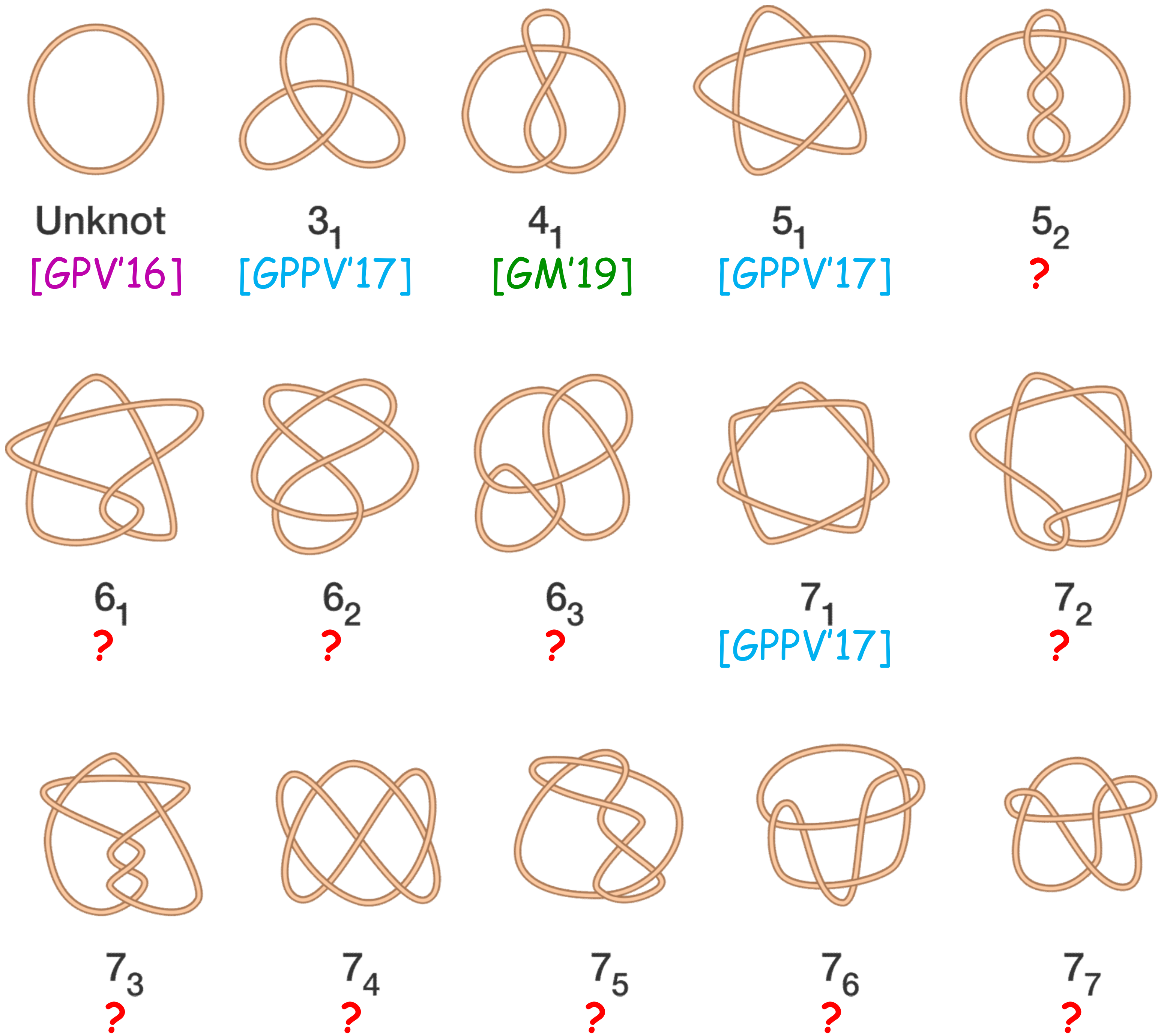
$$F_K(x,q) := \sum_{b \in \mathbb{Z}} x^b \, \widehat{Z}_b \left(S^3 \setminus K \right)$$

is well defined and invariant under the required braid moves (cf. Reidemeister moves).

