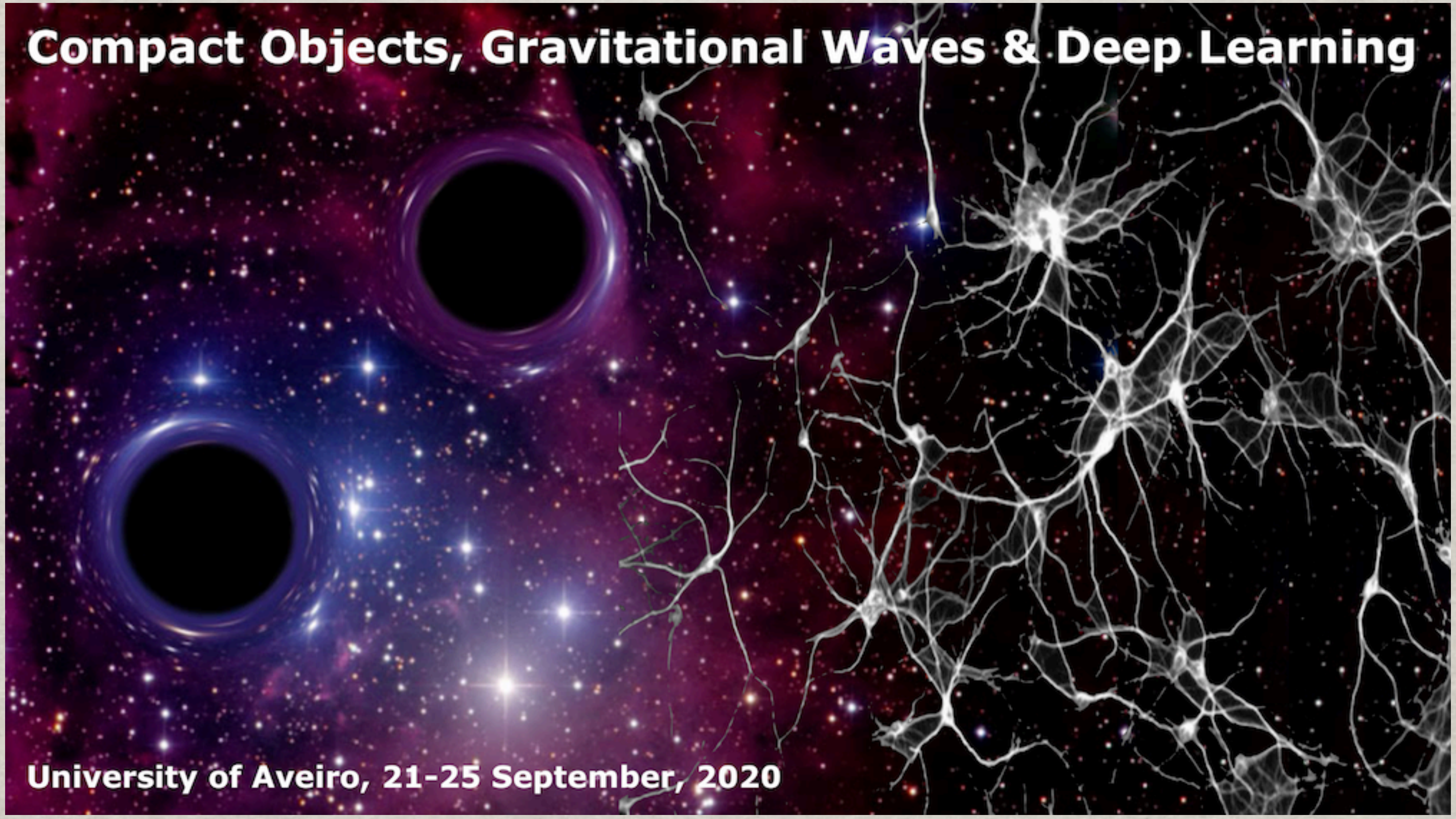
Black holes and exotic compact objects

C. Herdeiro
Departamento de Matemática and CIDMA, Universidade de Aveiro, Portugal



Plan of the lectures:

Lecture 1
Black holes: astrophysical evidence and a theory (brief) timeline

Lecture 2
Spherical black holes: the Schwarzschild solution

Lecture 3
Spinning black holes: the Kerr solution

Lecture 4
Exotic compact objects: the example of bosonic stars

Lecture 5
Non-Kerr black holes

The Kerr hypothesis

"In my entire scientific life, extending over forty-five years, the most shattering experience has been the realization that an exact solution of Einstein's field equations of general relativity, discovered by the New Zealand mathematician, Roy Kerr, provides the absolutely exact representation of untold numbers of massive black holes that populate the Universe."

S. Chandrasekhar, in *Truth and Beauty* (1987)

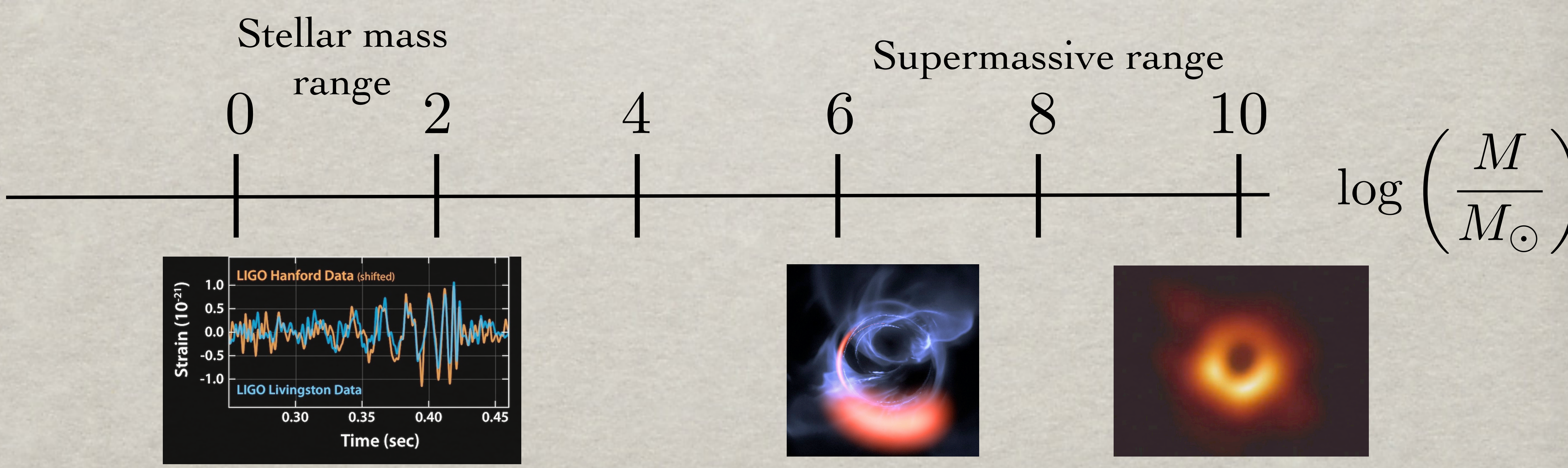
Testing the Kerr hypothesis is to keep some healthy skepticism:

1) are these untold numbers of massive black holes exactly represented by the Kerr metric ?

2) are these black holes all of the same type ?

3) are these objects really black holes ?

The **Kerr hypothesis**
is a very economical scenario:
the very same "object" spans (at least) 10 orders of magnitude!



1963: **Kerr's solution**

Phys. Rev. Lett. 11 (1963) 237-238

GRAVITATIONAL FIELD OF A SPINNING MASS AS AN EXAMPLE OF ALGEBRAICALLY SPECIAL METRICS

Roy P. Kerr*

University of Texas, Austin, Texas and Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio

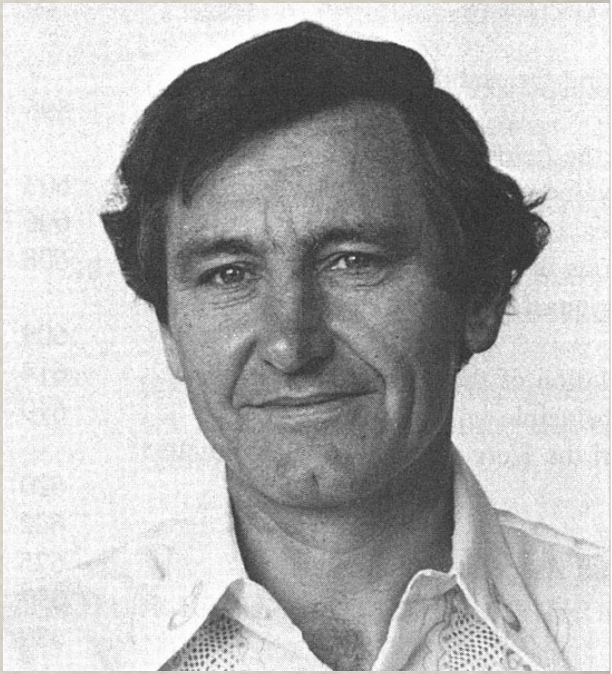
(Received 26 July 1963)

Goldberg and Sachs[1] have proved that the algebraically special solutions of Einstein's empty-space field equations are characterized by the existence of a geodesic and shear-free ray congruence, k_μ. Among these spaces are the plane-fronted waves and the Robinson-Trautman metrics[2] for which the congruence has nonvanishing divergence, but is hypersurface orthogonal.

where ζ is a complex coordinate, a dot denotes differentiation with respect to u, and the operator D is defined by

$$D = \partial/\partial\zeta - \Omega\partial/\partial u.$$

P is real, whereas Ω and m (which is defined to be $m_1 + im_2$) are complex. They are all independent of the coordinate r. Δ is defined by



Roy P. Kerr
(1934-)

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma}dt^2 - 2a\sin^2\theta\frac{(r^2+a^2-\Delta)}{\Sigma}dtd\phi + \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

$$\Sigma = r^2 + a^2\cos^2\theta$$
$$\Delta = r^2 - 2GMr + a^2$$

(in the coordinates introduced by Robert H. Boyer and Richard W. Lindquist, in 1967, J. Math. Phys. 8 (1967) 265)

Derivation of the Kerr metric.

One could try a direct "attack" to the Einstein equations using a sufficiently general ansatz in some "spheroidal" coordinate system.

The most generic ansatz adapted to axial symmetry and stationarity is:
(Carter CMP 17 (1970) 233 showed no generality is loss in assuming these fields commute)

$$
\begin{aligned}
ds^2 = \boxed{g_{tt}(r,\theta)dt^2} &+ \cancel{g_{tr}(r,\theta)dtdr} + \cancel{g_{t\theta}(r,\theta)dtd\theta} + \boxed{g_{t\varphi}(r,\theta)dtd\varphi} \\
&+ g_{rr}(r,\theta)dr^2 + \cancel{g_{r\theta}(r,\theta)drd\theta} + \cancel{g_{r\varphi}(r,\theta)drd\varphi} \\
&+ \cancel{g_{\theta\theta}(r,\theta)d\theta^2} + \cancel{g_{\theta\varphi}(r,\theta)d\theta d\varphi} \\
&\quad g_{rr}(r,\theta)r^2d\theta^2 \\
&+ \boxed{g_{\varphi\varphi}(r,\theta)d\varphi^2}
\end{aligned}
$$

Impose invariance under: $t \to -t$ and $\varphi \to -\varphi$

gauge freedom

vacuum

Papapetrou, Annals de l'I.H.P. Physique théorique, 4 (1966)

See e.g.
Herdeiro and Oliveira
Class. Quant. Grav. 36 (2019) 105015

Still, 3 unknown functions of two variables.
Einstein equations are prohibitively complicated... extra structure needed....

A hint: Kerr-Schild form for the Schwarzschild solution

Consider the Schwarzschild metric in outgoing Eddington-Finkelstein (EF) coordinates,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2 d\Omega_2$$

Its inverse is,

$$\left(\frac{\partial}{\partial s}\right)^2 = -2\frac{\partial}{\partial u}\frac{\partial}{\partial r} + \left(1 - \frac{2M}{r}\right)\left(\frac{\partial}{\partial r}\right)^2 + \frac{1}{r^2}\left[\left(\frac{\partial}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial \phi}\right)^2\right]$$

Which can be written simply as,

$$\boxed{g^{\mu\nu} = \eta^{\mu\nu} - \frac{2M}{r} l^\mu l^\nu}$$

Kerr-Schild form of the Schwarzschild solution

where,

$$\eta^{\mu\nu} = \begin{bmatrix} 0 & -1 & & \\ -1 & 1 & & \\ & & \frac{1}{r^2} & \\ & & & \frac{1}{r^2 \sin^2\theta} \end{bmatrix}$$

is the Minkoswki metric in retarded null coordinates, and $l = \frac{\partial}{\partial r}$ is null, $l^\mu l_\mu = 0$.

The null vector in the Kerr-Schild form

$$\eta^{\mu\nu} = \begin{bmatrix} 0 & -1 & & \\ -1 & 1 & & \\ & & \frac{1}{r^2} & \\ & & & \frac{1}{r^2 \sin^2\theta} \end{bmatrix}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - \frac{2M}{r} l^\mu l^\nu$$

$$l^\mu \partial_\mu = \frac{\partial}{\partial r}$$

has some special properties:

1) It is an affinely parameterised null geodesic:

$$l^\mu D_\mu l^\nu = 0$$

2) It is "shear-free":

$$D_{(\mu} l_{\nu)} D^{(\mu} l^{\nu)} - \frac{1}{2}(D_\mu l^\mu)^2 = 0$$

Exercise 3.1

Verify these properties.

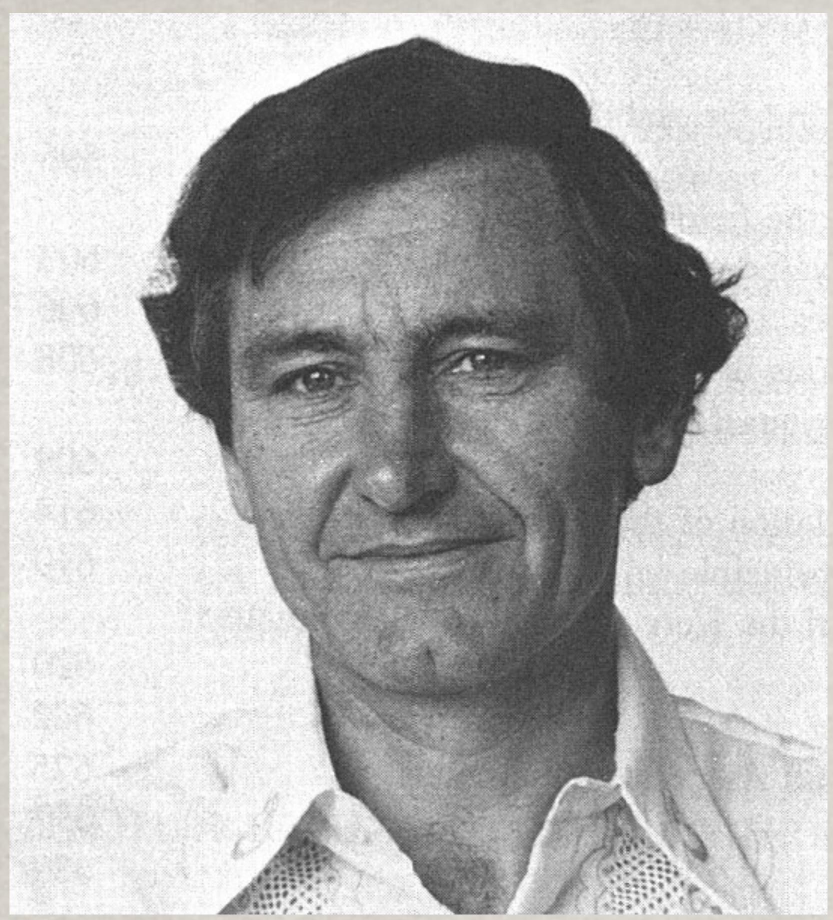
A crucial result for the derivation of the Kerr solution was:

J Goldberg, R Sachs, Acta Phys. Polon. Suppl. 22 (1962) 13

Goldberg-Sachs Theorem (1962): A vacuum solution of the Einstein field equations will admit a shear-free null geodesic congruence if and only if the Weyl tensor is algebraically special.

The Schwarzschild solution is therefore algebraically special.
It could be that its rotating generalization
would also be algebraically special.

Let us therefore look for an ansatz that reflects this.
The Kerr-Schild form is particularly appropriate.



