From modularity to resurgence

Outline:

1. Rogers-Ramanujan fns
2. q-Hypergeometric sums
3. Quantum modularity

Modular forms are fn with lots of symmetries. eg $SL_2\mathbb{Z}$

$$= \left\langle T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

Let $\mathfrak{h} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$

Then $SL_2\mathbb{Z} \circlearrowright \mathfrak{h}$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}$

Then we say $f : \mathfrak{h} \to \mathbb{C}$ is a modular form of weight k if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$

Enough for $SL_2\mathbb{Z}$ to check

$$f(\tau + 1) = f(\tau) \qquad f(-1/\tau) = \tau^k f(\tau).$$

The first implies a Fourier expansion

$$f(\tau) = f(q) \quad \text{where } q = e^{2\pi i \tau}.$$

Today we will consider similar kind of fn where $f(\tau+1) = f(\tau)$ &

interesting analytic fn

$$f(-1/\tau) = \tau^{k} \Omega(\tau) f(\tau)$$

conj. Borel resum. of certain series ?

These fn will arise in the form

$$f(q) = \sum_{n \in \mathbb{Z}^N} q^{Q(n)+L(n)} \prod (q;q)_{\lambda(n)}^{\pm}$$

where $(a;q)_n = \prod_{j=0}^{n-1} (1-aq^j)$.

This is the form of quantum sl_2 invariants of 3-mflds & knots.

① The Rogers-Ramanujan fns are two special fns!

$$G(q) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k}, \quad H(q) = \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_k},$$

where $(a;q)_n = \prod_{j=0}^{n-1} (1-aq^j)$

$$G(q) = 1+q+q^2+\cdots \qquad H(q) = 1+q^2+q^3+\cdots$$

$$G(q) = \prod_{j=0}^{\infty} \frac{1}{(1-q^{1+5j})(1-q^{4+5j})}$$

$$H(q) = \prod_{j=0}^{\infty} \frac{1}{(1-q^{2+5j})(1-q^{3+5j})}$$

Rogers-Ramanujan-identities.

These identities have beautiful consequences for the asymp. as $q \to 1$

For example, $e(x) = \exp(2\pi i x)$

$\& \; q = e(\tau)$, $\tilde{q} = e(-1/\tau)$

$$G\left(e\left(\frac{-1}{i \times 100}\right)\right) = 3.0007\ldots \times 10^{7}$$

$$\frac{G\left(e\left(\frac{-1}{i\times 100}\right)\right)}{e\left(\frac{-i\times 100}{60}\right)\frac{2\sin(2\pi/5)}{\sqrt{5}}} = 0.99895\ldots$$

$$G\left(e\left(\frac{-1}{i\times 100}\right)\right) - e\left(\frac{-i\times 100}{60}\right)\frac{2\sin(2\pi/5)}{\sqrt{5}}\, e\left(\frac{-1}{i\times 100\times 60}\right)$$

$\boxed{e(x) \sim q\,14}$

$$= 4.9337 \times 10^{-51}$$

i.e.

$$G\left(e\left(\frac{-1}{\tau}\right)\right) \sim e\left(\frac{-\tau}{60}\right)\frac{2\sin(2\pi/5)}{\sqrt{5}}\, e\left(\frac{-1}{60\tau}\right) + o(1)$$

$$G(\tilde{q}) \sim \frac{2\sin(2\pi/5)}{\sqrt{5}}\, \tilde{q}^{\frac{1}{60}}\, q^{-1/60}\left(1 + q + q^2 + \cdots\right)$$

$$\frac{2\sin(\pi/5)}{\sqrt{5}}\, \tilde{q}^{\frac{1}{60}}\, q^{11/60}\left(1 + q^2 + q^3 + \cdots\right)$$

In fact, from the Rogers - Ramanujan identities we can express these fns as Θ-fns to prove the identity for

$$g(q) = \begin{pmatrix} q^{-\frac{1}{60}} G(q) \\ q^{\frac{11}{60}} H(q) \end{pmatrix}$$

$$g(\tau+1) = \begin{pmatrix} e(-1/60) & 0 \\ 0 & e(11/60) \end{pmatrix} g(\tau)$$

$$g(-1/\tau) = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\frac{2\pi}{5} & \sin\frac{\pi}{5} \\ \sin\frac{\pi}{5} & -\sin\frac{2\pi}{5} \end{pmatrix} g(\tau)$$