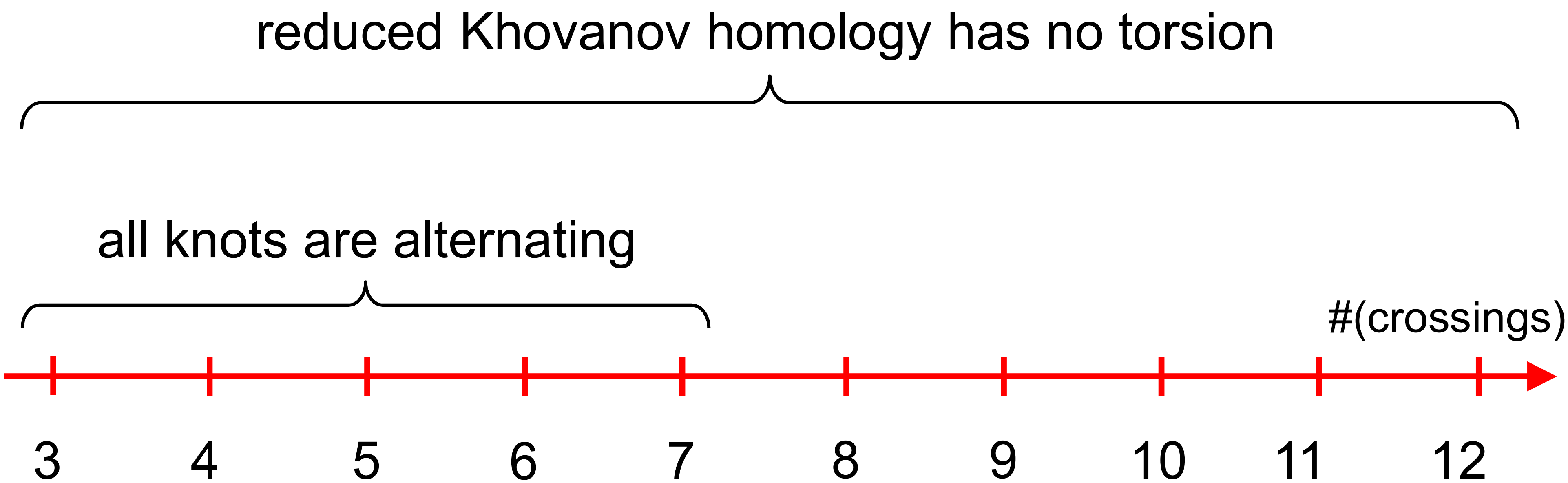
S.G., D.Pei, P.Putrov, C.Vafa
S.G., C.Manolescu
S.Park (2020, 2021)
J.Chae
:

