Exact WKB for the quantum periods

Qianyu Hao, UNIGE
(work with Alba Grassi, Andy Neitzke)

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In this talk, I will focus on perturbation series obtained from a second order ODE

$$\left(\epsilon^2\partial_z^2 + P(z)\right)\psi(z) = 0, \quad \text{where } V(z) = \Lambda^2/(2z^3) - \frac{u}{z^2} + 2m\Lambda/z - \Lambda^2.$$

This is a complexified Schrödinger equation, where $z \in C$, and C is a Riemann surface. u, m, Λ, ϵ are complex parameters of the theory.

Solutions to the ODE can be built by Borel summation of the all-order WKB series in the perturbation theory around $\epsilon = 0$

$$\psi(z) = \mathrm{e}^{\frac{1}{\epsilon}\int_{z_0}^{z}\sum_{n=0} Y^n(z)\epsilon^n \mathrm{d}z}.$$

[Dingle, Écalle, Nikolaev, Silverstone, Voros, $\cdots$]

This is one way in the scheme of the exact WKB method.

$Y(z,\epsilon) = \sum_{n=0} Y^n(z)\epsilon^n$ in the WKB ansatz $\psi(z) = \mathrm{e}^{\frac{1}{\epsilon}\int_{z_0}^{z} \sum Y(z,\epsilon)\mathrm{d}z}$ can be obtained by solving the Riccati equation

$$-Y(z,\epsilon)^2 - \epsilon\partial_z Y(z,\epsilon) + P(z) = 0, \text{ where } P(z) = V(z) - u$$

The classical term in the series is

$$Y^{0,(i)}(z) = \pm\sqrt{P(z)}, \quad \text{for } i = 1, 2.$$

After picking $Y^{0,(i)}(z,\epsilon)$, all $Y^{n,(i)}(z,\epsilon)$ are determined. $Y^{(i)}(z,\epsilon)$ produces 2 independent solutions $\psi^{(i)}$ of the ODE.

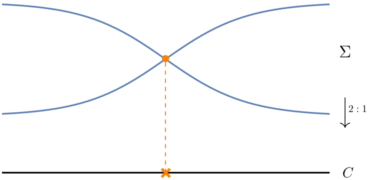
The Riccati equation swallowed the complexity of the 2nd-order ODE and we can recast it into a 1st-order ODE

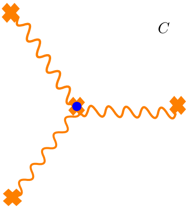
$$\left(\partial_z - \epsilon^{-1} Y^{(i)}\right) \psi^{(i)} = 0.$$

In this rewriting, we go from the Riemann surface C to the 2-fold cover of it defined by the classical part of the Riccati equation

$$\Sigma = \{(Y^0)^2 = P(z)\}.$$

The sheets of SW curve correspond to the distinct solutions $\psi^{(i)}$.



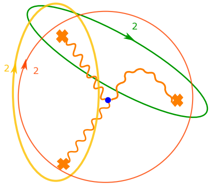


The main object in the exact WKB are the quantum periods Π_γ (or $\mathcal{X}_\gamma = \mathrm{e}^{\frac{\Pi_\gamma}{\epsilon}}$), also known as Voros symbols or spectral coordinates.

A definition of quantum period is the Borel summation of the series

$$\Pi_\gamma \equiv s(\Pi_\gamma^{\mathrm{WKB}}) = s(\sum_{n=0} \oint_\gamma Y^n(z) \mathrm{d}z \epsilon^n),$$

for some $\gamma \in \Gamma \subset H^1(\Sigma, \mathbb{Z})$.



The information of the monodromy or Stokes data is captured by quantum periods.

In this talk, we will show an exact quantization condition $\mathcal{X}_\gamma = -1$.

This ODE is an example of the QFT-ODE correspondence.

It corresponds to the 4d $\mathcal{N} = 2$ $SU(2)$ gauge theory with one flavor coupled to a surface defect. The moduli space of the surface defect is C.