Vacuum, axially symmetric,
stationary black hole
Kerr 1963

- Has two (macroscopic) degrees of freedom:
Mass "M" and Angular Momentum (per Mass) "a"

- It has remarkable mathematical properties:

a) Hidden symmetries, that allow separability of geodesic motion and of different types of perturbation;

b) An elegant geometrical structure: it is algebraically special.

$$ds^2 = -\frac{(\Delta - a^2 \sin^2\theta)}{\Sigma} dt^2 - 2a \sin^2\theta \frac{(r^2 + a^2 - \Delta)}{\Sigma} dt d\phi + \left(\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right) \sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2$$

$$\Sigma = r^2 + a^2 \cos^2\theta$$
$$\Delta = r^2 - 2GMr + a^2$$

Singularities:

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma}dt^2 - 2a\sin^2\theta\frac{(r^2+a^2-\Delta)}{\Sigma}dtd\phi$$
$$+\left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

1) Metric coefficients diverge when:

Outer or event horizon

$$\Delta = r^2 - 2Mr + a^2 = 0 \;\Leftrightarrow\; r = r_\pm \equiv M \pm \sqrt{M^2 - a^2}$$

Inner or Cauchy horizon

These are mere coordinate singularities that can be eliminated in EF type coordinates;

$$\Sigma = r^2 + a^2\cos^2\theta = 0 \quad\Leftrightarrow\quad r = 0 \text{ and } \theta = \frac{\pi}{2}$$

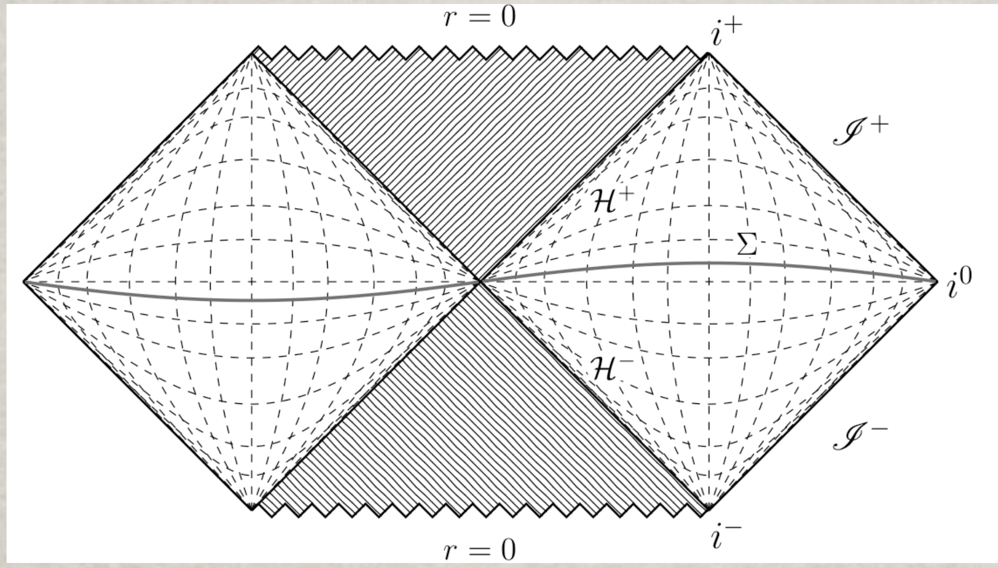
ring type singularity

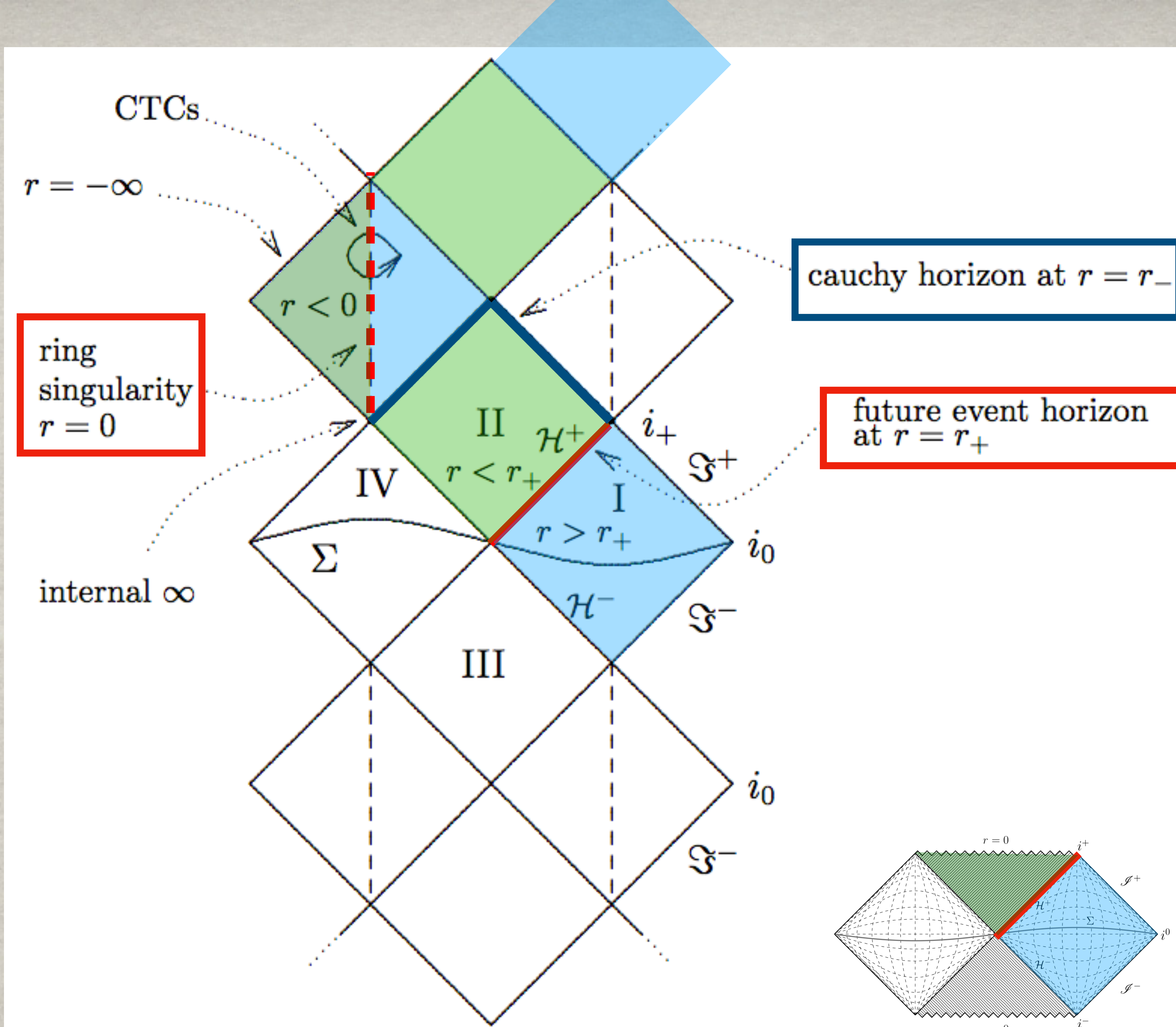
This is a physical curvature singularity; the Kretschmann scalar diverges:

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{8}{\Sigma^6}\left\{6M^2(r^6 - 15a^2r^4\cos^2\theta + 15a^4r^2\cos^4\theta - a^6\cos^6\theta)\right\}$$

2) The metric determinant is: $\det g = -\Sigma^2\sin^2\theta$

Carter-Penrose diagram
for the eternal Schwarzschild spacetime.

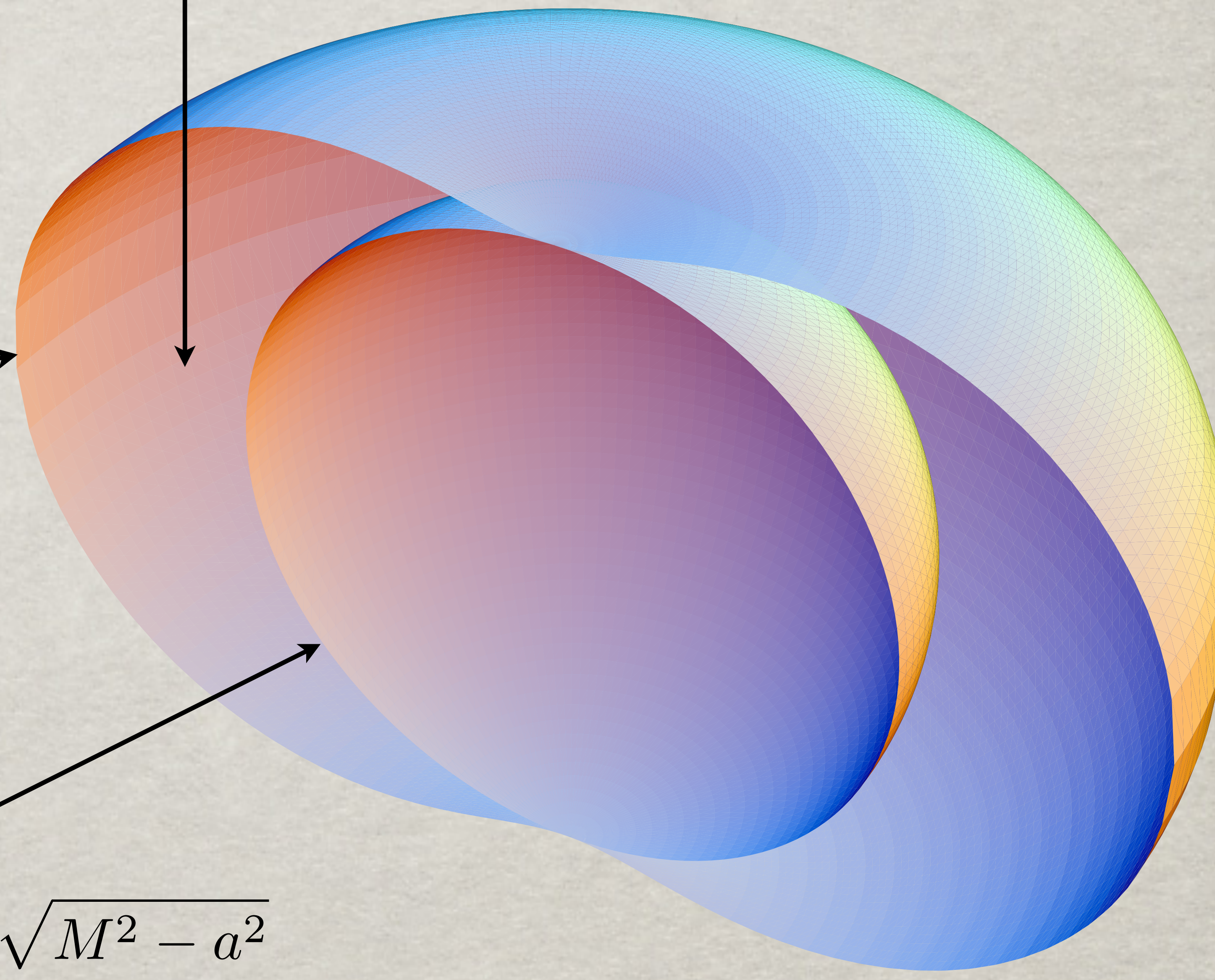




P. Townsend, arXiv:9707012

The Kerr solution has another remarkable property:

ergo-region



ergo-surface

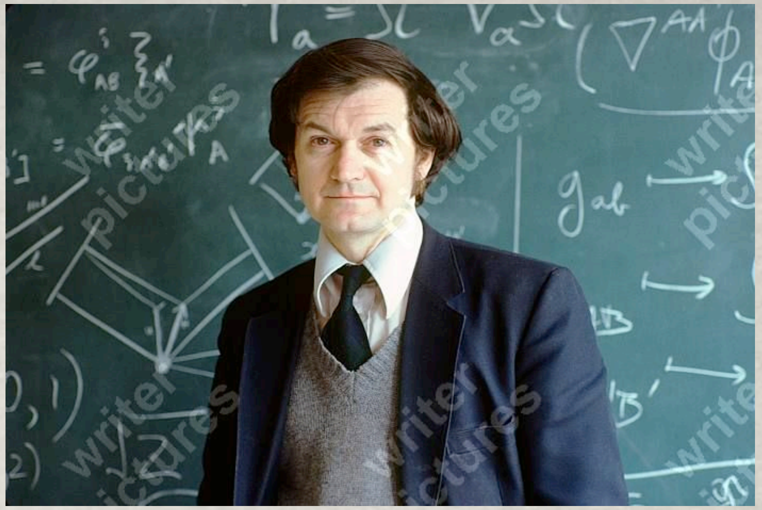
$$r_{\rm ergo} = M + \sqrt{M^2 - a^2\cos^2\theta}$$

$$r_+ = M + \sqrt{M^2 - a^2}$$

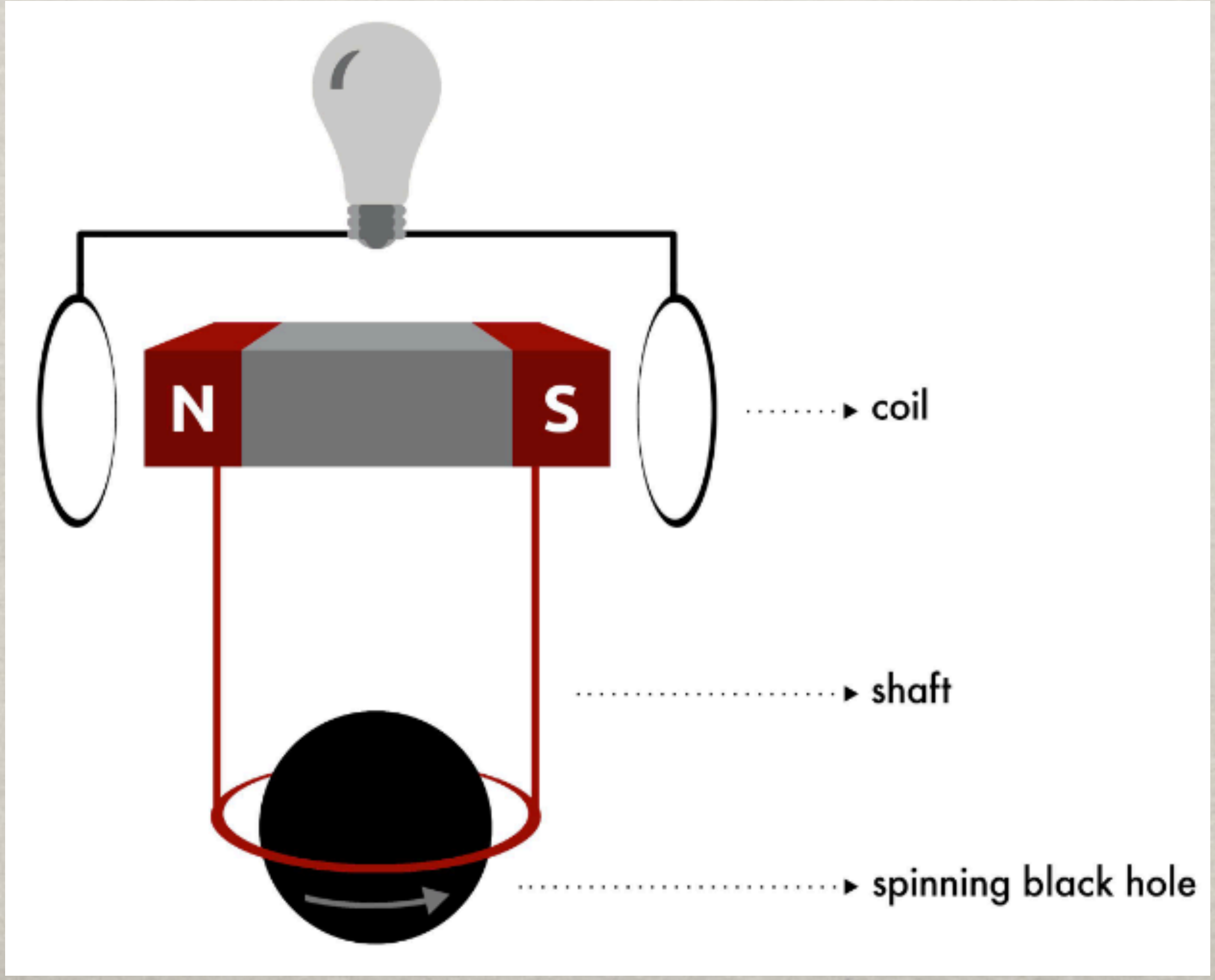
outer horizon

The Killing vector field $\partial/\partial t$ becomes spacelike outside the event horizon, in the “ergo-region”

It is possible to (classically!) extract energy from a rotating black hole
(Penrose 1969)



Electric circuit fed by the spin of a black hole



Brito et al. (2015)

The ability to extract
energy from a rotating
black hole will play an important role
in the existence of "hairy" black holes.