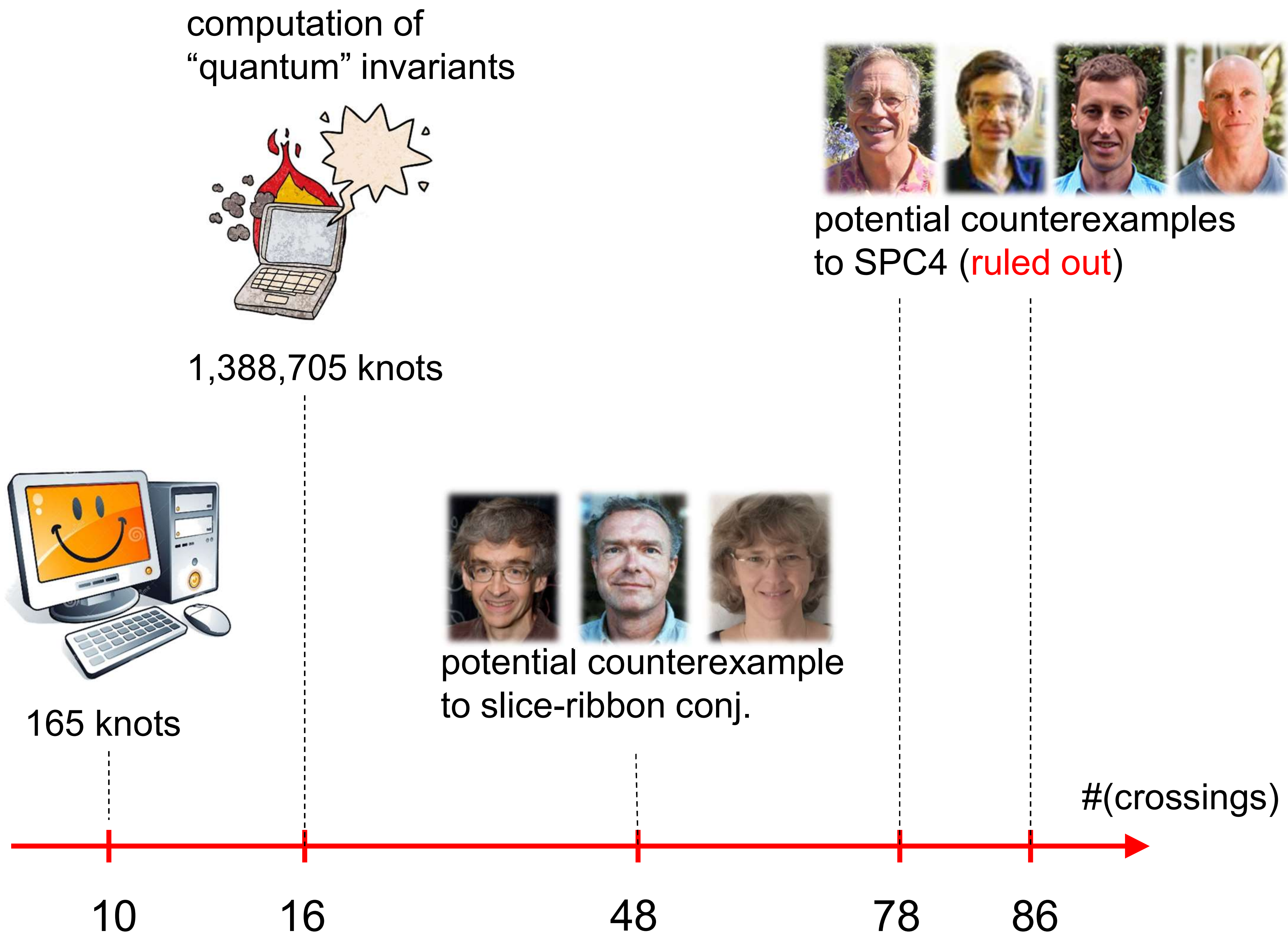


as of 2019:



Unknot
[GPV'16]
3_1
[GPPV'17]
4_1
[GM'19]
5_1
[GPPV'17]
5_2
?
6_1
?
6_2
?
6_3
?
7_1
[GPPV'17]
7_2
?
7_3
?
7_4
?
7_5
?
7_6
?
7_7
?

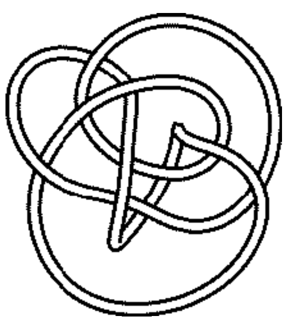

reduced Khovanov homology has no torsion
all knots are alternating
#(crossings)
3
4
5
6
7
8
9
10
11
12


computation of
"quantum" invariants
1,388,705 knots
potential counterexamples
to SPC4 (ruled out)
165 knots
potential counterexample
to slice-ribbon conj.
#(crossings)
10
16
48
78
86

$M_3 = S^3_{-1/2}($$)$:

$$\widehat{Z}(q) = q^{-\frac{1}{2}}(1 + q^2 + 3q^3 + 4q^4 + 6q^5 + 8q^6 + 12q^7 + \ldots$$
$$\ldots + 20179997428388332001212q^{500} + \ldots)$$

