RESURGENCE IN 3-DIMENSIONAL TOPOLOGY

NUMBER THEORY SEMINAR

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• I will talk about resurgence of certain exact and perturbative quantum invariants of 3-dimensional spaces.

<u>Good examples</u> of 3-dimensional manifolds $= S^3 - K$, or if you wish $K \subset S^3$ knot.

<u>An example</u> of an exact invariant is $\langle K \rangle_N$ = Kashaev invariant

Here $N = 1, 2, 3, \ldots$

<u>Fact</u> <u>Ie</u> $\langle K \rangle_N$ grows exponentially wrt N

(G-Le)

$$|\langle K \rangle_N| \le \exp\left(\frac{N v_8}{2\pi} \#(\text{crossings of } K)\right)$$

$v_8 = 3.66\ldots$ = volume of regular ideal octahedron (Catalan's constant)

This bound is asymptotically sharp.

Def $G_K(z) = \sum_{N=1}^{\infty} \langle K \rangle_N z^N$

is the germ of an analytic

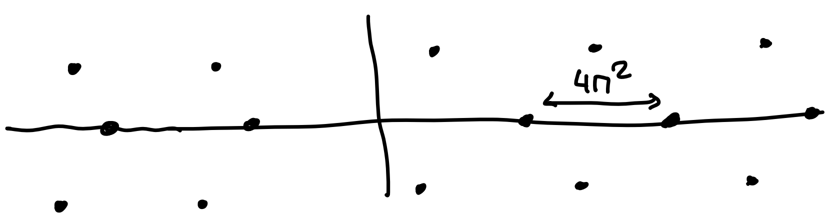
Conj (G 2008) $G_K(z)$ is a resurgent function.

Complements

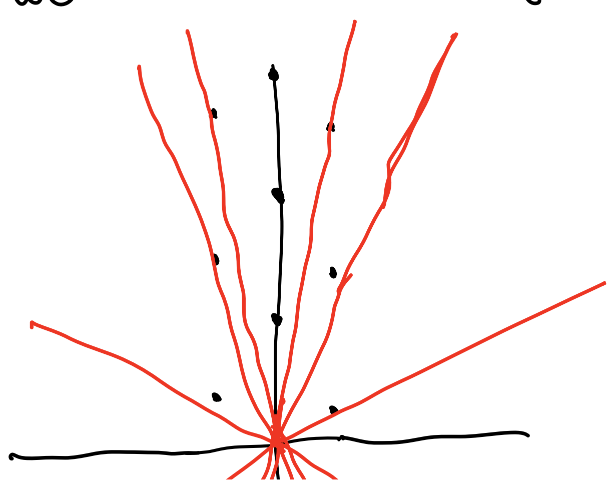
Singularities in Borel plane are $\exp\left(-\frac{\Lambda_K}{2\pi i}\right)$

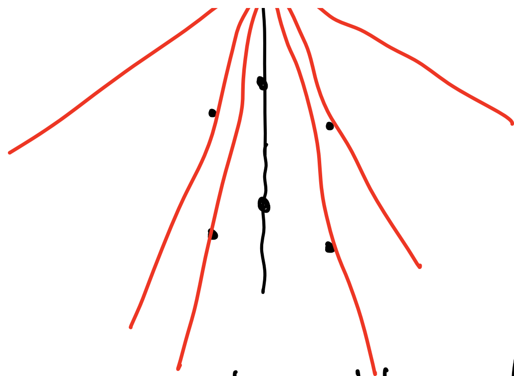
$$\Lambda_K = v(CS(\rho) + 4\pi^2 \mathbb{Z})$$

ρ = boundary parabolic reps



If we rotate 90° we get

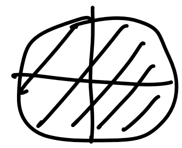




the peacock patterns (Gu–G–Mariño)

Moreover, branch behavior is known.

Thus, if you wish, you can take $\widetilde{\mathbb{C}-\Lambda_K}$ (universal cover)

and uniformize it to  $|z|<1$

Note that Λ_K is readily computable.