The Seiberg-Witten curve Σ describing the IR physics can be obtained from the chiral ring of the surface defect.

$$\sigma^2 = P(z)$$

The ODE can be obtained by turning on the Ω-background with parameter ϵ corresponding to the rotation along the surface defect.[Jeong, Nekrasov, Shatashvili, $\cdots$]

The Ω-background quantizes the structure of chiral operators.[Neitzke, Shehper, $\cdots$]

We can construct a vector bundle over the moduli space C from chiral operators. It comes with a natural $\epsilon-$connection obtained by keeping track of the Q_ϵ invariant chiral operators, e.g.

$$\epsilon\partial_z + \begin{pmatrix} 0 & P(z) \\ 1 & 0 \end{pmatrix}.$$

It is straight forward to recast an equation for the flat section of the ϵ-connection into an oper or ODE, e.g.

$$\epsilon^2\partial_z^2 + P(z)$$

The existence of the QFT-ODE correspondence has also been shown from different perspectives. [Alday, Cecotti, Gaiotto, Gukov, Jeong, Moore, Neitzke, Nekrasov, Tachikawa, Vafa, Verlinde, $\cdots$]

A SW theory produces:

- A finite rank charge lattice Γ with a skew pairing $\langle\cdot,\cdot\rangle \to \mathbb{Z}$(EM and flavor charge lattice).
- A homomorphism $Z : \Gamma \to \mathbb{C}$(central charges).

In the dictionary of ODE-QFT correspondence, classical periods correspond to the central charges.

Quantum periods have the physical interpretations as IR line defect VEVs. [Gaiotto, Moore, Neitzke]

The QFT-ODE correspondence has been generalized to the correspondence between 5d $\mathcal{N}=1$ theories with the insertion of a codim-2 defect and difference equations [Aganagic, Cherkis, Cheng, Dijkgraaf, Elliott, Grassi, Hatsuda, Huang, Krefl, Marino, Nekrasov, Pestun, Shatashvili, Vafa, $\cdots$].

The simplest example is the 5d theory obtained by compactifying M-theory on $\mathbb{C}^3$. The quantization of Seiberg-Witten curve corresponds to a difference equation

$$\left(\mathrm{e}^{\hat{p}} - 1 - \mathrm{e}^{\hat{z}} q^{-\frac{1}{2}}\right) \psi(z,\epsilon) = 0, \quad [\hat{z},\hat{p}] = \epsilon$$

where

$$\mathrm{e}^{\hat{p}} \psi(z,\epsilon) = \psi(z+\epsilon,\epsilon).$$

[Garoufalidis, Kashaev]

In this talk, we will show the known exact WKB methods for the quantum period. In particular, the QFT-ODE correspondence provides a beautiful and analytic way to resum it.

Because of the singularities, quantum periods are piecewise analytic function of $(m, \Lambda, u, \epsilon)$. I will explain a convenient way to relate Borel summations in different chambers.

Some generalizations of the exact WKB to 5d $\cdots$

Outline

Borel summation

Wronskians

GMN TBA integral equations

Nekrasov-Shatashvili free energy

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Borel summation $s(Y)(z,\epsilon)$ of

$$Y(z,\epsilon) = \sum Y^n(z)\epsilon^n$$

includes two steps

- Borel transform

$$\mathcal{B}Y(z,\zeta) = \sum \frac{Y^n(z)}{n!}\zeta^n$$

  $\mathcal{B}Y(z,\zeta)$ has a finite radius of convergence, but the analytic continuation has singularities.

- Laplace transform

$$s(Y)(z,\epsilon) = \frac{1}{\epsilon}\int_0^{\infty e^{\mathrm{i}\arg(\epsilon)}} \mathcal{B}Y(z,\zeta)e^{-\zeta/\epsilon}d\zeta.$$

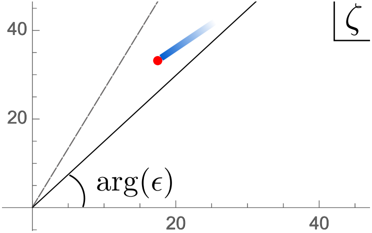
  Singularities of $\mathcal{B}Y(z,\zeta)$ are responsible for the Stokes phenomenon.

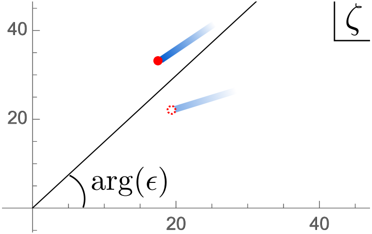
Jumps can happen when

- $\arg(\epsilon)$ changes (integral contour rotates)
- z or other parameters of the theory, $\wp = (u, m, \Lambda)$, change (singularities move)

such that a singularity is on the integral contour at a critical moment, i.e.

$$\arg(\epsilon) = \arg(\zeta_{\mathsf{sing}}(z, \wp)).$$

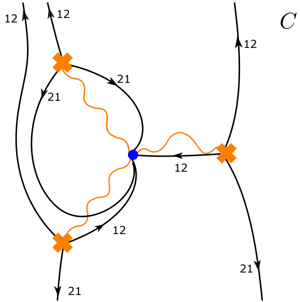




$s(Y)(z, \epsilon, \wp)$ jumps at codim-1 walls in the parameter space of the theory parametrized by $(z, \epsilon, \wp)$. They correspond to the singularities of $\mathcal{B}Y(z, \epsilon)$.