$M_3 = -S^3_{+5}(\mathbf{10_{145}})$:



$$\begin{aligned}
b = 2 : &\quad q^{14/5}\left(-1 + q + 2q^2 + 4q^3 + \ldots\right) \\
b = 1 : &\quad q^{11/5}\left(-1 - q - 4q^2 - 7q^3 + \ldots\right) \\
b = 0 : &\quad 2q^4 + 2q^5 + 4q^6 + 8q^7 + 14q^8 + \ldots \\
b = -1 : &\quad q^{11/5}\left(-1 - q - 4q^2 - 7q^3 + \ldots\right) \\
b = -2 : &\quad q^{14/5}\left(-1 + q + 2q^2 + 4q^3 + \ldots\right)
\end{aligned}$$

Conjecture:

"conformal weight"

$$\chi_b(q) \;=\; \widehat{Z}_b(q) \;=\; q^{\Delta_b} \sum_n a_n q^n$$

Character of a logarithmic Vertex Algebra
(that depends on M_3, but not on b)

Corollary:

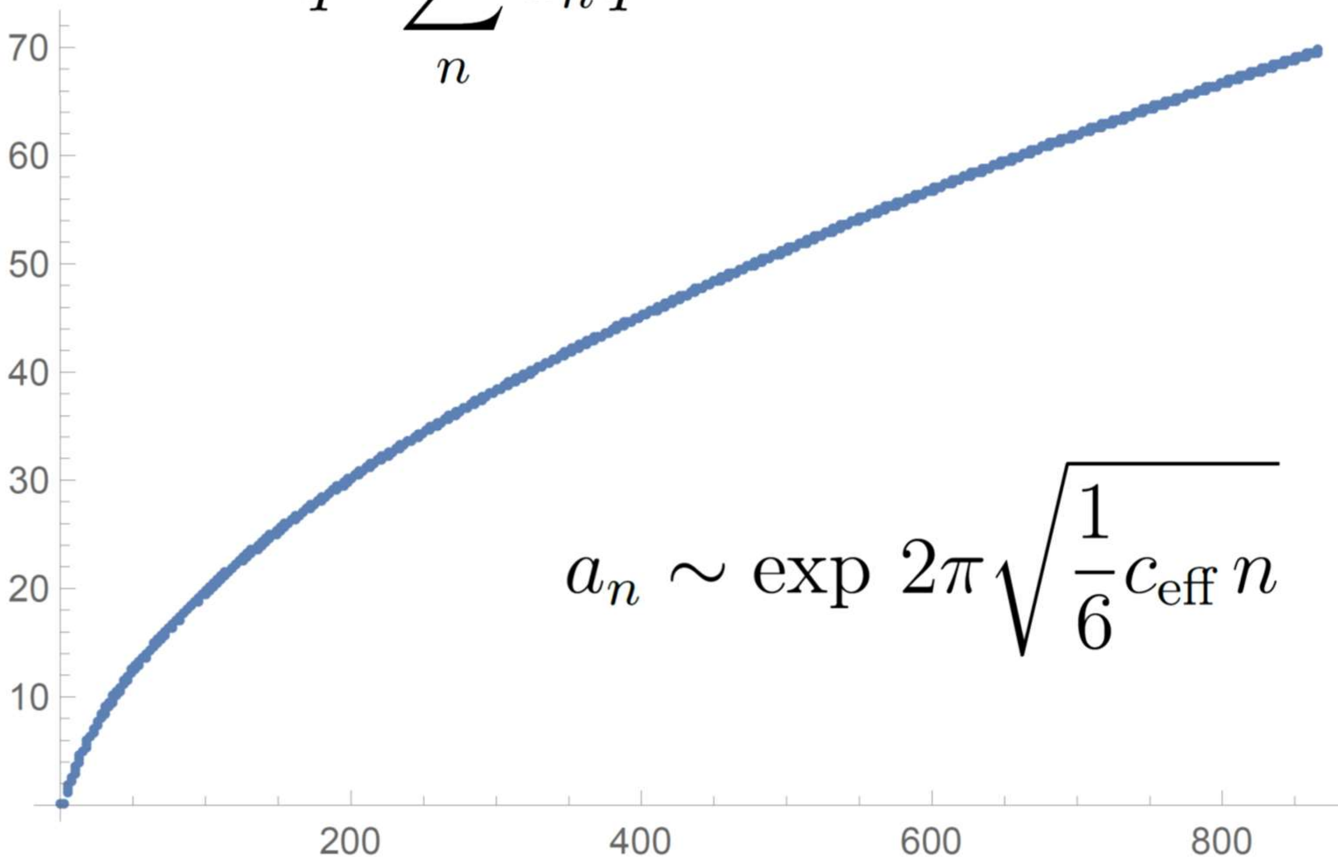
$$a_n \sim \exp\, 2\pi \sqrt{\frac{1}{6} c_{\text{eff}}\, n}$$