Note Above conjecture was formulated prior to the work on asympt expansion of Kashaev invariant (eg of Dimofte-Gukov-Lenells-Zagier)

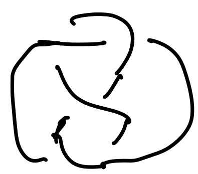
Note Conjecture also formulated for closed 3-manifolds M where instead of Kashaev inv, one use WRT

$$Z_N(M) \text{ which is } O(N^{poly}).$$

Campbell Wheeler studies the ramifications of resurgence for $M=-1/2$ surgery on 4_1 knot.

So what are the implications of $G_K(z)$ being resurgent?

Test it for simplest hyperbolic knot (4₁ knot)



$$\langle 4_1 \rangle_N = \sum_{n=0}^{N-1} (q;q)_n (q^{-1};q^{-1})_n \qquad \boxed{q = e^{2\pi i/N}}$$

$$(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1}) \quad n \geq 0$$

$$= \sum_{n=0}^{\infty} (q;q)_n (q^{-1};q^{-1})_n$$

(an element of the Habiro ring)

So, let us look at asymptotics of coeffs of $G_{4_1}(z)$ $\left(\begin{array}{l}\text{equivalent singularities} \\ \text{of } G_{4_1}(z)\end{array}\right)$

Volume Conjecture (Kashaev)

$$\frac{1}{N}\log|\langle K\rangle_N| \sim \mathrm{vol}(K)$$

For 4_1 knot, we find

$$\langle 4_1\rangle_N \sim e^{\frac{\mathrm{vol}(4_1)}{2\pi}N} \cdot N^{3/2} \cdot \Phi\left(\frac{2\pi i}{N}\right)$$

where

$$\mathrm{vol}(4_1) = 2\,\mathrm{Im}(\mathrm{Li}_2(e^{2\pi i/6})) \sim 2.02$$

and

$$\Phi(h) = \frac{1}{\sqrt[4]{3}}\left(1 + \frac{11}{72\sqrt{-3}}h + \frac{697}{2(72\sqrt{-3})^2}h^2 + \frac{724351}{30(72\sqrt{-3})^3}h^3 + \dots\right)$$

Note. Terms are alg numbers in $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(e^{2\pi i/6})$ except for leading term whose square, up to a power of i, is also. So coeffs are exact.

Note $\Phi(h)$ is a factorially divergent formal power series

Note We can compute $\langle 4_1\rangle_N$ $N=20000$ with 10000 digits of precision, use Richardson transform and from it extract first 150 coeffs of $\Phi(h)$

Note In [Dimofte-G, 2008] we gave a formula for $\Phi(h)$ using ideal triangulations, NZ solutions and formal Gaussian integration.

Note $\Phi(h)$ can also be computed from a stationary phase approximation to a finite dim (1-dim for 4_1) state-integral

Doing so, Jie Gu got easily 300 terms

Alright, we now have good control on $\Phi(h)$.

In a paper with Don Zagier (Knots, perturbative series and quantum modularity) we went a bit further.

For 4_1, when $N=100$

$$\langle K \rangle_{100} = 8198518838051 2462.93100549954341$$

$$100^{3/2} \hat{\Phi}\left(\frac{2\pi i}{100}\right)^{opt} = 81985188380512461.9269943535808$$

$$\langle K \rangle_{100} - 100^{3/2} \hat{\Phi}\left(\frac{2\pi i}{100}\right)^{opt} = 1.0040111 4185$$

Repeating the experiment with $N=100$ being replaced by $100+10n$, $n=1..20$ we found out that this difference has the asymptotic expansion

$\Phi^{(0)}(h) = 1 - h^2 + \frac{47}{12} h^4 + \dots$ with $h = \frac{2\pi i}{N}$

Moreover (discovery)

$$\Phi^{(0)}(h) := \sum_{n=0}^{\infty} (q;q)_n (q^{-1};q^{-1})_n = 1 - h^2 + \frac{47}{12} h^4 + \dots$$

when $q = e^h$

<u>Note</u> $(q;q)_n = (1-e^h)\dots(1-e^{nh})$

$= (-1)^n n!\, h^n + O(h^{n-1})$

Thus the series $\Phi^{(0)}(h)$ is <u>well</u>-defined and factorially divergent.