Stokes graph $\mathcal{W}$ is the projection of the codim-1 wall to C.

$s(Y)(z, \epsilon)$ exists only away from $\mathcal{W}$.



Similarly, Borel summation applies to $\Pi_\gamma(\epsilon)$ or its exponential $\mathcal{X}_\gamma(\epsilon) = \mathrm{e}^{\Pi_\gamma(\epsilon)}$.

In the QFT-ODE correspondence, singularities of quantum periods correspond to central charges of BPS states Z_γ's.[Grassi, Gu, H, Marino, Neitzke, $\cdots$]

$$\zeta_{\mathsf{sing}} = Z_\gamma$$

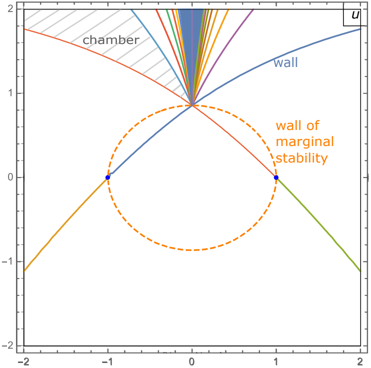
$\mathcal{X}_\gamma(\epsilon, \wp)$ is piecewise analytic.

$\mathcal{X}_\gamma(\epsilon, \wp)$ jumps at codim-1 walls in the parameter space of the theory, $(\epsilon, \wp)$.

The walls separate the parameter space into chambers.

Each wall corresponds to a BPS state.

E.g. walls on the u plane for $SU(2)$ SYM



Jumps of quantum periods are given by the Kontsevich-Soibelman transformations.[Delabaere, Gaiotto, Moore, Neitzke, Pham, $\cdots$] E.g.

$$\mathcal{X}'_\mu(\epsilon) = (1 + \mathcal{X}_\gamma(\epsilon))^{\langle \mu, \gamma \rangle \Omega(\gamma)} \mathcal{X}_\mu(\epsilon).$$

- γ labels the charge for the BPS state corresponding to the wall crossed.
- $\Omega(\gamma)$ is a piecewise invariant of the theory which counts BPS states with charge γ supported at the wall.

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The first order ODE

$$\left(\partial_z - \epsilon^{-1} Y^{(i)}\right) \psi^{(i)} = 0$$

defines $GL(1)$ connection ∇^{ab} of a line bundle $\mathcal{L}$ over Σ.

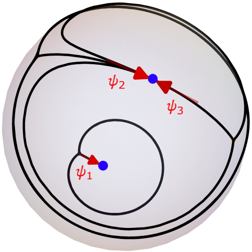
The solution $\psi^i(z)$ is a flat section of ∇^{ab}.

Since the solution jumps at Stokes graph, ∇^{ab} naively don't extend across the lift of $\mathcal{W}^{\arg(\epsilon)}$ to Σ. In order to extend it we need to impose the gluing map

$$\begin{pmatrix} \psi_1^L \\ \psi_2^L \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} \psi_1^L \\ \psi_2^L \end{pmatrix} = \begin{pmatrix} \psi_1^R \\ \frac{[\psi_1^L, \psi_2^L]}{[\psi_1^L, \psi_2^R]} \psi_2^R \end{pmatrix},$$

where $[\psi_i, \psi_j] = \det \begin{pmatrix} \psi_i, \psi_j \\ \psi_i', \psi_j' \end{pmatrix}$ is the Wronskian.

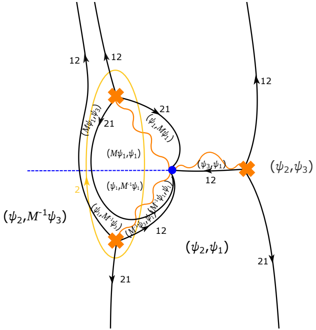
We use exponentially decaying solutions ψ_1, ψ_2, ψ_3 along Stokes directions and monodromy M to build the basis of each chamber in the complement of $\mathcal{W}^{\arg(\epsilon)}$.