M.Cheng, S.Chun, F.Ferrari, S.G., S.Harrison
M.Cheng, S.Chun, B.Feigin, F.Ferrari, S.G., S.Harrison, D.Passaro
:

Surprise #1:

$$\widehat{Z}(q) = q^{-\frac{1}{2}}(1 + q^2 + 3q^3 + 4q^4 + 6q^5 + 8q^6 + 12q^7 + \ldots$$
$$\ldots + 20179997428388332001212q^{500} + \ldots)$$

$$= q^{\Delta} \sum_n a_n q^n$$



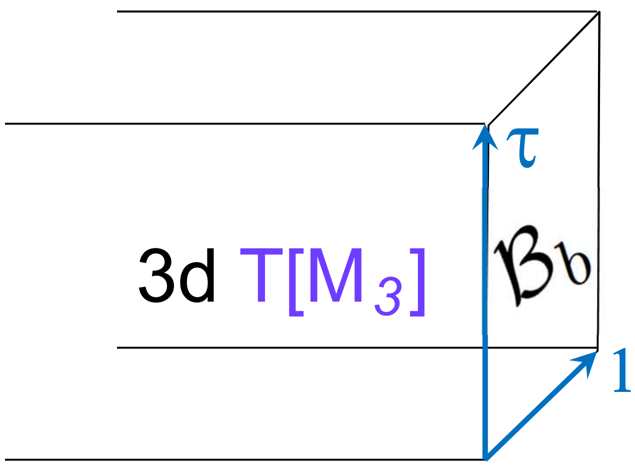
$$a_n \sim \exp 2\pi \sqrt{\frac{1}{6} c_{\text{eff}}\, n}$$

John Cardy

Surprise #2:

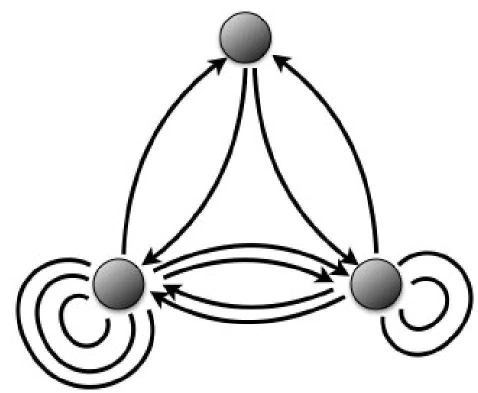
$$C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\widehat{Z}_b(M_3, q) = \sum_{d_i \geq 0} \frac{1}{(q)_{\boldsymbol{d}}} q^{\frac{1}{2}\boldsymbol{d}\cdot C\cdot\boldsymbol{d}+(\text{terms linear in } \boldsymbol{d})}$$



BPS spectrum

BPS quiver



P.Kucharski

T.Ekholm, A.Gruen, S.G., P.Kucharski, S.Park, M.Stosic, P.Sulkowski

Advanced Studies in Pure Mathematics 61, 2011
New Structures and Natural Constructions in Mathematical Physics
pp. 161–184

Fermionic formulas for characters of $(1,p)$ logarithmic model in $\Phi_{2,1}$ quasiparticle realisation

Boris Feigin, Evgeny Feigin and Il'ya Tipunin

The main result of the paper is formulated as follows.