To know more, do not miss lecture 5 !!

Probing the Kerr black hole

with

light

Kerr space-time:
BH with mass M and angular momentum J

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\sin^2\theta}{\Sigma}\left(adt - (r^2+a^2)d\phi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

$$\Sigma \equiv r^2 + a^2\cos^2\theta \ , \qquad \Delta \equiv r^2 - 2MR + a^2$$

Geodesics are Liouville integrable due to the existence of hidden symmetry (Killing tensor)

Convenient to use the Hamilton-Jacobi formalism Carter 1968

$$\frac{1}{2}g^{\mu\nu}\partial_\mu S \partial_\nu S = -\frac{\partial S}{\partial \lambda}$$

$$\partial_\mu S = p_\mu = g_{\mu\nu}\frac{dx^\nu}{d\lambda}$$

"Obvious" separation of variables

$$S = \frac{1}{2}m^2\lambda - Et + j\phi + f(r,\theta)$$

Convenient to use the Hamilton-Jacobi formalism Carter 1968

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"Obvious" separation of variables

$$S = \frac{1}{2}m^2\lambda - Et + j\phi + f(r,\theta)$$

"Non-obvious" separation of variables

$$f(r,\theta) = f_r(r) + f_\theta(\theta)$$

From the t, ϕ momentum equations:

$$\frac{dt}{d\lambda} = \frac{1}{\Sigma}\left\{\frac{r^2+a^2}{\Delta}\left[E(r^2+a^2) - ja\right] + a(j - aE\sin^2\theta)\right\}$$

$$\frac{d\phi}{d\lambda} = \frac{1}{\Sigma}\left\{\frac{j}{\sin^2\theta} - aE + \frac{a}{\Delta}\left[E(r^2+a^2) - ja\right]\right\}$$

Use the r, θ momentum equations in the H-J equation; the latter separates:

$$\left(\frac{d\theta}{d\lambda}\right)^2 = \frac{\Theta(\theta)}{\Sigma^2} \qquad \Theta(\theta) \equiv Q - \cos^2\theta\left[\frac{j^2}{\sin^2\theta} + a^2(m^2 - E^2)\right]$$

Q= Carter's constant

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{R(r)}{\Sigma^2} \qquad R(r) \equiv \left[E(r^2+a^2) - ja\right]^2 - \Delta\left[m^2r^2 + (j - aE)^2 + Q\right]$$

Four first-order differential equations:

$$\Sigma\dot{r} = \pm\sqrt{\mathcal{R}}\ ,$$

$$\Sigma\dot{\theta} = \pm\sqrt{\Theta}\ ,$$

$$\Sigma\dot{t} = \frac{E}{\Delta}[(r^2+a^2)^2 - a^2\Delta\sin^2\theta] - \frac{2Mar}{\Delta}j\ ,$$

$$\Sigma\dot{\varphi} = \frac{2MaEr}{\Delta} + j\frac{(\Delta - a^2\sin^2\theta)}{\Delta\sin^2\theta}\ ,$$

$$\mathcal{R} \equiv H^2 - \Delta[Q + (aE-j)^2 + m^2r^2]\ , \qquad H \equiv E(r^2+a^2) - ja\ ,$$

$$\Theta \equiv Q - \cos^2\theta\left(a^2(m^2-E^2) + \frac{j^2}{\sin^2\theta}\right)\ ,$$

Given suitable initial conditions for a photon (m=0), the trajectory can be obtained by numeric integration.

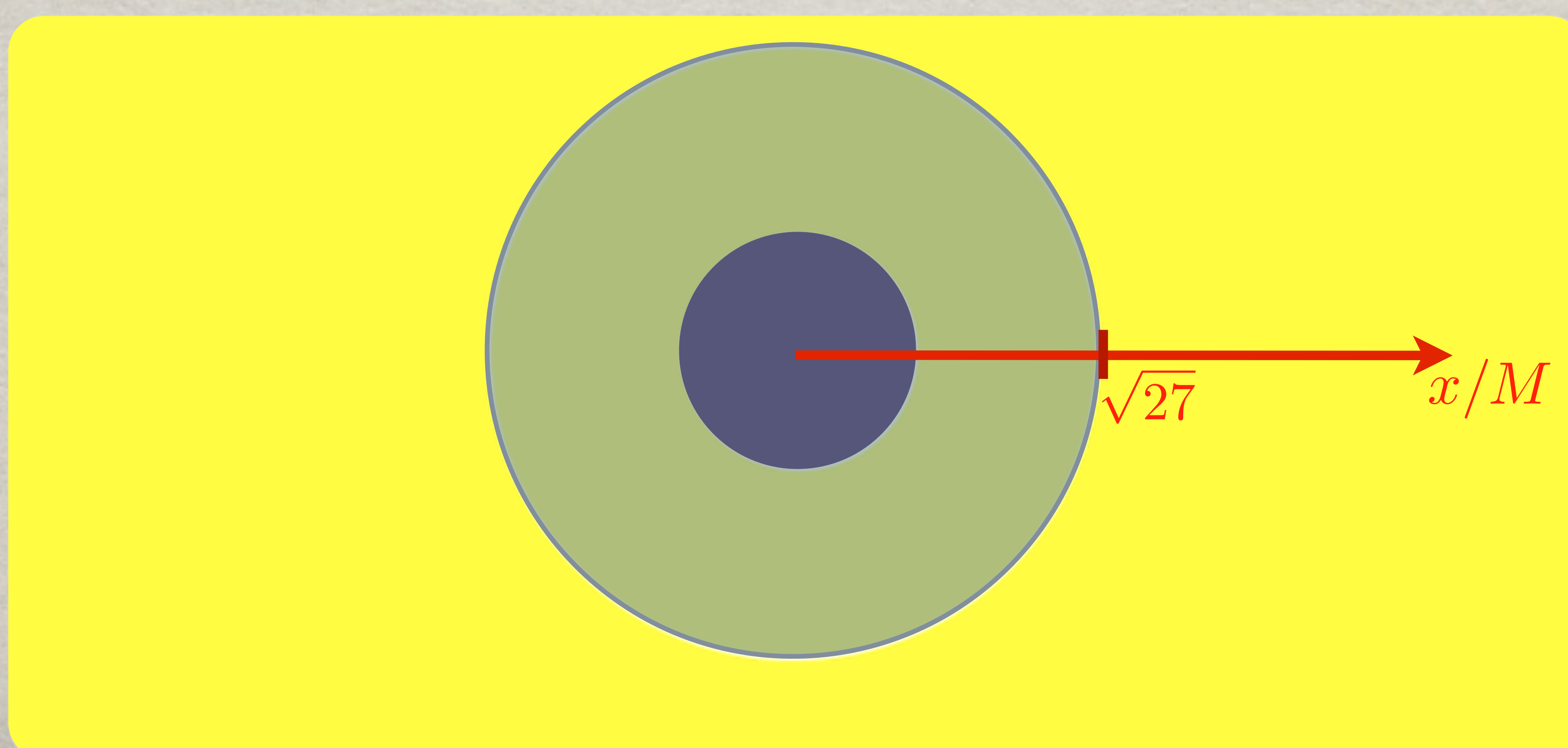
What is the form of the BH **shadow** ?

In the Schwarzschild case, the shadow edge is determined by the light ring.
All test motions are planar.
Null geodesic motion is determined by a single impact parameter:

$$\eta = \frac{j}{E}$$

For the light ring:

$$\eta = \sqrt{27}M$$



For the Kerr case, test motions are **not** planar. Null geodesic motion is determined by two impact parameters:

$$\eta = \frac{j}{E} \qquad \chi = \frac{Q}{E^2}$$

Light rings only exist on the equatorial plane ($\chi = 0$). More generically there are "spherical" orbits, that have "r=constant" in Boyer-Lindquist coordinates. Teo, GRG 35 (2003) 1909

Using the geodesic equations, the impact parameters of a spherical photon orbit at a certain radial coordinate R=r/M obeys:

$$\eta = -\frac{R^3 - 3R^2 + a^2 R + a^2}{a(R-1)}, \qquad \chi = -\frac{R^3(R^3 - 6R^2 + 9R - 4a^2)}{a^2(R-1)^2}.$$

Exercise 3.3

Obtain these expressions.

Spherical photon orbits exist for $R \in [r_1, r_2]$ where these radii are defined as the roots of χ:

$$r_1 = 2\left\{1 + \cos\left(\frac{2}{3}\arccos\left[-\frac{|a|}{M}\right]\right)\right\}, \quad r_2 = 2\left\{1 + \cos\left(\frac{2}{3}\arccos\left[\frac{|a|}{M}\right]\right)\right\}.$$

Co-rotating light ring — Counter-rotating light ring

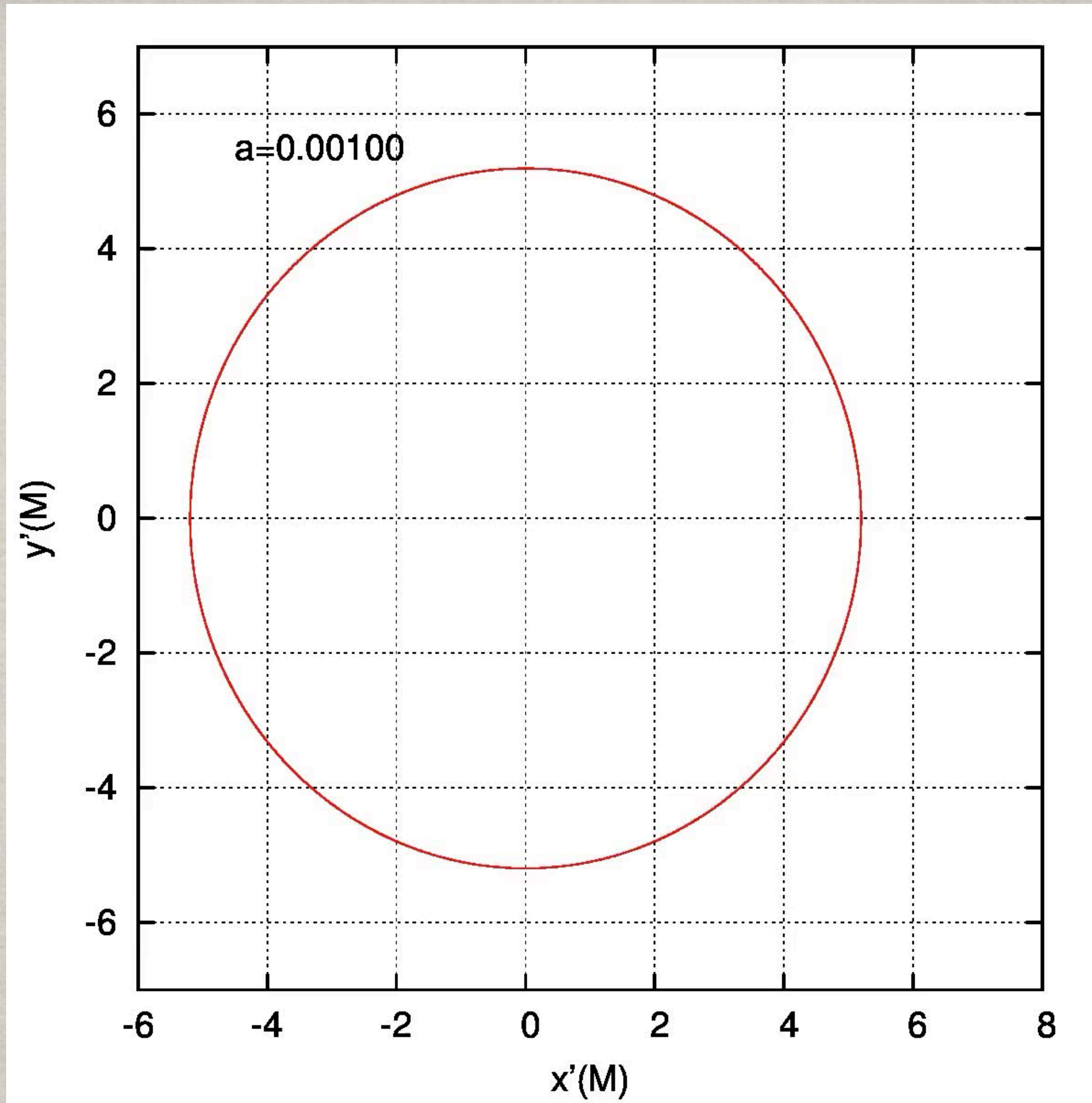
In the Kerr case the spherical orbits (including the light rings) determine the shadow edge, as we shall see next.

In a more generic stationary black hole spacetime,
where the geodesic motion is not necessarily integrable
the shadow edge is determined by a set of bound photon orbits dubbed
fundamental photon orbits (FPOs)
Cunha, C.H., Radu, PRD 96 (2017) 024039

It is possible to classify the different types of FPOs.
Cunha, C.H., Radu, PRD 96 (2017) 024039

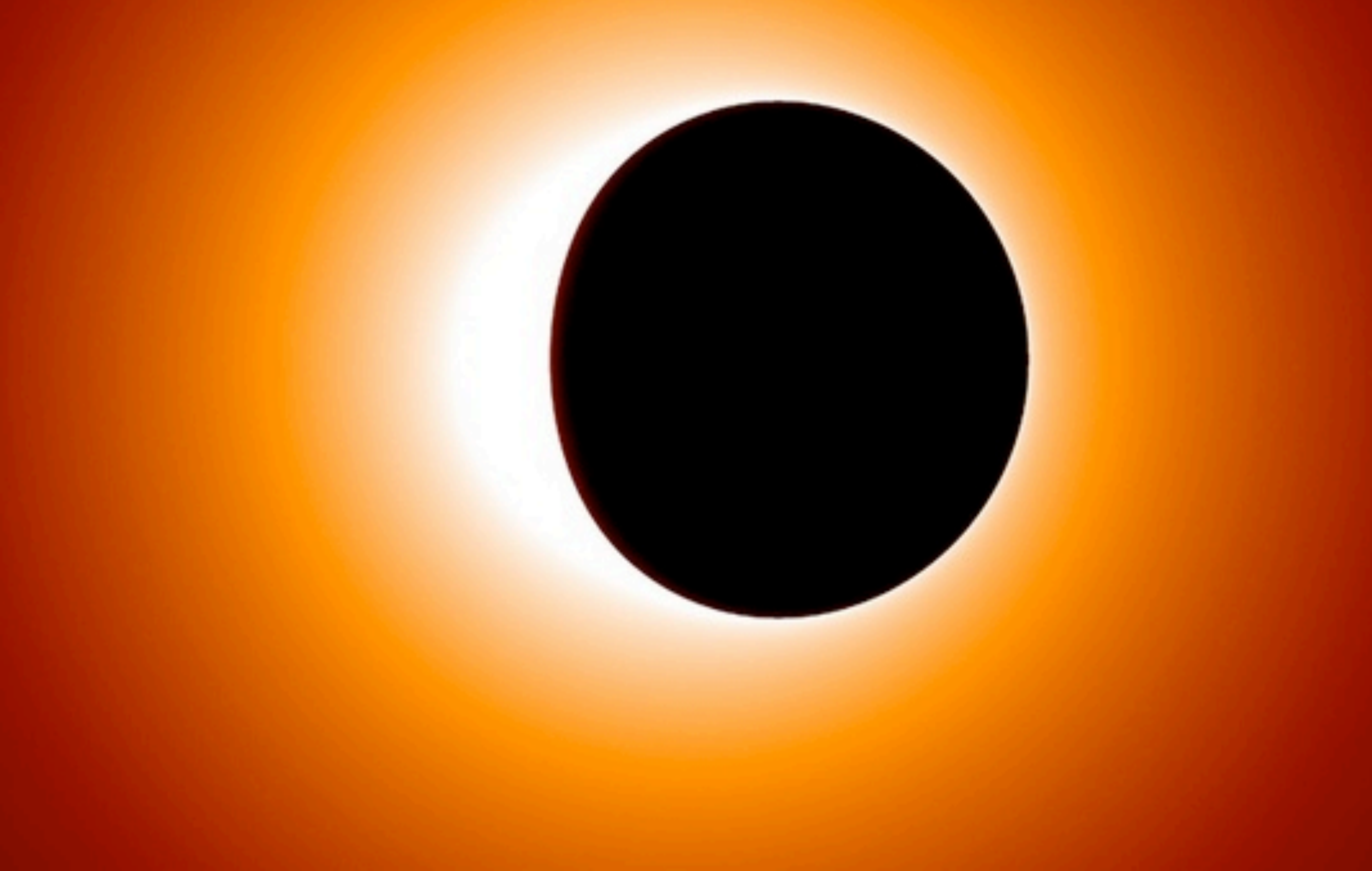
In the Kerr case, the spherical orbits are the FPOs.
Teo, GRG 35 (2003) 1909

As we shall see in **Lecture 5**, more complex structures are possible.