(2) We can apply a similar asymptotic analysis to a similar sum. Take

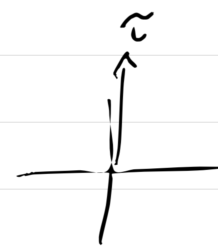
$$F_m(q) = \sum_{k=0}^{\infty} \frac{q^{2k(k+1)+mk}}{(q;q)_k}$$

This fn satisfies the recursion

$$F_m(q) - F_{m+1}(q) = q^{4+m} F_{m+4}(q)$$

Taking the classical limit of this equation gives the polynomial

$$1 - x = x^4$$



This has solutions

$x^{(1)} = -1.22\ldots \quad x^{(2)} = 0.724\ldots$

$x^{(3)} = 0.248\ldots - 1.03\ldots i \quad x^{(4)} = \overline{x^{(3)}}$

$$V^{(j)} = \underbrace{\mathrm{Li}_2(x^{(j)})}_{\left(\sum_{n=1}^{\infty} \frac{z^n}{n^2}\right)} - \frac{\pi^2}{6} + 2\log(x^{(j)})^2 - \underbrace{k^{(j)}}_{-2,0,1,-1}\log(x^{(j)})$$

Then suppose $\tau \to \infty i$ along $i\mathbb{R}$

$$F_0(e(-1/\tau)) \sim e\left(\frac{V^{(2)}}{(2\pi i)^2}\tau\right)\frac{1}{\sqrt{\delta^{(2)}}}\left(1 + A_1^{(2)}\frac{2\pi i}{\tau} + \cdots\right)$$

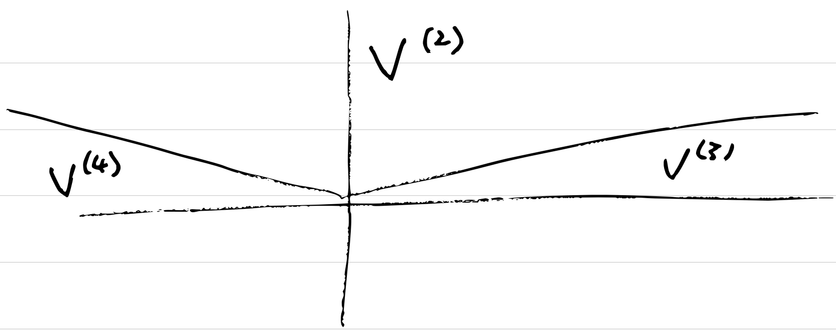
where $\delta = 4 - 3x$

$$A_1 = \frac{-64 + 100x + 18x^2 - 54x^3}{24\,\delta^3}$$

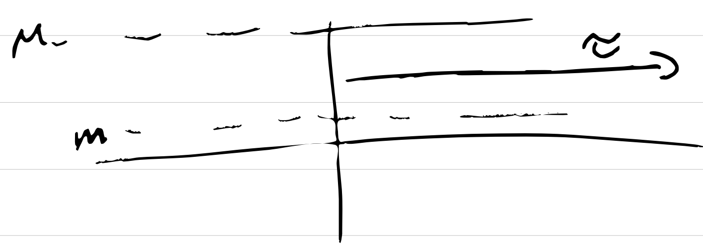
$\vdots$

While if $\tau \to i\infty$ on a small angle just above the reals

$$F_0(e(-1/\tau)) \sim e\left(\frac{V^{(3)}}{(2\pi i)^2}\tau\right)\frac{1}{\sqrt{S^{(4)}}}\left(1 + A_1^{(3)}\frac{2\pi i}{\tau} + \cdots\right)$$



If $|\tau| \to \infty$ on an angle bounded away from 0 or π then $q \to 0$ exponentially. However, we can take $|\tau| \to \infty$ & $m < \mathrm{Im}(\tau) < M$



There are not really any good asymptotics satisfied by F_0 as τ behaves this way. However, it can be fixed. Indeed, as $\tau \to \infty$ with compact. Im part

$$\frac{F_0(\tilde{q})}{F_1(q)} \sim e\left(\frac{V^{(3)}}{(2\pi i)^2}\tau\right)\frac{1}{\sqrt{\delta^{(3)}}}\left(1 + A_1^{(3)}\frac{2\pi i}{\tau} + \cdots\right)$$

This leads to the idea of...