Theorem 1.1. *The characters* (1.7) *can be written in the form*

(1.8)

$$\chi_{s,p}(q) = q^{\frac{s^2-1}{4p}+\frac{1-s}{2}-\frac{c}{24}} \sum_{n_+,n_-,n_1,\dots,n_{p-1}\geq 0} \frac{q^{\frac{1}{2}\mathbf{n}\mathcal{A}\cdot\mathbf{n}+\mathbf{v}_s\cdot\mathbf{n}}}{(q)_{n_+}(q)_{n_-}(q)_{n_1}\dots(q)_{n_{p-1}}}$$

$U_q(\mathfrak{g})$

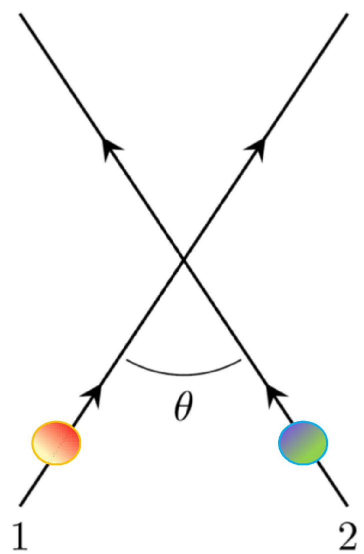
Quantum groups

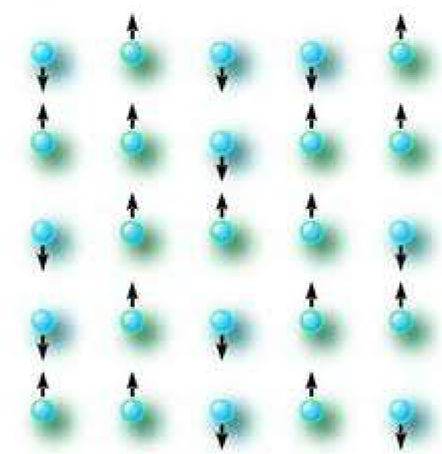
Kazhdan-Lusztig

Vertex Algebras

Integrable lattice models

2d CFT





H.Bethe (1931)

:

A.Zamolodchikov, Al.Zamolodchikov (1979)

A.Zamolodchikov (1989)

Al.Zamolodchikov (1990)

F.Smirnov (1990)

N.Reshetikhin, F.Smirnov (1990)

➡ Yangian symmetry, Bethe ansatz equation, ...

arXiv:hep-th/9301046

Fermionic Sum Representations for Conformal Field Theory Characters

R. Kedem,[1] T.R. Klassen,[2] B.M. McCoy,[1] and E. Melzer[1]

1. Introduction

Recently it was found [1] that characters (or branching functions) of the coset conformal field theories $\frac{(G^{(1)})_1 \times (G^{(1)})_1}{(G^{(1)})_2}$, G a simply-laced Lie algebra, can be represented in the form

$$\sum_{\mathbf{m}}^{Q} \frac{q^{\frac{1}{2}\mathbf{m}B\mathbf{m}^t}}{(q)_{m_1}\cdots(q)_{m_r}} \quad , \tag{1.1}$$

Rogers-Ramanujan

S.Kerov, A.Kirillov, N.Reshetikhin (1986)
A.Kirillov, N.Reshetikhin (1988)

R.Kedem, B.McCoy (1993)
R.Kedem, T.Klassen, B.McCoy, E.Melzer (1993)
S.Dasmahapatra, R.Kedem, T.Klassen, B.McCoy, E.Melzer (1993)
R.Kedem, B.McCoy, E.Melzer (1993)
A.Berkovich, B.McCoy, A.Schilling, S.Warnaar (1997)
:

E. Frenkel, A. Szenes (1993)
W.Nahm, A.Recknagel, M.Terhoeven (1993)

<u>Conjecture (“mirror symmetry”):</u>

$$\widehat{Z}(M_3, q) = \chi(q) \quad \longleftrightarrow \quad \widehat{Z}(-M_3, q) = \chi(q^{-1})$$