Animation: Pedro Cunha

Shadow of an extremal Kerr black hole (equatorial plane observation, spin upwards)



We can now assess the contribution of the FPOs to the shadow's edge.

j~0.82

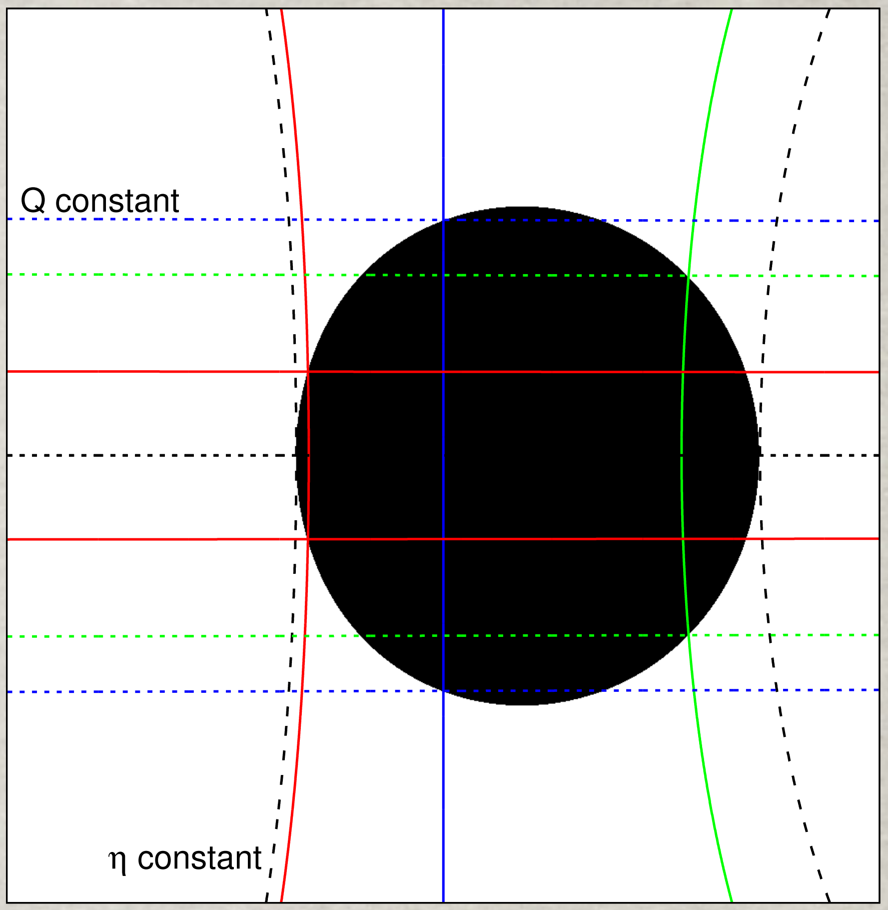
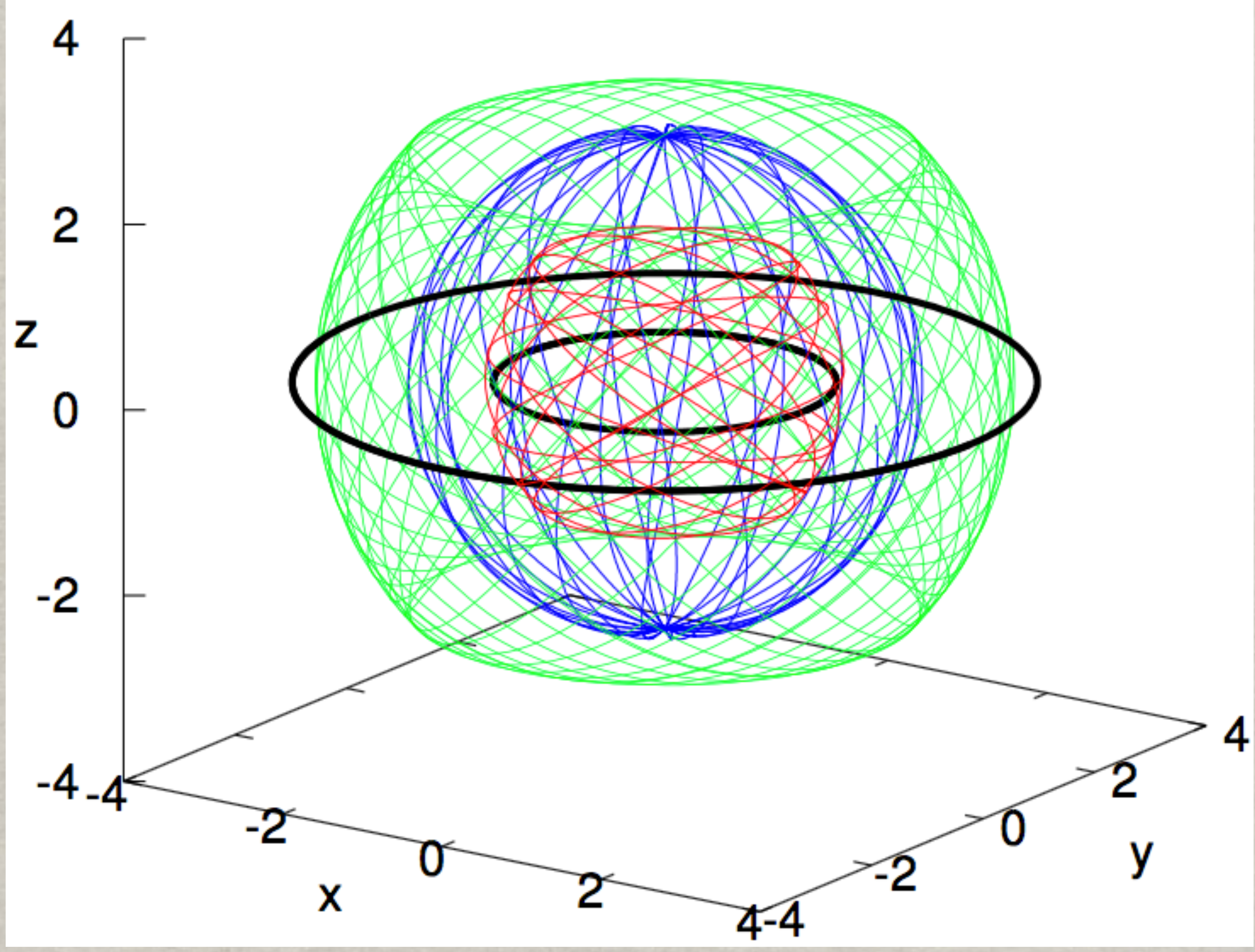
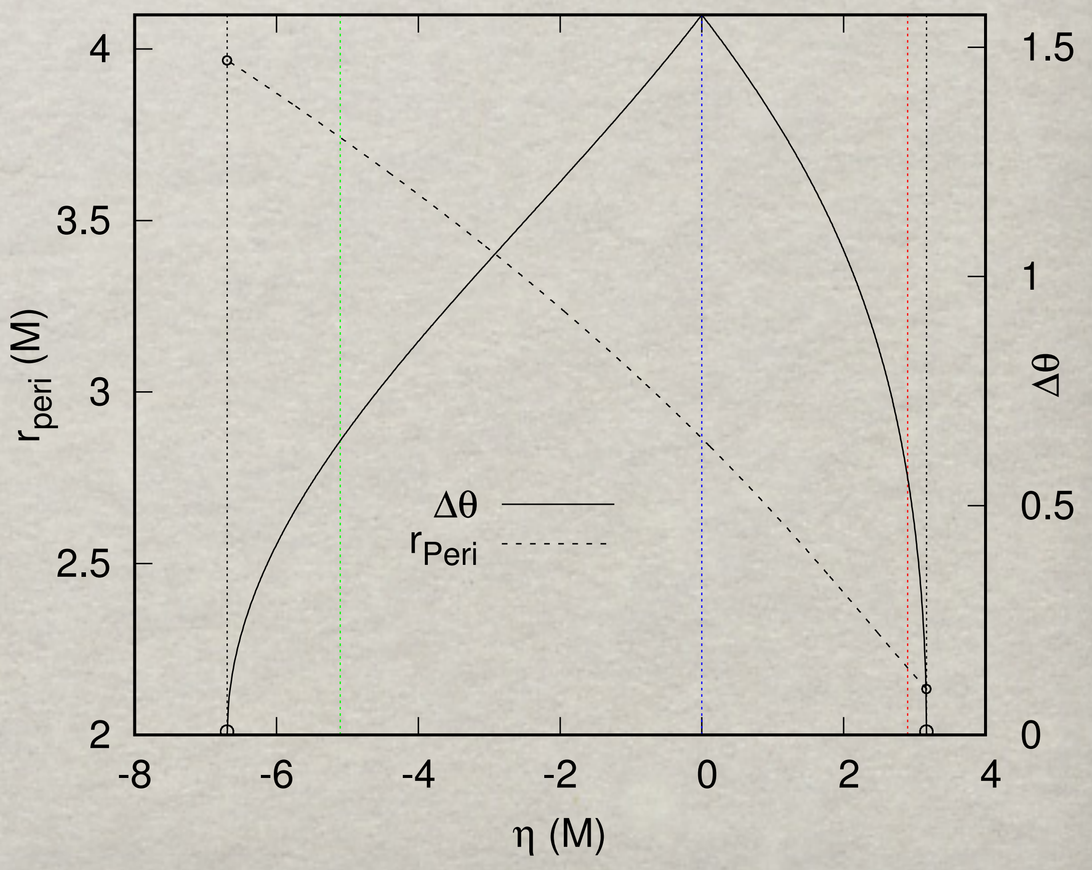


Illustration of the **Fundamental Photon Orbits** which are the **Spherical Photon Orbits** in the Kerr case

Cunha, C.H., Radu, PRD 96 (2017) 024039





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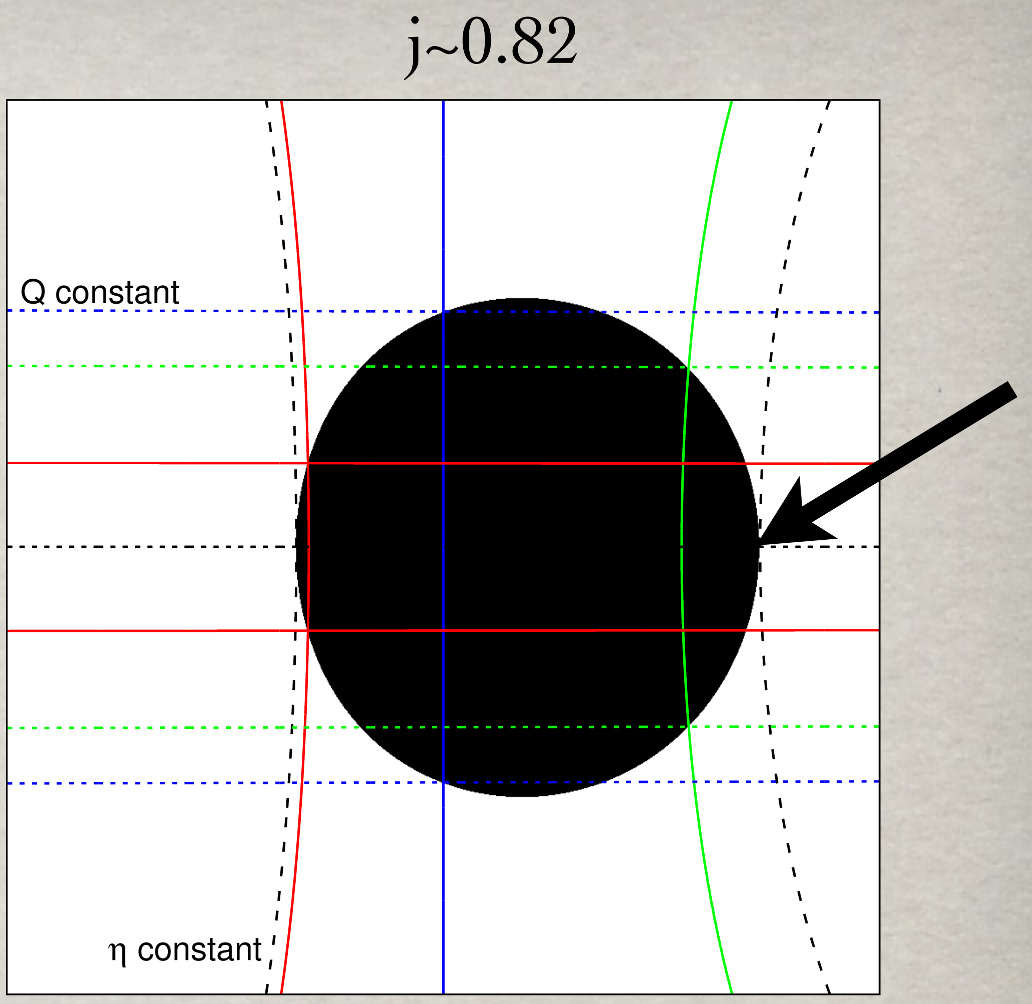