③ Quantum modularity QMF

Defn 1: f is a quantum modular form (QMF) if $f(\tau+1) = f(\tau)$ & as $\tau \to \infty$ with cpct im. part

$$f(-1/\tau) \sim \phi\left(\frac{2\pi i}{\tau}\right) f(\tau)$$

asymp. series with $e\left(\frac{V}{(2\pi i)^2}\tau\right)$ term.

Let $\hat{\Phi}_m^{(j)}(\tau) = e\left(\frac{V^{(j)}}{(2\pi i)^2}\tau\right)\frac{1}{\sqrt{\delta^{(j)}}}\left(1 + A_{1,m}^{(j)}\frac{(2\pi i)}{\tau} + \cdots\right)$

Then we have using similar methods to Bettin-Drappeau.

Thm [W]: As $\tau \to +\infty$

$$F_m(-1/\tau) \sim \hat{\phi}_m^{(3)}(\tau)\, F_1(\tau)$$

i.e. F_1 is a QMF.

However, following work of Garoufalidis, Gu, Kashaev, Mariño, Zagier, we can refine this both conjectually & analytically.

Firstly, we note that numerically

$$\frac{A_{k,m}^{(3)}}{\sqrt{\delta^{(3)}}} \sim \frac{1}{2\pi i}\sum_{\ell \geq 0}\frac{\Gamma(k-\ell)}{(V_2 - V_3)^{k-\ell}}\frac{A_\ell^{(2)}}{\sqrt{\delta^{(2)}}}$$

Conj: $\hat{\phi}_m^{(j)}$ is 1-Borel resummable with ~~Stokes rays~~ with log sing in Borel plane at

$$\boxed{\begin{aligned} &s\left(\sum a_i x^i\right) \\ &= \int_0^\infty \exp(-\zeta) \sum \frac{a_i x^i \zeta^i}{\Gamma(i+1)} d\zeta \end{aligned}} \qquad \frac{V^{(i)} - V^{(j)}}{2\pi i} + 2\pi i k \quad {\scriptstyle k \in \mathbb{Z}}$$

Moreover, on the Stokes ray i, k $\in \mathbb{Z}$.

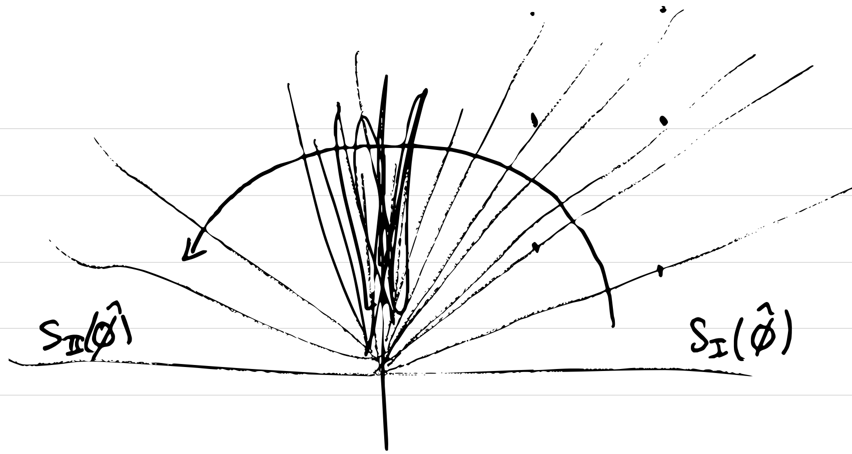
$$S_+\left(\hat{\phi}_m^{(j)}\right) - S_-\left(\hat{\phi}_m^{(j)}\right) = S_{j,i,k}\, q^k\, s\left(\hat{\phi}_m^{(i)}\right)$$

Therefore if we take a matrix

$$s(\hat{\phi}) = \left(S(\phi_m^{(j)})\right)_{\substack{j=1,\dots,4 \\ m=0,\dots,3}} \quad \text{we have}$$

$$S_+(\hat{\phi}) = S_-(\phi) \underbrace{\begin{pmatrix} 1 & 0 & q^k S & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{= \sigma_{i,j,k}(q)}$$

Therefore, we can compute the difference between analytic cont. of Borel resum. by taking prod. of $\sigma_{i,j,k}$.