Character of a log-VOA

Character of a “mirror” log-VOA

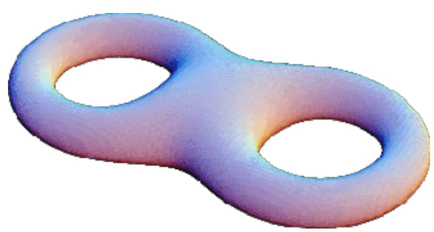
$$\text{WRT}: \quad \mathcal{H}(\Sigma) = \text{quantization of } \mathcal{M}_{\text{flat}}(G, \Sigma)$$

finite-dimensional compact

$$\text{GPPV}: \quad \mathcal{H}(\Sigma) = \text{quantization of } \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$$

infinite-dimensional non-compact

$$\omega = \frac{1}{\hbar} \int_{\Sigma} \text{Tr}\, \delta A \wedge \delta A$$

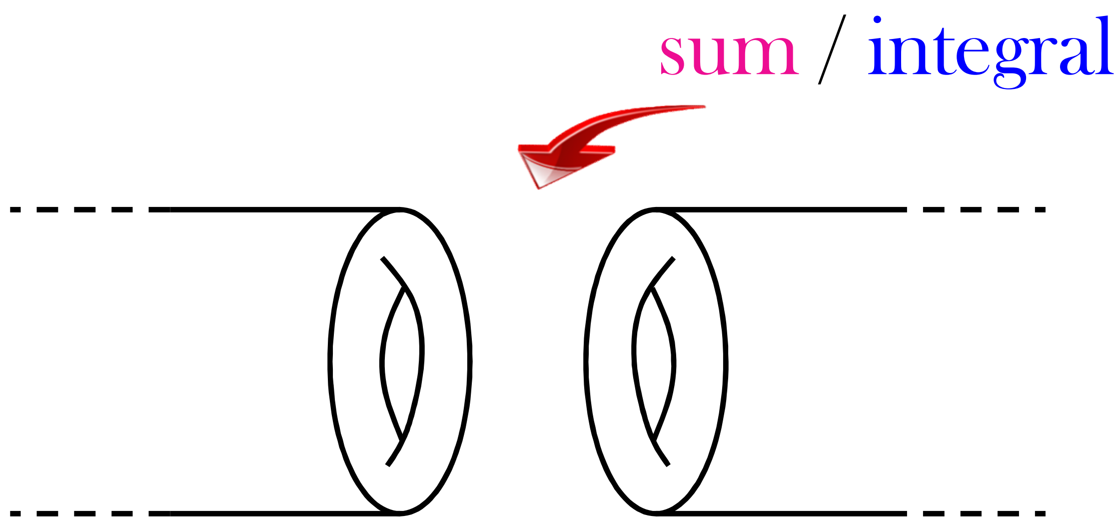


$$\text{WRT}: \quad \mathcal{H}(\Sigma) = \text{quantization of } \mathcal{M}_{\text{flat}}(G, \Sigma)$$

finite-dimensional compact

$$\text{GPPV}: \quad \mathcal{H}(\Sigma) = \text{quantization of } \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$$

infinite-dimensional non-compact

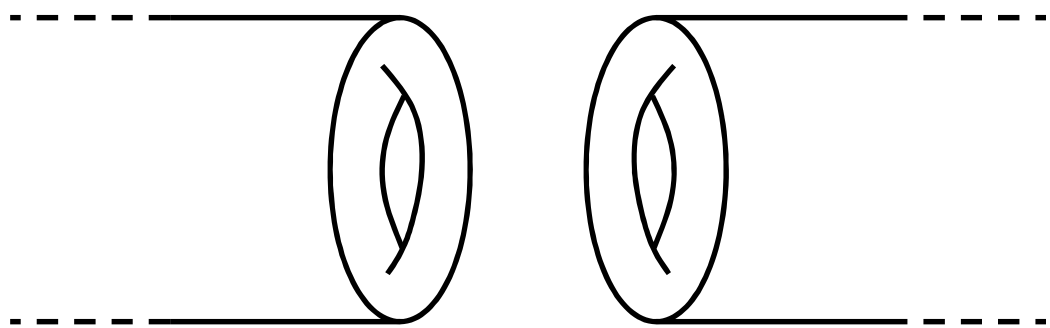


Surgery formulae:

$\mathrm{rank}(G) = 1$

$$\text{WRT}: \quad \mathcal{H}(T^2) = \mathbb{C}\left[\frac{\Lambda}{W \times k\Lambda^\vee}\right] \Rightarrow \sum_{n=1}^{k-1}$$

$$\text{GPPV}: \quad \mathcal{H}(T^2) = \mathbb{C}\left[\frac{\Lambda \times \Lambda^\vee}{W}\right] \Rightarrow \sum_{(m,n)\in\mathbb{Z}^2}$$