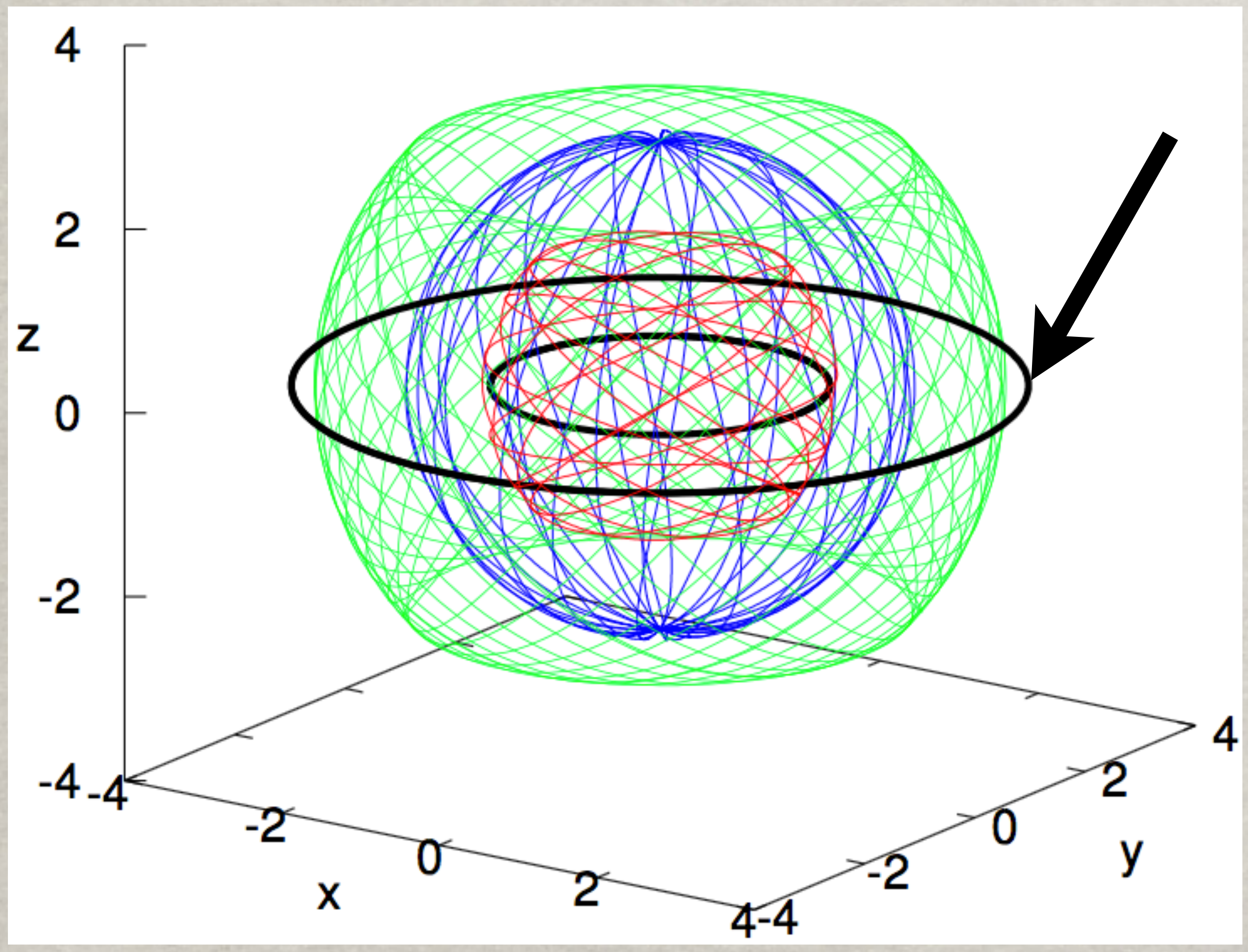

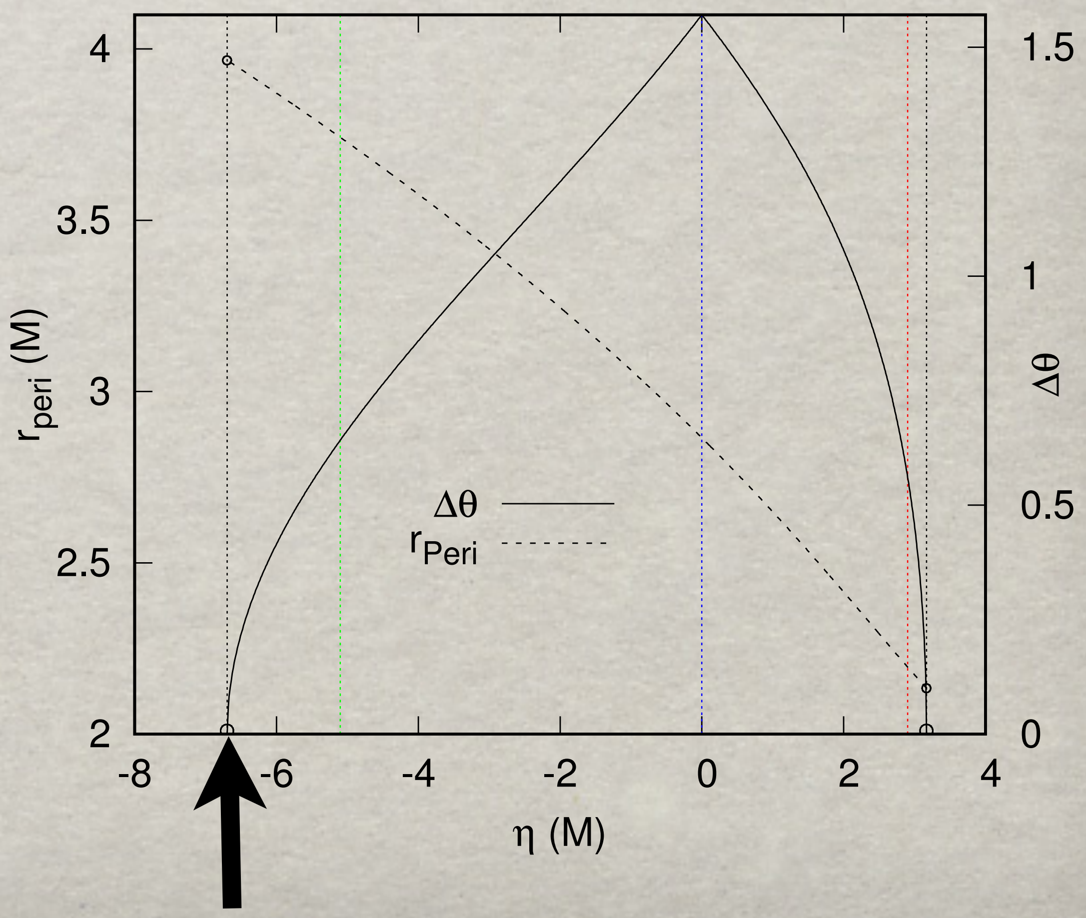

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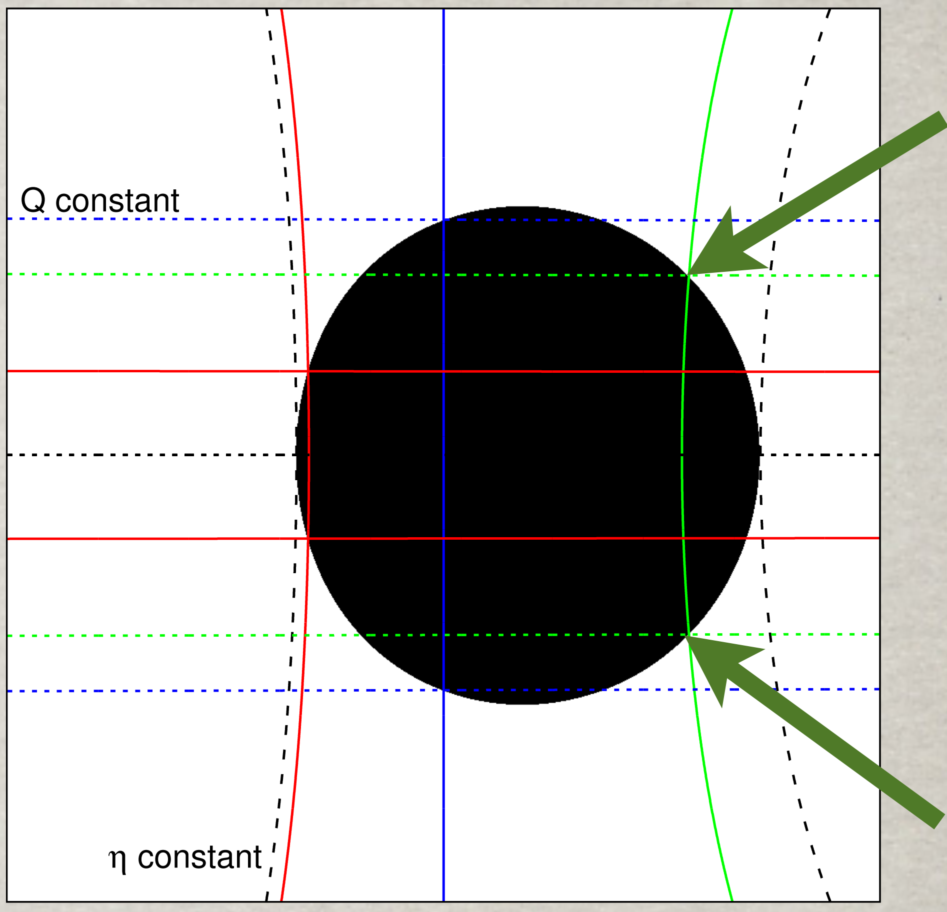

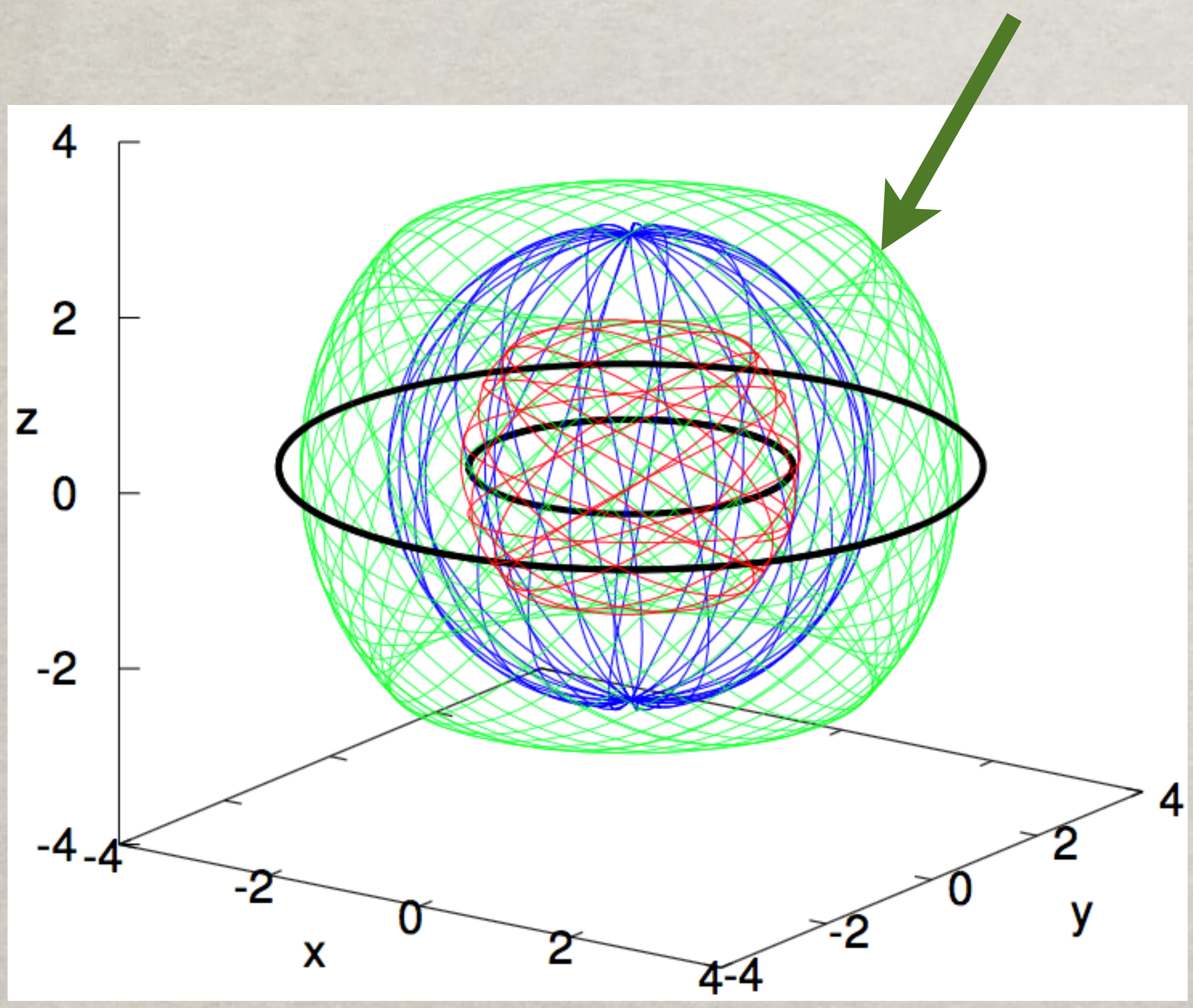

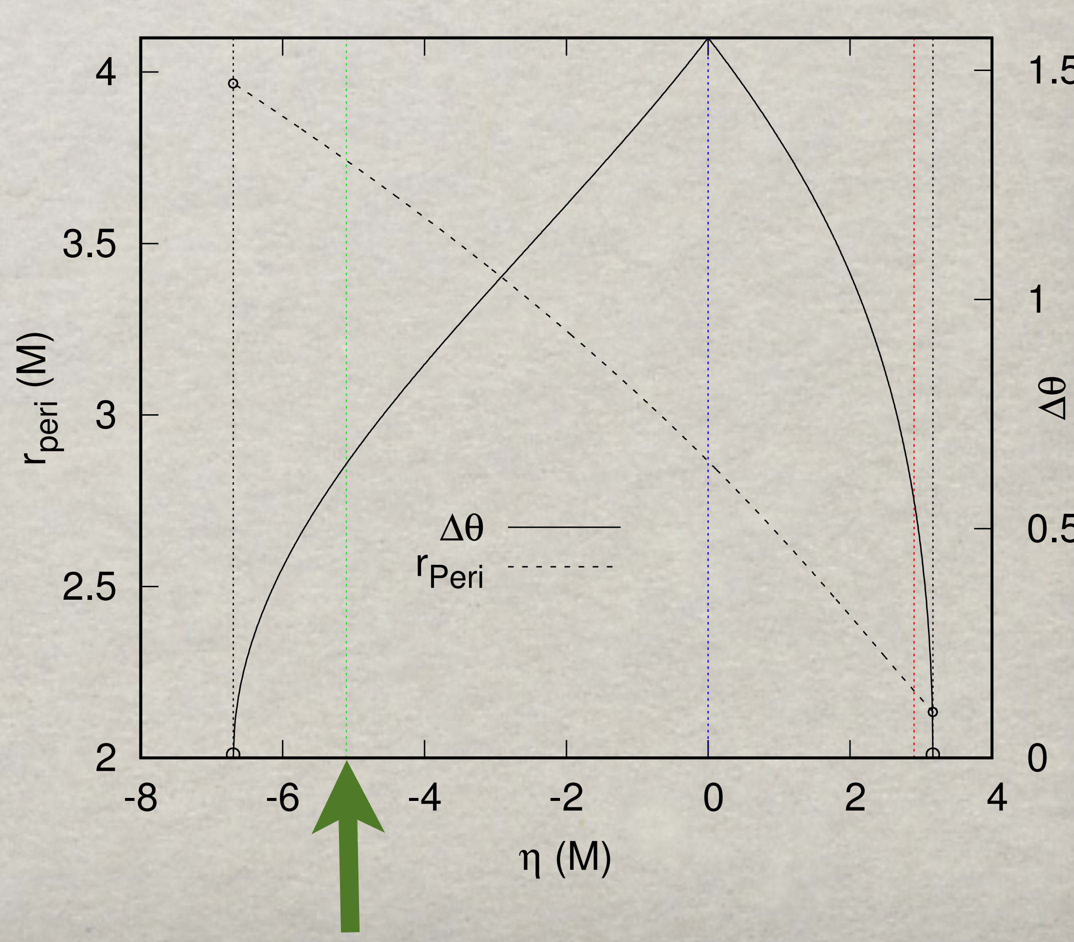

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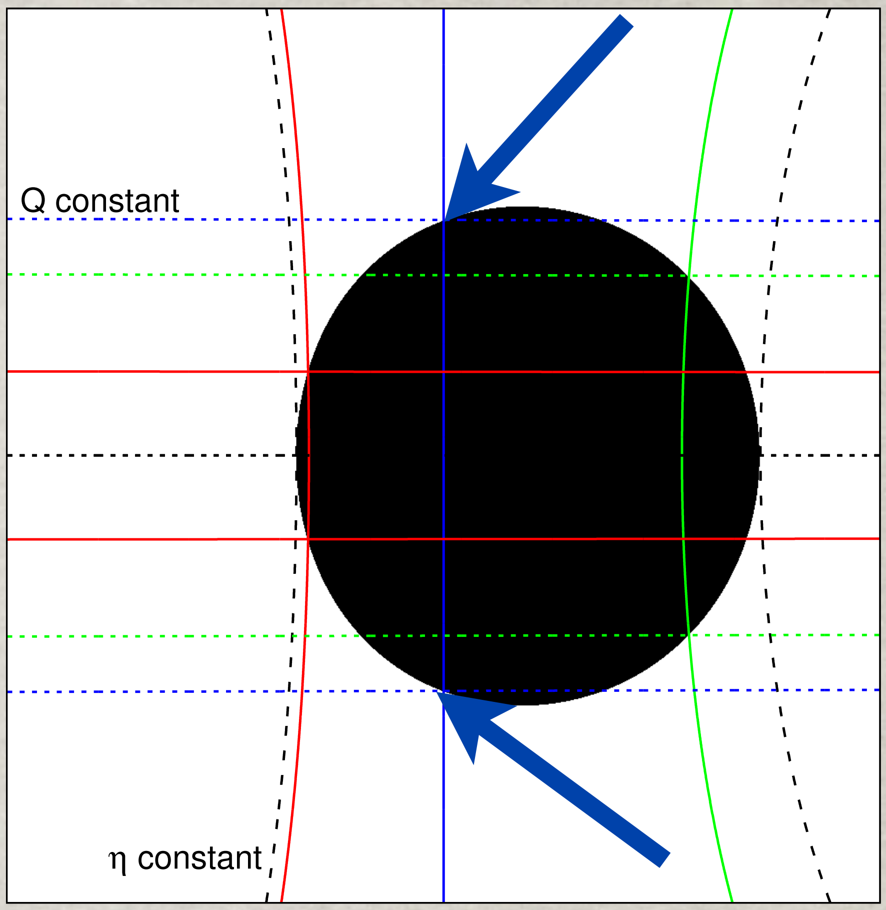