We get an ∞-prod. Instead can compute $S_I \hat\phi$ & $S_{II}(\hat\phi)$ as the power of q separates the Stokes.

To find a numerical guess for S_I S_{II} we make use of the theorem.

<u>Thm</u> [W] There exists (explicit)

$$f(q) = \begin{pmatrix} F_0(q) & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot \\ F_3(q) & \cdot & \cdot & \cdot \end{pmatrix}$$

s.t $f(\tilde q) = \Omega_s(\tau)\, f(q) \begin{pmatrix} 1 & & & \\ & \tau & & \\ & & \tau^2 & \\ & & & \tau^4 \end{pmatrix}$
where Ω is analytic on $\mathbb{C} - \mathbb{R}_{\leq 0}$

<u>Def^n</u> 2 f is QMF if $f\left(\frac{a\tau+b}{c\tau+d}\right) = \Omega_\gamma(\tau) f(\tau)$
with $\Omega_\gamma(\tau)$ hol. on $\mathbb{C} - (\mathrm{sgn}(c))\mathbb{R}_{\leq -d/c}$

Sim. $f(\tilde{q}) = \Omega_{-S}(\tau)\, f(\tau) \begin{pmatrix} 1 & & & \\ & -\tau & & \\ & & \tau^2 & \\ & & & \tau^4 \end{pmatrix}$

where $\Omega_{-S}(\tau)$ is analytic for $\tau \in \mathbb{C} - \mathbb{R}_{\geq 0}$.

Rmk: To not get trivial therefore need a basis with different weights / automorphy at $S, -S$.

Conj.^{2}: $S_{I}(\hat{\phi}) = \Omega_S(\tau) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}^{-1}$

$$S_{II}(\hat{\phi}) = \Omega_{-S}(\tau) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 1 & 0 & 0 \end{pmatrix}^{-1}$$

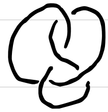
Cor Conj 1 + Conj 2 $\Rightarrow$ Stokes constants in upper half plane are given by

$$S_I(\hat{\phi})^{-1} S_{II}(\hat{\phi}) - I$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} f(q) \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} f(q)^{-1} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 1 & 0 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -q-2q^2 & 1+q+q^2 & 1-q^2 & -1-q \\ q^2 & -q-q^2 & -q & q+q^2 \\ -q-q^2 & 1-q^2 & -q-2q^2 & 2q^2 \\ q & -1+q+q^2 & 2q+q^2 & -q-2q^2 \end{pmatrix} + O(q^3)$$

One can perform similar computations for examples coming from 3-mflds.

Ex: $4_1(1,2)$  $=4_1$ $\quad$ $(1,2)$ glue in solid torus to S^3-4_1 with a twist.

$$WRT_m(q) = \sum_{0\le l\le k} (-1)^k q^{-\frac{1}{2}k(k+1)+l(l+1)+mk} \times \frac{(q)_{2k+1}}{(q)_l (q)_{k-l}}$$

$$\hat{Z}_m(q) = (q)_\infty^2 \sum_{k,j=0}^{\infty} \frac{q^{k(2k+1)+jk+j-mk-m}}{(q)_j (q)_{2k} (q)_{k+j}}$$

Thm [W] $\exists$ 8×8 matrix of q-ser.

$$Z(q) = \begin{bmatrix} \hat{Z}(q) & \cdots & \cdots \\ \vdots & \ddots & \\ \vdots & & \ddots \end{bmatrix}$$

s.t.

$$Z(\tilde{q}) = \Omega_{\pm S}(\tau) Z(q)$$

analytic on $\mathbb{C} - (\pm\mathbb{R}_{\le 0})$

sim comp gives conj.

$\rightsquigarrow$ Stokes. the closest being

$$\begin{pmatrix} -1 & & 1 & & & -1 & -1 & -1 \\ -1 & & 1 & & & -1 & -1 & -1 \\ 1 & & -1 & & & 1 & 1 & 1 \\ 1 & & -1 & & & 1 & & \\ 1 & & -1 & & & 1 & & \end{pmatrix}$$