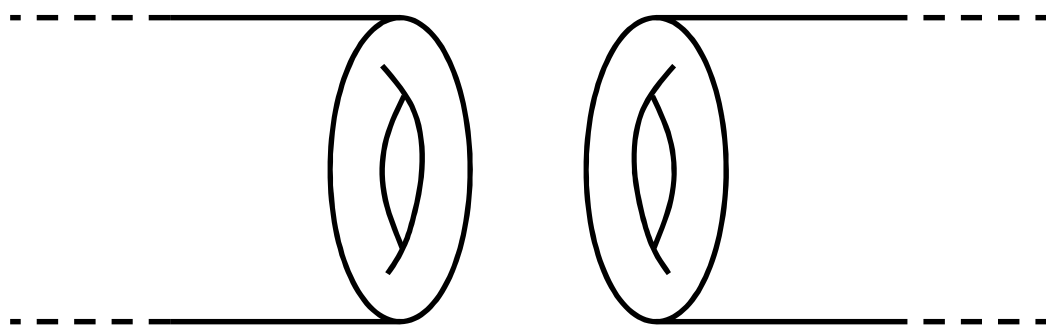
Surgery formulae:

WRT :

$$\left(\frac{k}{2}\right)^{g-1} \sum_{n=1}^{k-1} \left(\sin\frac{\pi n}{k}\right)^{2-2g}$$

GPPV :

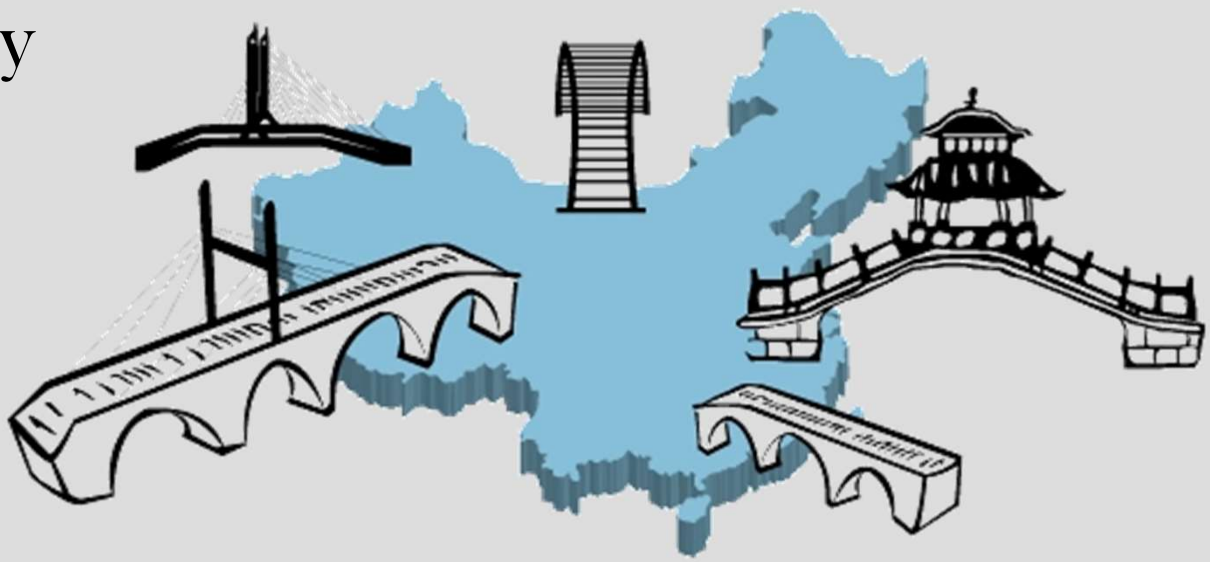
$$\sum_{n} \oint_{|x|=1} \frac{dx}{2\pi i x} \longleftrightarrow \sum_{(m,n)\in\mathbb{Z}^2}$$



topology

enumerative
geometry

mathematical
physics



gauge
theory

vertex algebra

quantum groups