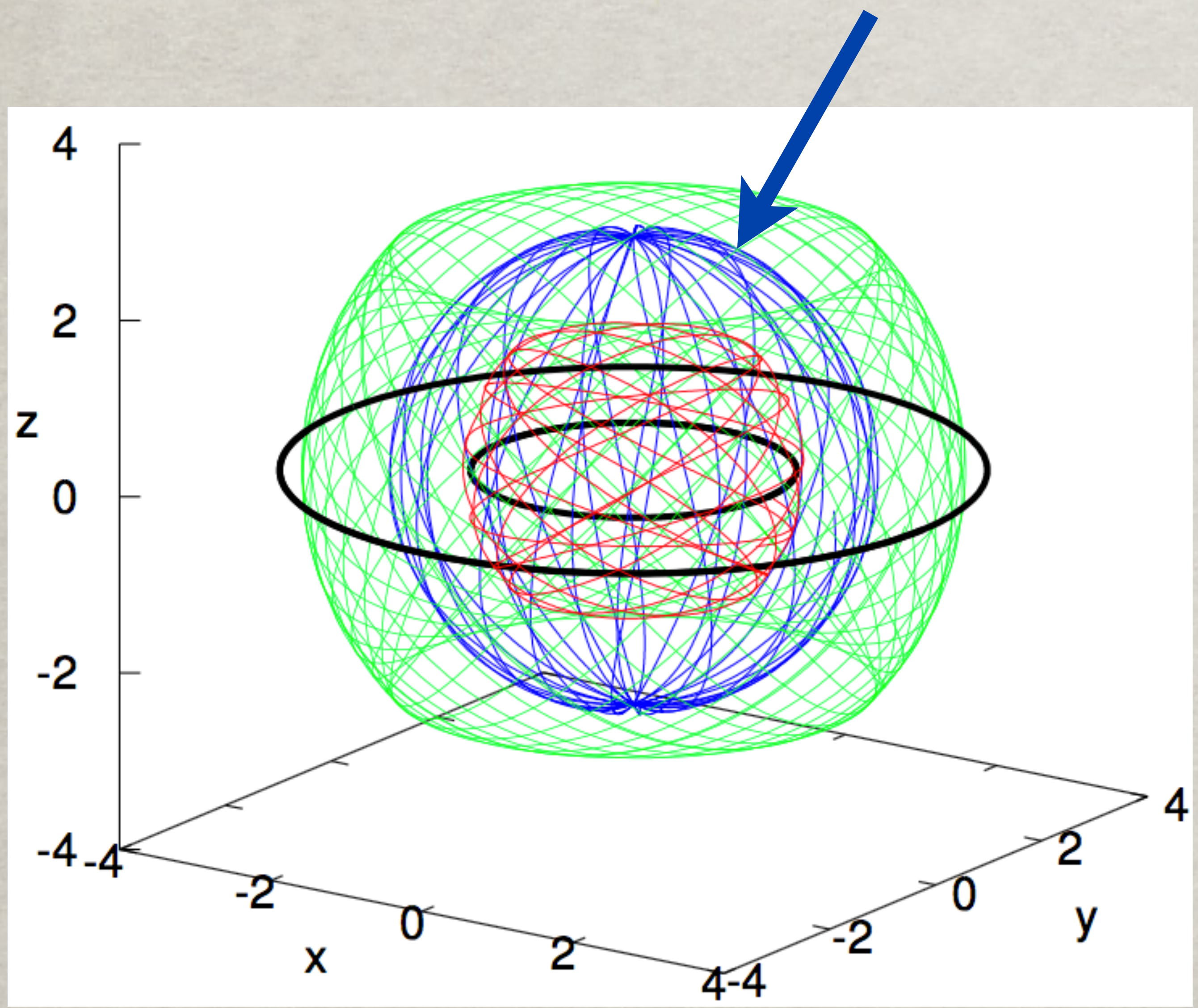

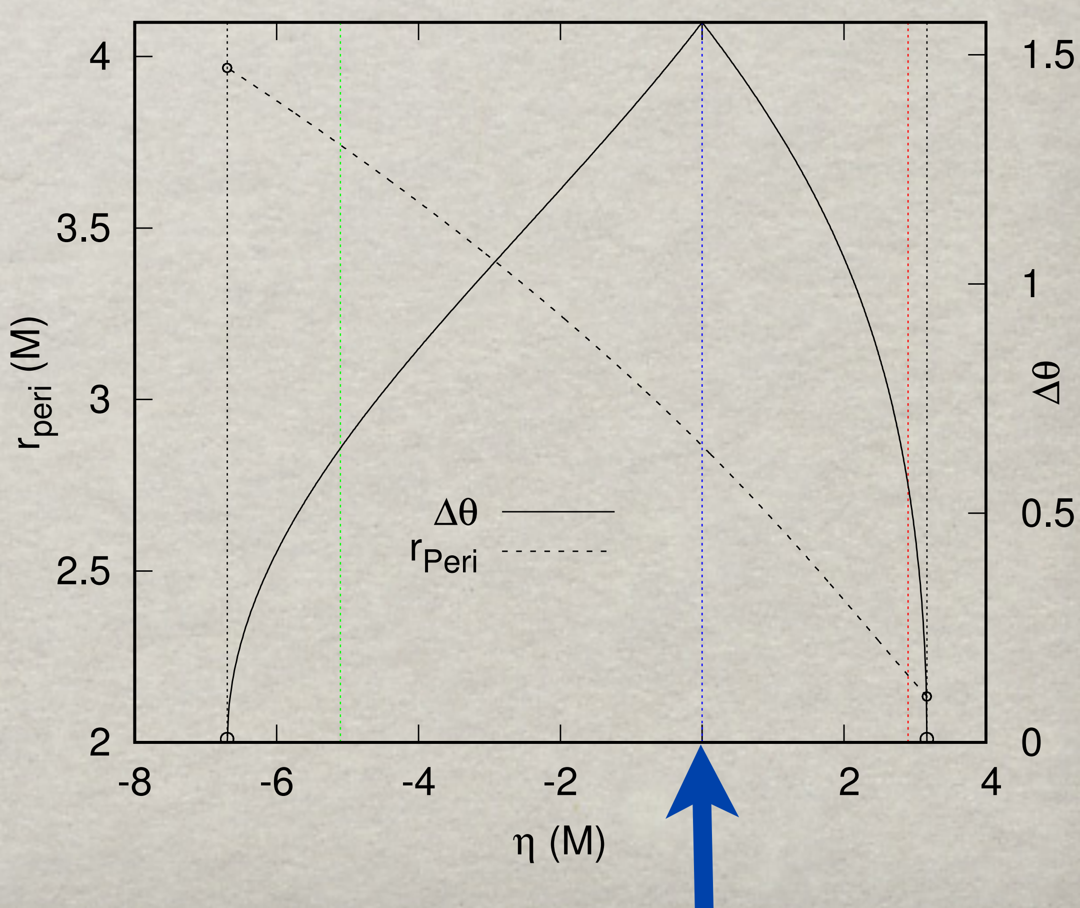

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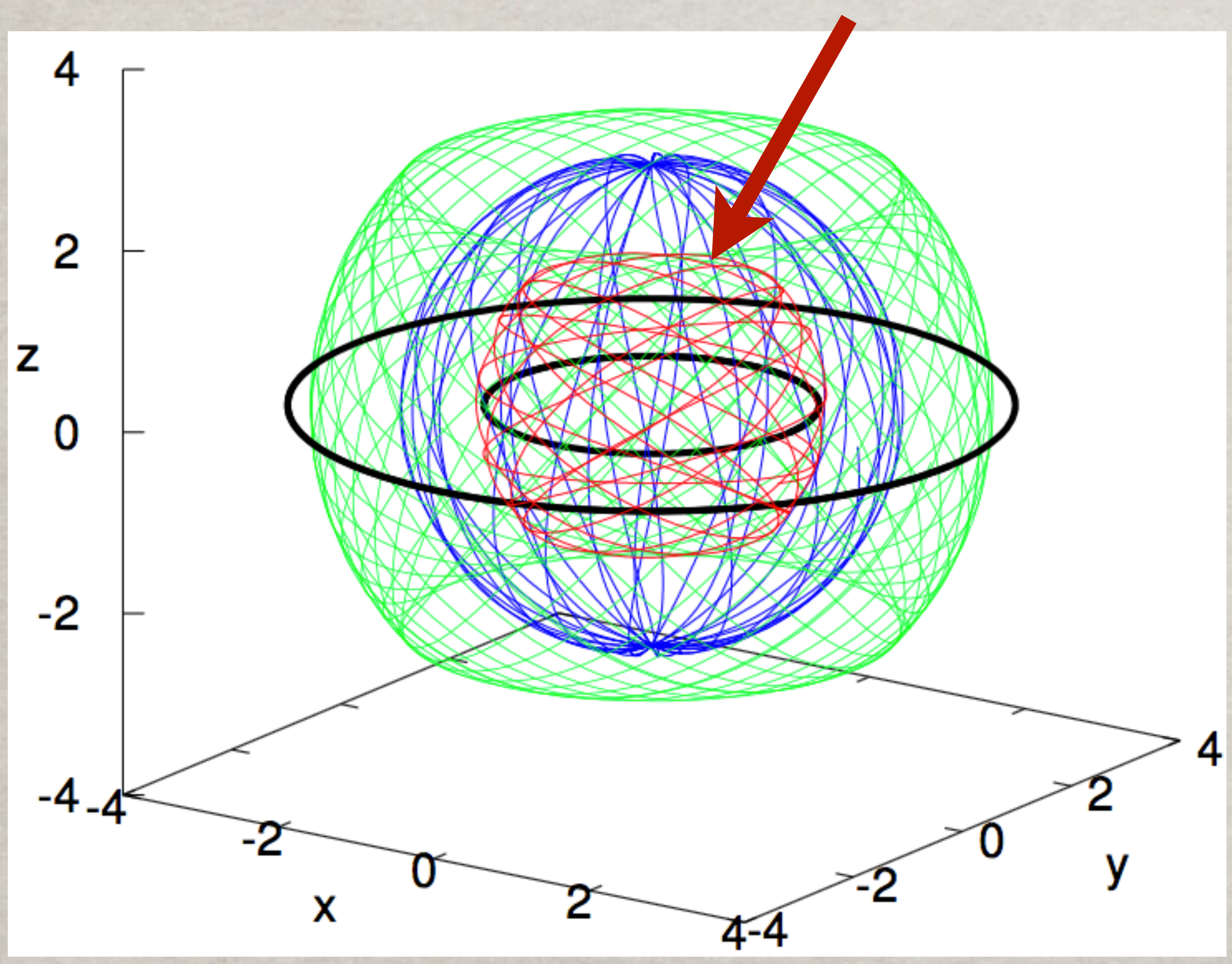

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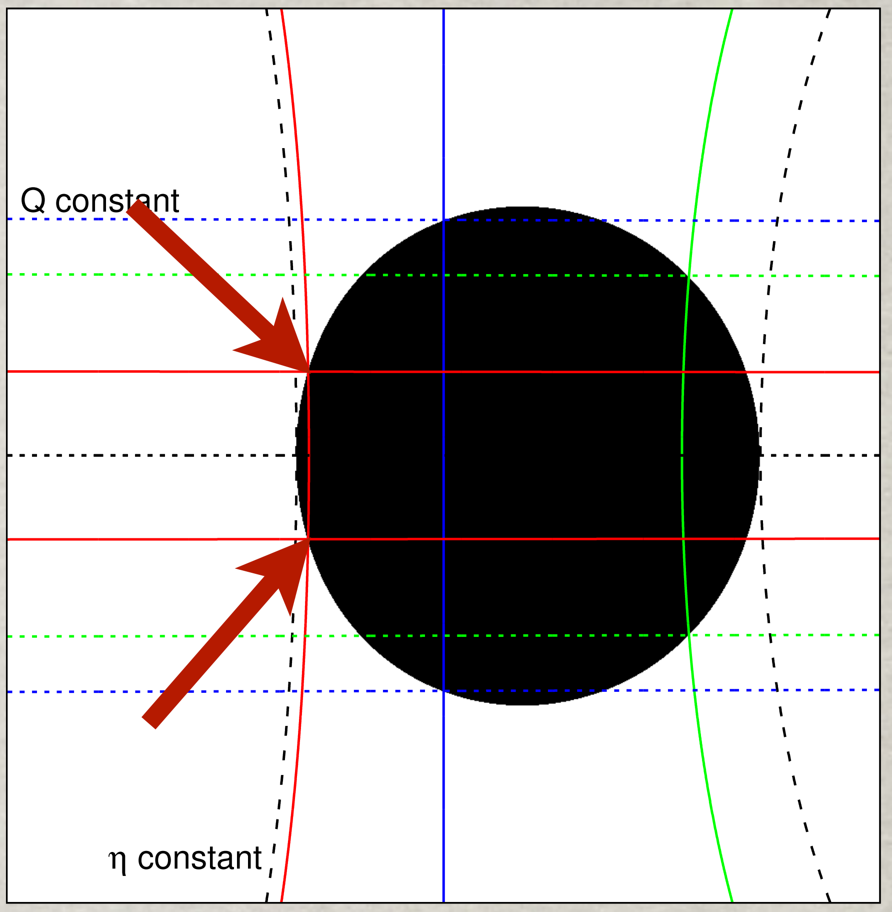