Quantum periods are equivalently defined by

$$\mathcal{X}_\gamma = \mathrm{Hol}_\gamma \nabla^{\mathrm{ab}}.$$

[Hollands, Iwaki, Nakanishi, Neitzke, $\cdots$]
E.g.



$$\mathcal{X}_\gamma = \frac{[\psi_3, \psi_1]}{[M\psi_1, \psi_1]} \frac{[\psi_1, \psi_2]}{[M^{-1}\psi_3, \psi_2]}$$

To go from the ϵ-connection to ∇^{ab}, we have used the Stokes graph $\mathcal{W}$.

$\mathcal{W}$ consists of points z satisfying

$$\arg(\epsilon) = \arg(-Z^{\wp}(z)),$$

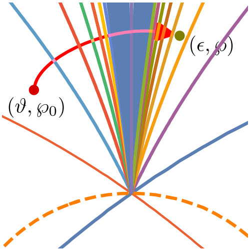
Define $\mathcal{W}^{\vartheta,\wp_0}$ for a generic ϑ replacing $\arg(\epsilon)$ on the l.h.s., and on the r.h.s. we can use any $\wp_0$.

$$\vartheta = \arg(-Z^{\wp_0}(z)),$$

The generalized $\mathcal{W}^{\vartheta,\wp_0}$-abelianization when we use the same canonical form of gluing across $\mathcal{W}^{\vartheta,\wp_0}$ produces us $\nabla^{\mathsf{ab},\vartheta,\wp_0}$.

$$\mathcal{X}_\gamma^{\vartheta,\wp_0}(\epsilon,\wp) = \mathsf{Hol}_\gamma \nabla^{\mathsf{ab},\vartheta,\wp_0}.$$

$\mathcal{X}_\gamma^{\vartheta,\wp_0}$ is the analytic continuation of $\mathcal{X}_\gamma$ in the chamber containing $(\vartheta, \wp_0)$.



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Riemann-Hilbert problem

Quantum periods are solutions solving some Riemann-Hilbert problem. [Bridgeland, Gaiotto, Moore, Neitzke, Voros, $\cdots$]

- Asymptotics:

$$\mathcal{X}_\gamma(\epsilon) \sim \mathrm{e}^{\frac{Z_\gamma}{\epsilon}}, \quad \epsilon \to 0$$

- Piecewise analytic:

$$\mathcal{X}_\mu(\epsilon)' = (1 + \mathcal{X}_\gamma(\epsilon))^{\langle \mu, \gamma \rangle \Omega(\gamma)} \mathcal{X}_\mu(\epsilon), \quad \arg(\epsilon) = \arg(-Z_\gamma)$$

- $\ldots$

The RH problem can be rewritten as TBA-like integral equations

$$\mathcal{X}_\mu(\epsilon) = \exp\left(\frac{Z_\mu}{\epsilon} + \frac{1}{4\pi \mathrm{i}} \sum_{\mu \in H_1(\Sigma, \mathbb{Z})} \int_{\epsilon' \in \mathbb{R}_- Z_\gamma} \frac{\mathrm{d}\epsilon'}{\epsilon'} \frac{\epsilon' + \epsilon}{\epsilon' - \epsilon} \log\left((1 + \mathcal{X}_\gamma(\epsilon'))^{\langle \mu, \gamma \rangle \Omega(\gamma)} \right) \right)$$

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For convenience, we choose a particular basis (γ_e, γ_m) for the charge lattice Γ.

For a 4d $\mathcal{N} = 2$ SUSY, Nekrasov-Shatashvili free energy

$$F^{\text{inst}}(\Pi_{\gamma_e}, m, \epsilon, \Lambda) = \sum c_n(\Pi_{\gamma_e}, m, \epsilon)\Lambda^{3n},$$

is a computable object which has the property that

$$\Pi_{\gamma_m} = \Pi_{\gamma_e} \log \Lambda + \epsilon \log \frac{\Gamma(1 + \frac{2\Pi_{\gamma_e}}{\epsilon})}{\Gamma(1 - \frac{2\Pi_{\gamma_e}}{\epsilon})} + \epsilon \log \frac{\Gamma(\frac{1}{2} + \frac{m - \Pi_{\gamma_e}}{\epsilon})}{\Gamma(\frac{1}{2} + \frac{m + \Pi_{\gamma_e}}{\epsilon})} + \partial_{\Pi_{\gamma_e}} F^{\text{inst}}.$$

F^{inst} provides an alternative analytic resummation of the all-order WKB series of quantum periods. [Mironov, Morozov, Nekrasov, Rosly, Shatashvili, $\cdots$]

Comparison of the four methods

We have introduced 4 definitions for the quantum periods.

The 4 methods are all efficient.