OK. Now let us repeat the experiment calling $\Phi(h) \rightsquigarrow \Phi^{(\sigma_1)}(h)$ $\Phi^{(0)}(h) \rightsquigarrow \Phi^{(\sigma_0)}(h)$

Now

$$100^{-3/2} \langle K \rangle_{100} - \hat{\Phi}^{(\sigma_1)}\left(\frac{2\pi i}{100}\right)^{opt} - \Phi^{(\sigma_0)}\left(\frac{2\pi i}{100}\right)^{opt} \sim \Phi^{(\sigma_2)}\left(\frac{2\pi i}{100}\right)$$

to all 49 digits of precision

where $\Phi^{(\sigma_2)}(h) = \Phi^{(\sigma_1)}(-h)$.

What are $\Phi^{(\sigma)}(h)$ for $\sigma = \sigma_1, \sigma_0, \sigma_2$?

They are the 4_1 3 boundary parabolic representations.

<u>Thus</u> $\langle 4_1 \rangle_N$ gives rise to 3 asymptotic (resurgent) series $\Phi^{(\sigma_j)}(h)$ $j=0,1,2$

<u>Next step</u> Asymptotics of the coefficients of $\phi^{(\sigma_j)}(\hbar)$

Write $\phi^{(\sigma)}(\hbar) = \sum_{n=0}^{\infty} A^{(\sigma)}(n)\,\hbar^n$

Then

$$A^{(\sigma_1)}(n) \sim \frac{3}{2\pi i} \sum_{\ell \geq 0} A^{(\sigma_2)}(\ell)\, \frac{(n-1-\ell)!}{(2V(4_1))^{n-\ell}}$$

$$A^{(\sigma_2)}(n) \sim \frac{-3}{2\pi i} \sum_{\ell \geq 0} A^{(\sigma_1)}(\ell)\, \frac{(n-1-\ell)!}{(-2V(4_1))^{n-\ell}}$$

$$A^{(\sigma_0)}(n) \sim \sqrt{2\pi} \sum_{\ell \geq 0} A^{(\sigma_1)}(\ell)\, \frac{\Gamma(n-\ell+\frac{3}{2})}{(-V(4_1))^{n-\ell+3/2}} - \sqrt{2\pi} \sum_{\ell \geq 0} A^{(\sigma_2)}(\ell)\, \frac{\Gamma(n-\ell+\frac{3}{2})}{(V(4_1))^{n-\ell+3/2}}$$

So

$$A^{(\sigma)}(n) \sim (2\pi)^{\kappa_\sigma - 1} \sum_{\sigma' \neq \sigma} M_K(\sigma,\sigma') \sum_{\ell \geq 0} A^{(\sigma')}(\ell)\, \frac{\Gamma(n-\ell+\kappa_\sigma)}{(V(\sigma)-V(\sigma'))^{n-\ell+\kappa_\sigma}}$$

where $\kappa_\sigma = \begin{cases} 3/2 & \sigma = \sigma_0 \\ 0 & \sigma \neq \sigma_0 \end{cases}$

$$M_{4_1} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & -3 \\ 0 & 3 & 0 \end{pmatrix} \qquad 3 \times 3$$