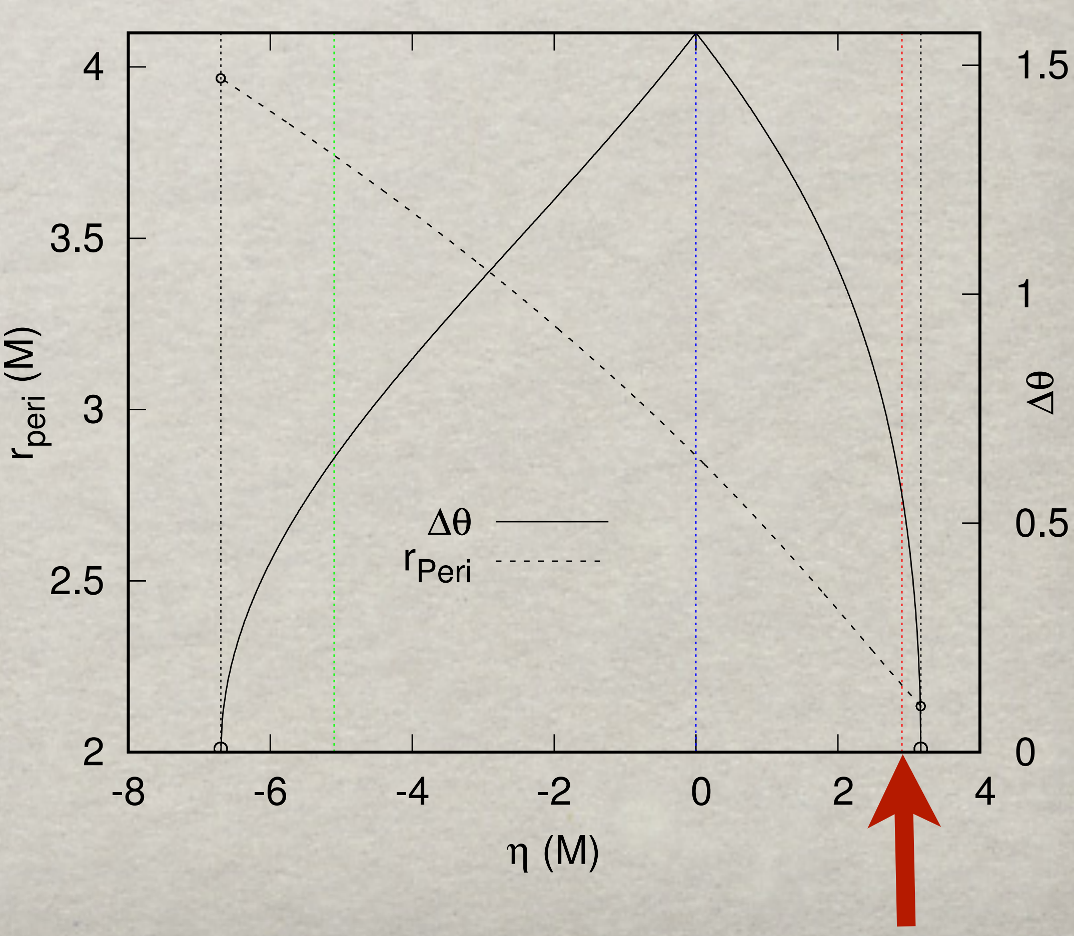

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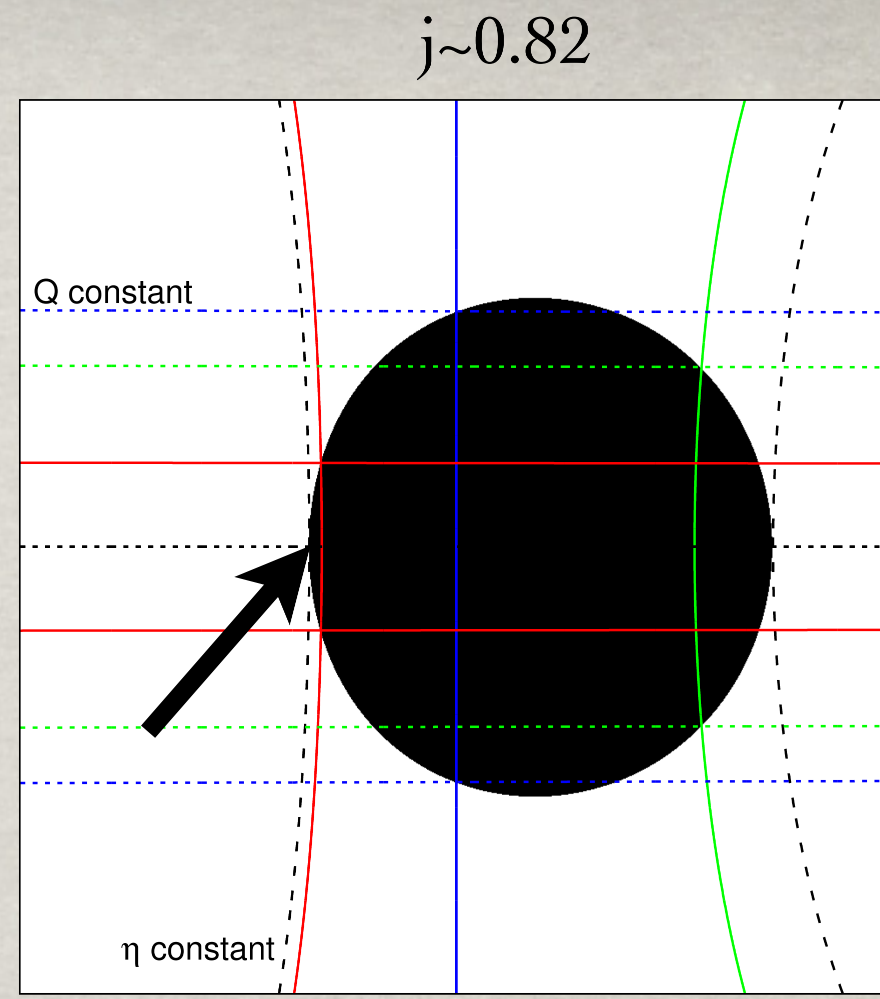
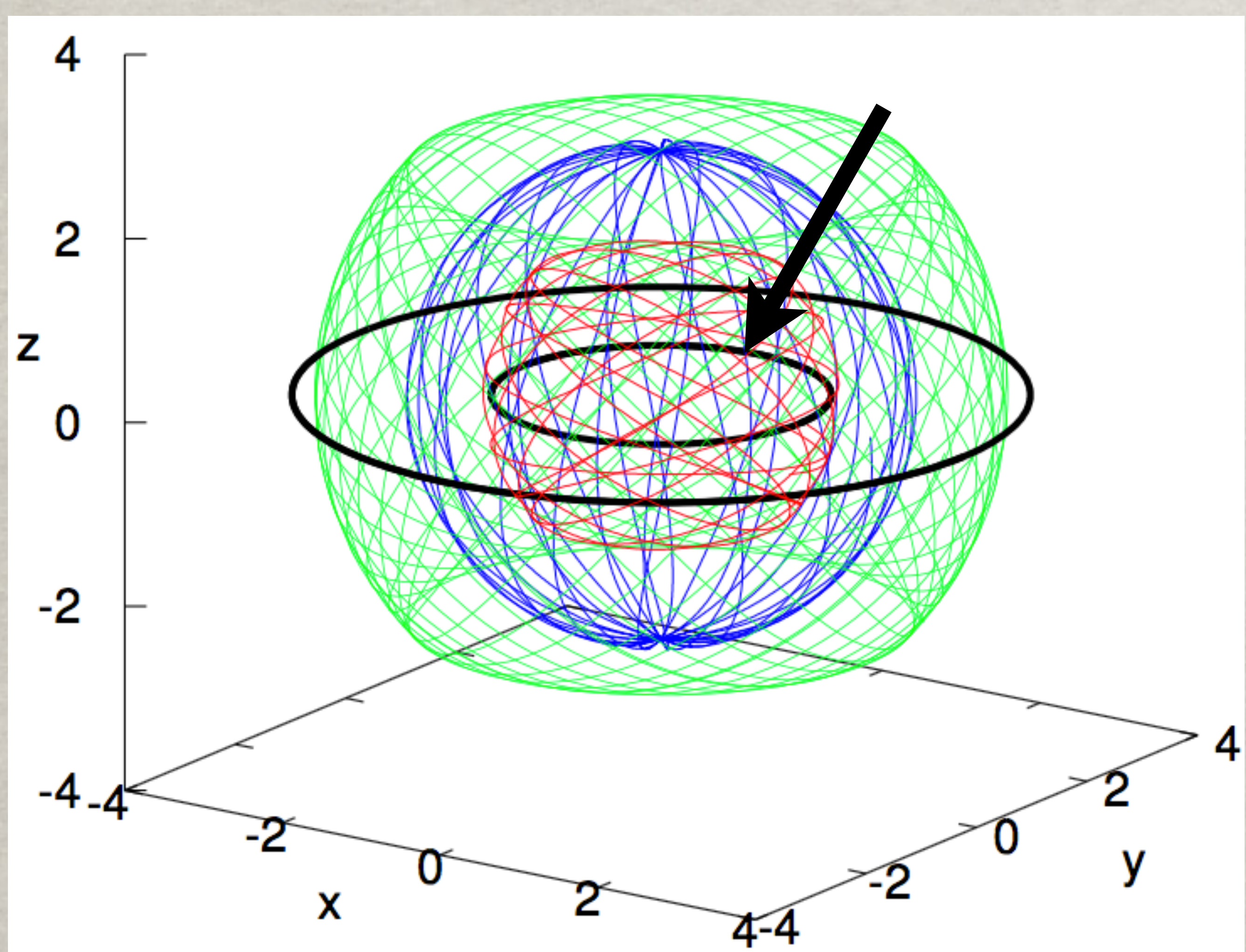
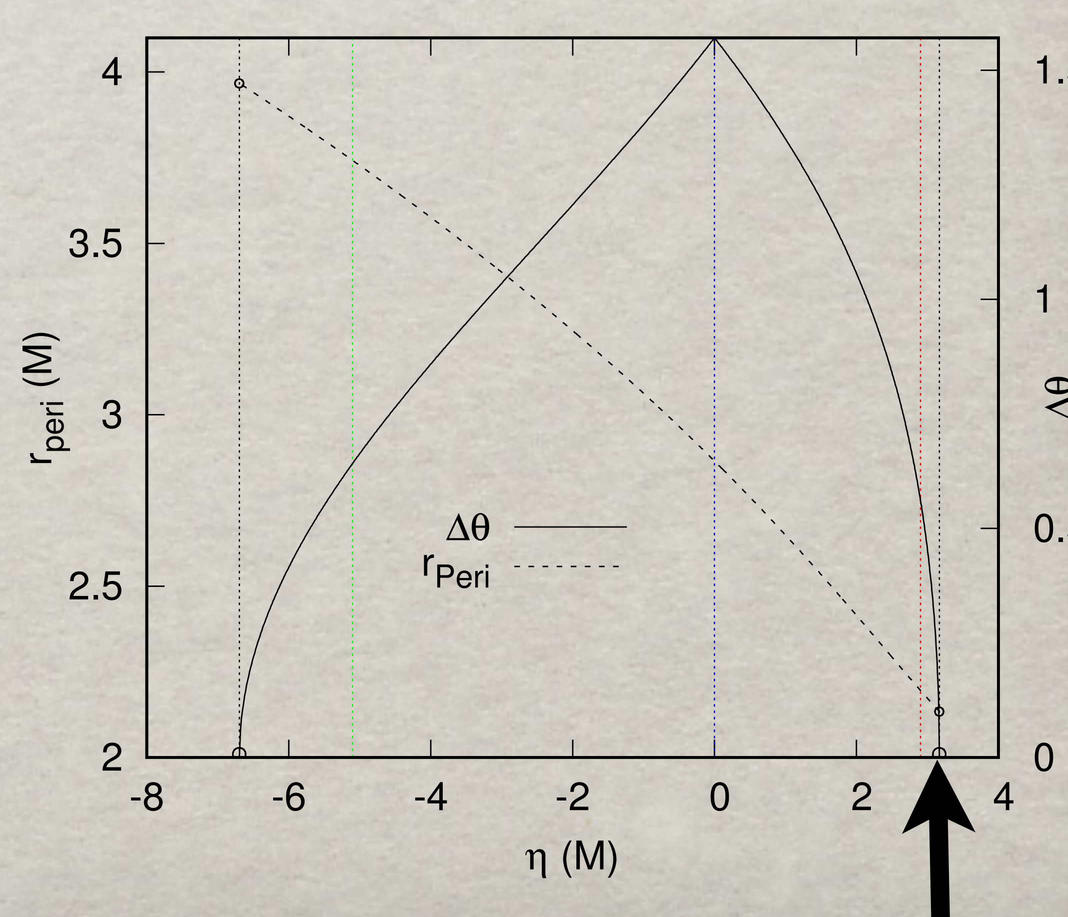






Lensing - Setup of ray tracing

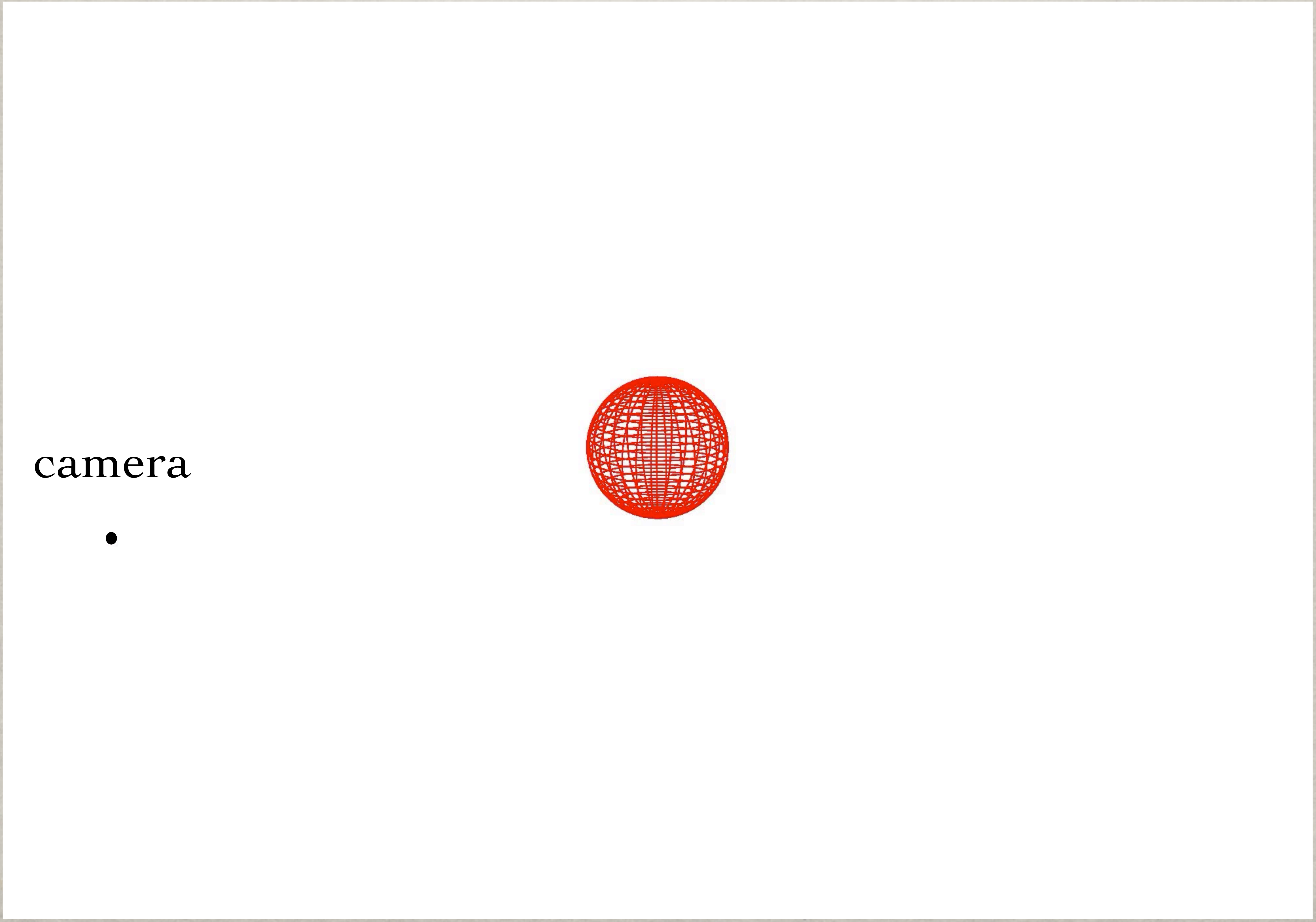
The most naive approach would be to evolve the light rays directly from the source and detect which ones reach the observer. However this procedure is inefficient since most rays would not reach.

A better approach is to evolve the light rays from the observer backward in time and identify their origin: *backward ray-tracing*.

The information carried by each ray is then assigned to a pixel in a final image, which embodies the optical perception of the observer.

Technique: backwards ray-tracing

(animation: Pedro Cunha)



Ray tracing - Schwarzschild and Kerr examples

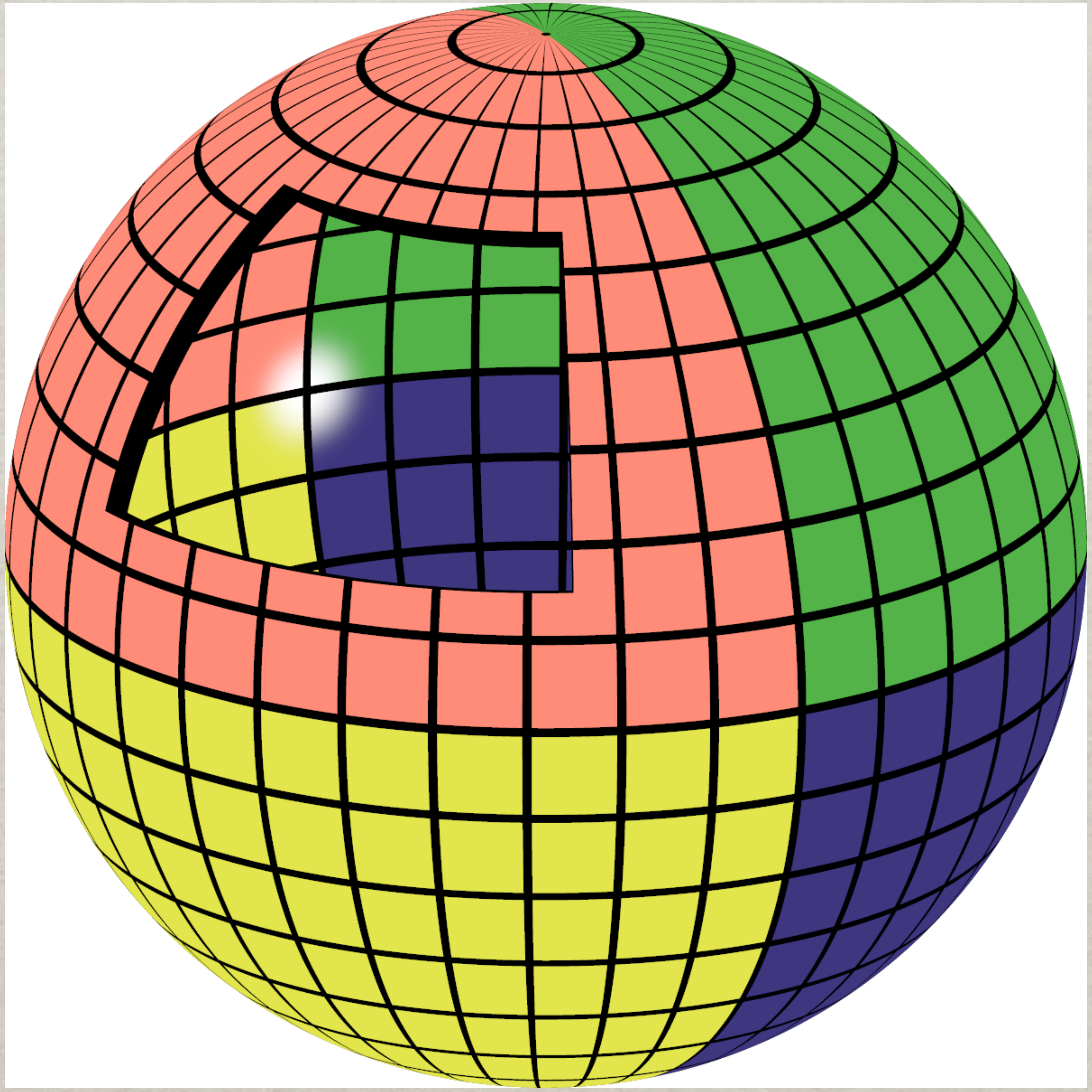
To study lensing,
the integration from the observer position ends on some chosen "far away" light source (or on the black hole).