They are proposed to be equivalent once we consider appropriate transformation.[Hollands, Gaiotto, Iwaki, Moore, Nakanishi, Neitzke, Nekrasov, Rosly, Shatashvili, $\cdots$]

- TBA equations always produce solutions matching with Borel summation result.

- Wronskians method can be used as a bridge to produce the transformations.
- Quantum periods obtained from NS free energy is the analytic continuation from the instanton locus.

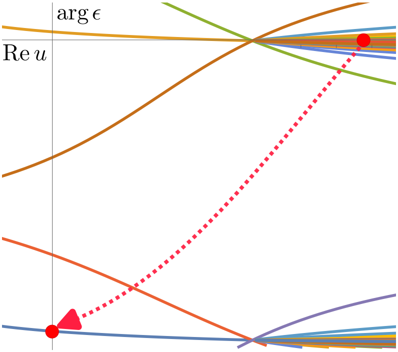
Instanton locus lies on a wall outside the wall of marginal stability. Borel summation is defined by median summation.

Using NS free energy and Wronskians, we can get analytic expressions for the quantum periods in (almost) all the parameter space.

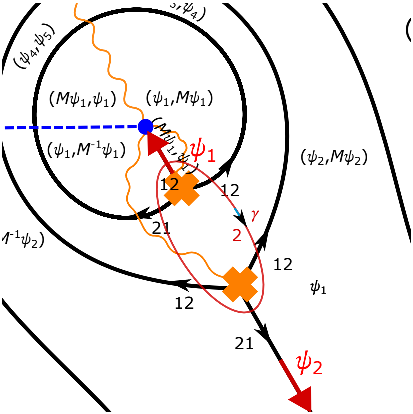
E.g. An analytic expression for a quantum period in strong coupling region

$$\mathcal{X}_\gamma = \frac{e^{-\Pi^{\mathbf{NS}}_{\gamma_m}+\frac{\Pi^{\mathbf{NS}}_{\gamma_e}}{2}-\frac{2\pi m}{h}}\left(\left(e^{\Pi^{\mathbf{NS}}_{\gamma_m}}+1\right)^2 e^{\frac{\Pi^{\mathbf{NS}}_{\gamma_e}}{2}+\frac{2\pi m}{h}}+e^{\Pi^{\mathbf{NS}}_{\gamma_m}}\left(e^{\Pi^{\mathbf{NS}}_{\gamma_m}+\Pi^{\mathbf{NS}}_{\gamma_e}}+e^{\Pi^{\mathbf{NS}}_{\gamma_e}}+1\right)+1\right)}{\left(e^{\Pi^{\mathbf{NS}}_{\gamma_e}}-1\right)^2},$$

where Π^{NS}_γ's are the analytic expressions obtained directly from NS partition function.

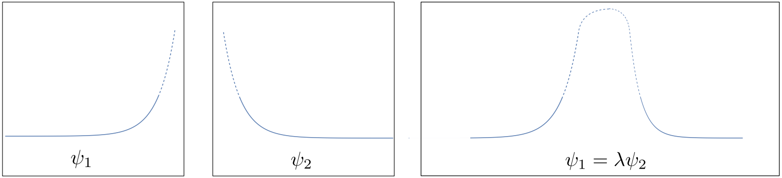


Exact quantization condition



Restricted to the instanton locus of the parameter space, two of the Stokes directions align.

If we stay on this line, we have a real Schrödinger equation with



For a bound state, we need

$$\psi_1 = \lambda\psi_2, \quad \text{for some } \lambda$$

Substitute this condition in to the Wronskians exrepssion

$$\mathcal{X}_\gamma = \frac{[\psi_1, \psi_5][\psi_4, \psi_2]}{[\psi_5, \psi_2][\psi_4, \psi_1]} = -1.$$

We reproduce a well known exact quantization condition.

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Some updates on the generalization to 5d

- Obtaining the all-order WKB series is in general hard. [Dingle, Morgan]
  Some recent progress has been made via holomorphic anomaly. [Gu, Marino]
- The analogous to the Stokes graph is the exponential network in 5d.[Banajee, Longhi, Romo]
  There is a new type of walls appearing in 5d we have studied the corresponding jump of solutions.[Alim, Grassi, H, Hollands, Neitzke, Tulli].
- GMN TBA method has been studied in a very special chamber of the pure $SU(2)$ SYM. [Del Monte, Longhi]
- Refined topological string/ Nekrasov partition function produces a perturbative series. [Alim, Grassi, H, Hatsuda, Hollands, Huang, Marino, Neitzke, Teschner, Tulli, $\cdots$]

Thank you!