Incidentally the next 2 simplest hyp knots are 5_2 and $(-2,3,7)$

Trace field $\quad \alpha^3 - \alpha^2 + 1 = 0$

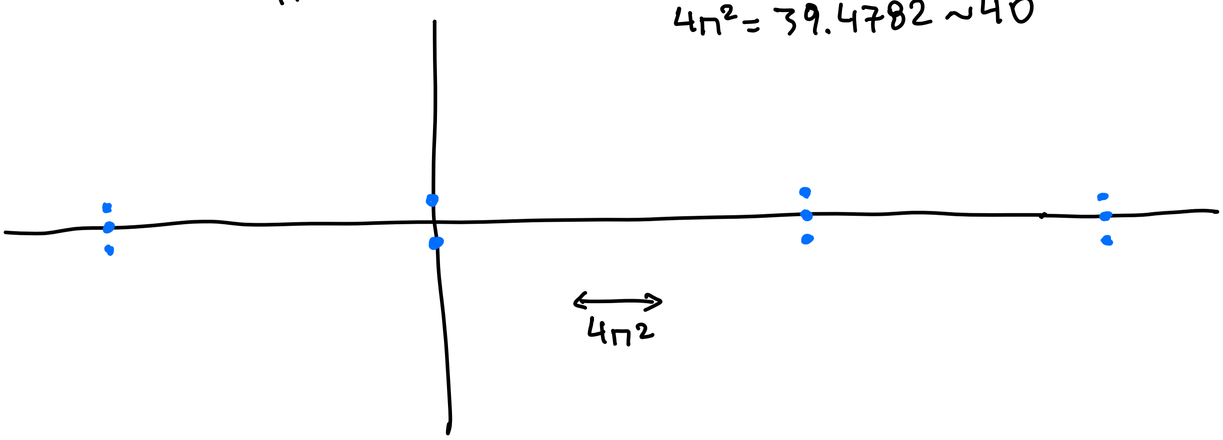
$$M_{5_2} = \left(\begin{array}{c|ccc} 0 & 1 & 1 & -1 \\ \hline 0 & 0 & 4 & -3 \\ 0 & -4 & 0 & 3 \\ 0 & 3 & 3 & 0 \end{array}\right) \qquad 4\times 4$$

$$M_{(-2,3,7)} = \left(\begin{array}{c|cccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & -1 & -1 & & & & \\ 0 & -2 & -2 & & & 0 & \\ 0 & -2 & -2 & & & & \\ 0 & -2 & -2 & & & & \end{array}\right) \qquad 7\times 7$$

Next step

What lies beyond the asymptotics of $\Phi^{(\sigma_1)}_{4_1}(\hbar)$?

$V(\sigma_1) = i * 2.02\ldots$
$V(\sigma_0) = 0$
$V(\sigma_2) = -i * 2.02\ldots$
$4\pi^2 = 39.4782 \sim 40$



In other words

Compute the Stokes constants for all transseries corrections.

In other words

Compute the Stokes phenomenon of Borel transform of $\Phi_{4_1}^{(\sigma_i)}(\hbar)$.

This was achieved in

G-Gu-Mariño: The resurgent structure of quantum knot invariants.

Key features The transseries corrections of $\Phi^{(\sigma)}(\hbar)$ involve

$S \cdot \tilde{q} \cdot \Phi^{(\sigma')}(\hbar)$

$\operatorname{Im}\tau > 0 \to \operatorname{Re}\hbar < 0$

$\hbar = 2\pi i\tau$

$\tau = \frac{\hbar}{2\pi i}$

$\tilde{\tau} = -\frac{1}{\tau} = -\frac{2\pi i}{\hbar}, \quad 2\pi i\tilde{\tau} = \frac{4\pi^2}{\hbar}$

$q = e^{\hbar}$

$\tilde{q} = e^{4\pi^2/\hbar}$

where $S \in \mathbb{Z}$

Thus the transseries have the form

$g^{(\sigma)}(\tilde{q})\,\Phi^{(\sigma)}(\hbar)$ where $g^{(\sigma)}(q) \in \mathbb{Z}[[q]]$

See equations (79), (80) of GGM paper

There, two Stokes matrices appear

$$S^{+}(q) = \begin{pmatrix} & \\ & \end{pmatrix} \qquad S^{+}(0) = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$S^{-}(q) = \begin{pmatrix} & \\ & \end{pmatrix} \qquad S^{-}(0) = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

In the course of the calculation of the transseries of $\Phi^{(\sigma_j)}_{4_1}(\hbar)$ for $j=1,2$ the following $\tilde{q}$ series were found (eqn (71) of GGM)

$$M_I(q) = \begin{pmatrix} 1-q-2q^2-2q^3-2q^4\ldots & -\text{sgminus} \\ 1-7q-14q^2-8q^3-2q^4\ldots & \text{sGminus} \end{pmatrix}$$

$$\text{sgminus} = 1-2q-3q^2-2q^3-q^4+3{}^{5}+5q^6+11q^7+13q^8+17q^9+17q^{10}+17q^{11}+\ldots$$

$$\text{sGminus} = 1+10q+15q^2-2q^3-19q^4-69q^5-85q^6-145q^7-137q^8-133q^9-73q^{10}+23q^{11}+\ldots$$

$$\text{sgminus}(q)\cdot G(q) + \text{sGminus}(q)\, g(q) - 2 = 0$$

Actually $\text{sgminus} = g^{(1)}$

$\text{sGminus} = G^{(1)}$

where

<u>Conjecture</u>

$$M_I(q) = \begin{pmatrix} g^{(0)} & -g^{(0)} - g^{(-1)} \\ G^{(0)} & -G^{(0)} - G^{(-1)} \end{pmatrix}$$

$$g^{(m)}(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2+nm}}{(q;q)_n^2}$$

$$G^{(m)}(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2+nm}}{(q;q)_n^2}\left(2m + E_1(q) + 2\sum_{j=1}^{n} \frac{1+q^j}{1-q^j}\right)$$

$E_1(q) = 1 - 4\sum_{n=1}^{\infty} q^n/(1-q^n)$ first Eisenstein series

Thm $g^{(m)}, G^{(m)}$ is a fundamental solution to

ⓐ $y_{m+1}(q) - (2-q^m) y_m(q) + y_{m-1}(q) = 0, \; m \in \mathbb{Z}$

ⓑ $\det\begin{pmatrix} g^{(m)} & G^{(m)} \\ g^{(m+1)} & G^{(m+1)} \end{pmatrix} = 2$

Hence full resurgence structure of $\Phi^{(0)}_{4_1}(\hbar)$ is governed by a 2×2 fundamental solution to a linear q-difference eqn.

Same for $\Phi^{(0)}_{5_2}(\hbar)$ 3×3.

Extension to 3×3: done in G-Gu-Mariño-Wheeler for 4_1.

4×2 for 5_2.

But how was $g^{(m)}(q), G^{(m)}(q)$ guessed?

This requires to go back to history 12 years ago.

Another part of the story

(complementary)

The second paper with Don Zagier

Knots and their related q-series.

On the train back from Diablerets 2011

Don and I computed the radial asymptotics

of the series

$$g(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_n^2} \qquad q = e^{2\pi i \tau}$$

$\tau \in i\mathbb{R}_+ \to 0$ and much to our surprise

we found out that

$$g(e^{2\pi i\tau}) \sim \sqrt{\tau}\left(\hat{\Phi}(2\pi i\tau) - i\hat{\Phi}(-2\pi i\tau)\right)$$

Thus g is related to 4_1. But why?

There is an Andersen-Kashaev state integral whose factorization produces

<u>Thm</u> (G Kashaev)

$$2i\left(\frac{\tilde{q}}{q}\right)^{\frac{1}{24}} \int_{\mathbb{R}+i\varepsilon} \Phi_b(x)^2 e^{-\pi i x^2} dx$$

$$= \sqrt{\tau}\, G(q)\, g(\tilde{q}) - \frac{1}{\sqrt{\tau}}\, g(q)\, G(\tilde{q})$$

where

$$G(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n^2} \left(2m + E_1(q) + 2\sum_{j=1}^{n} \frac{1+q^j}{1-q^j}\right)$$

$$E_1(q) = 1 - 4\sum_{n=1}^{\infty} q^n/(1-q^n)$$ first Eisenstein series

Thm There is a 2x2 matrix of state (GZagier) integrals that bilinearly factorizes in terms of q ~~times~~ $\tilde{q}$ series $g^{(m)}, G^{(m)}$ and extends past cut plane.

This is a holo quantum modular form responsible for the complete description of the resurgent structure of the matrix of $\phi^{(\sigma,m)}(h)$ series.

But all this is conjectural, and numerically checked.

The story extends to closed 3-manifolds and the example of $M = -1/2$ surgery on 4_1 is studied in Wheeler's thesis for the full 8×8 matrix (1 trivial connection plus 7 $PSL_2\mathbb{C}$ other ones)