This could be a sky full of stars.

But academic "celestial spheres" are interesting for observing lensing patterns.

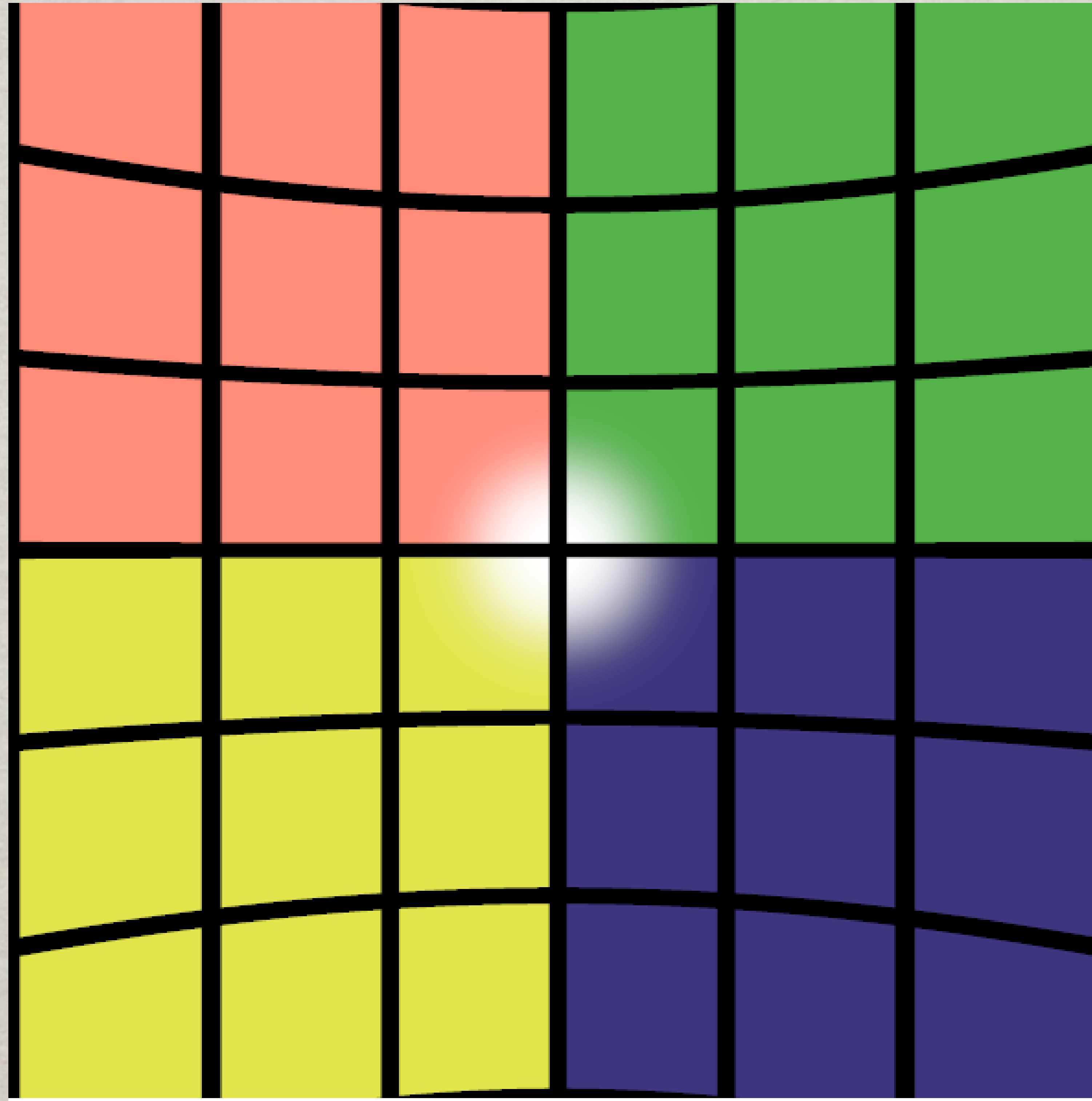
Light source is a "painted on" sphere at infinity:

- four colored quadrants with a superimposed grid;
- bright reference spot in the direction towards which we point the camera.



A. Bohn et al. CQG32(2015)065002

Visualization from camera (60° field of view): Minkowski

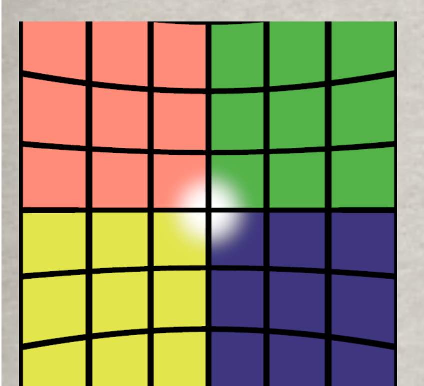


10° by 10° squares

- no deflection of light;
- bowing of the grid lines is an expected geometric effect of viewing a latitude-longitude grid.

Visualization from camera (60° field of view): Schwarzschild

White dot on grid at "infinity" has been lensed into an Einstein ring



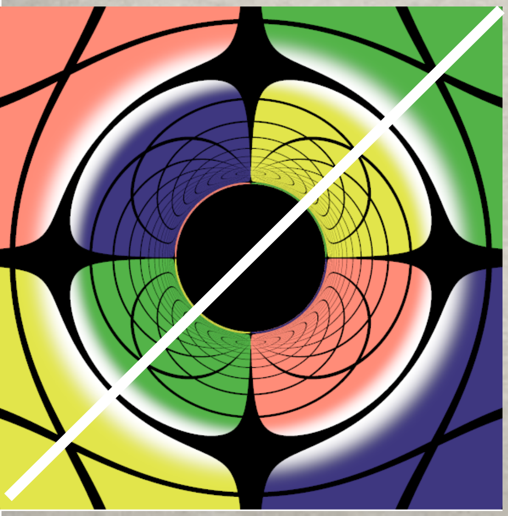
Regions inside the Einstein ring: photons deflected by larger angles than Einstein ring photons => inverted image of reference grid

Second Einstein ring corresponding to light from a source behind the camera

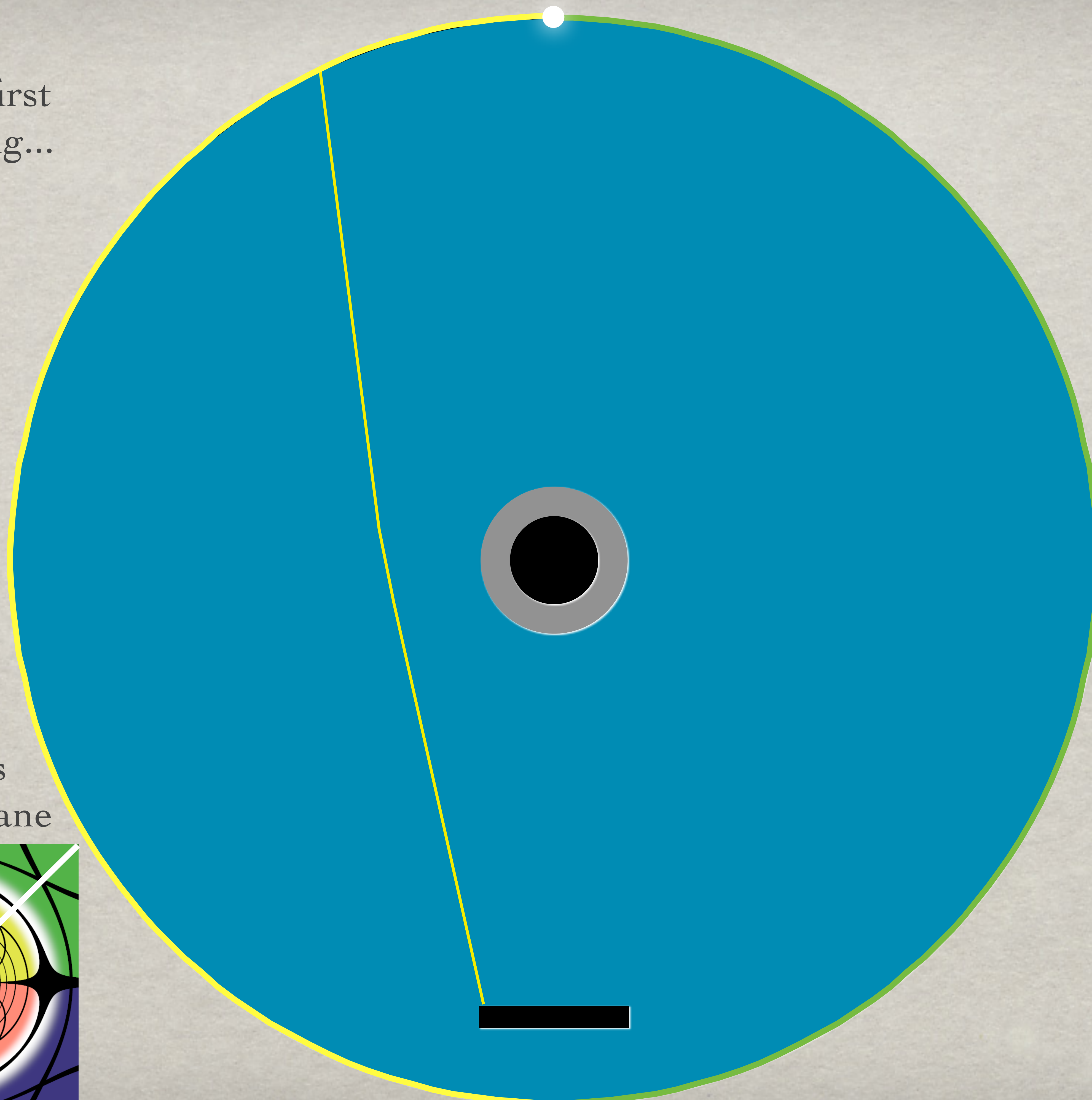
There will be an infinite number of Einstein rings

Origin,
on the sphere at "infinity",
of the camera image.

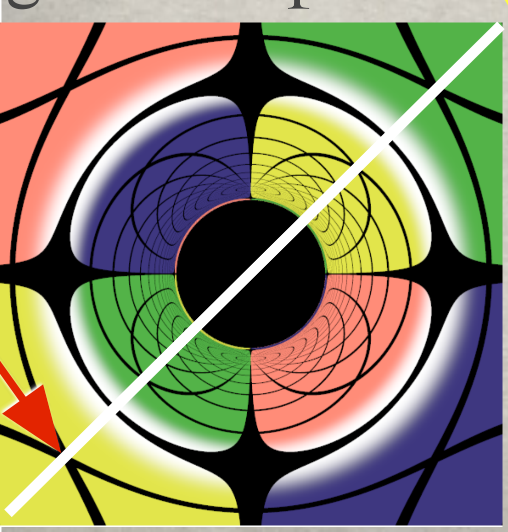
... on this
geodesic plane



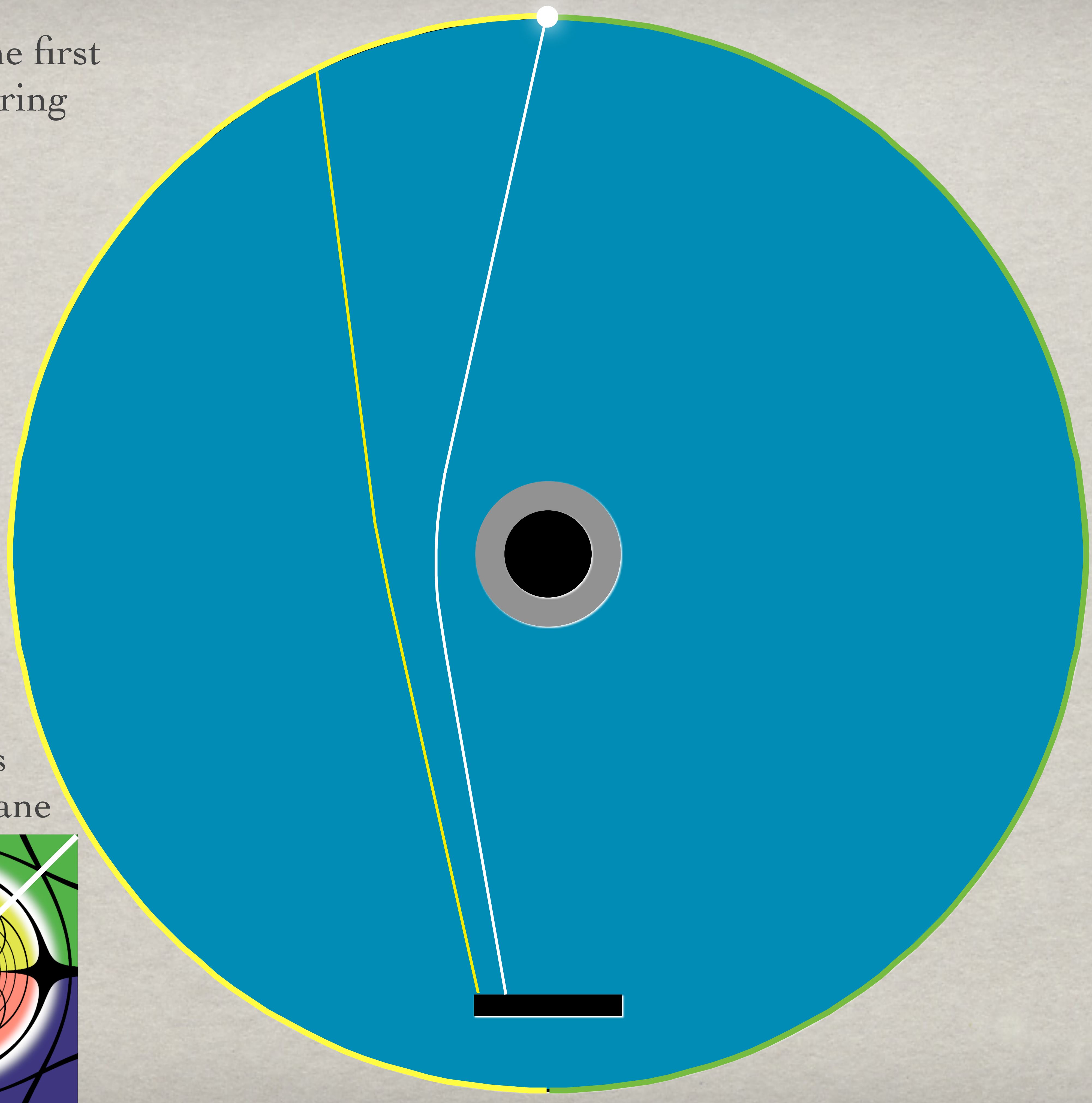
Point
outside of first
Einstein ring...



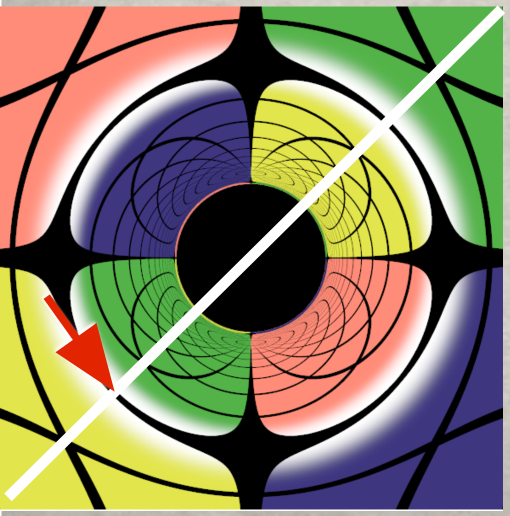
... on this
geodesic plane



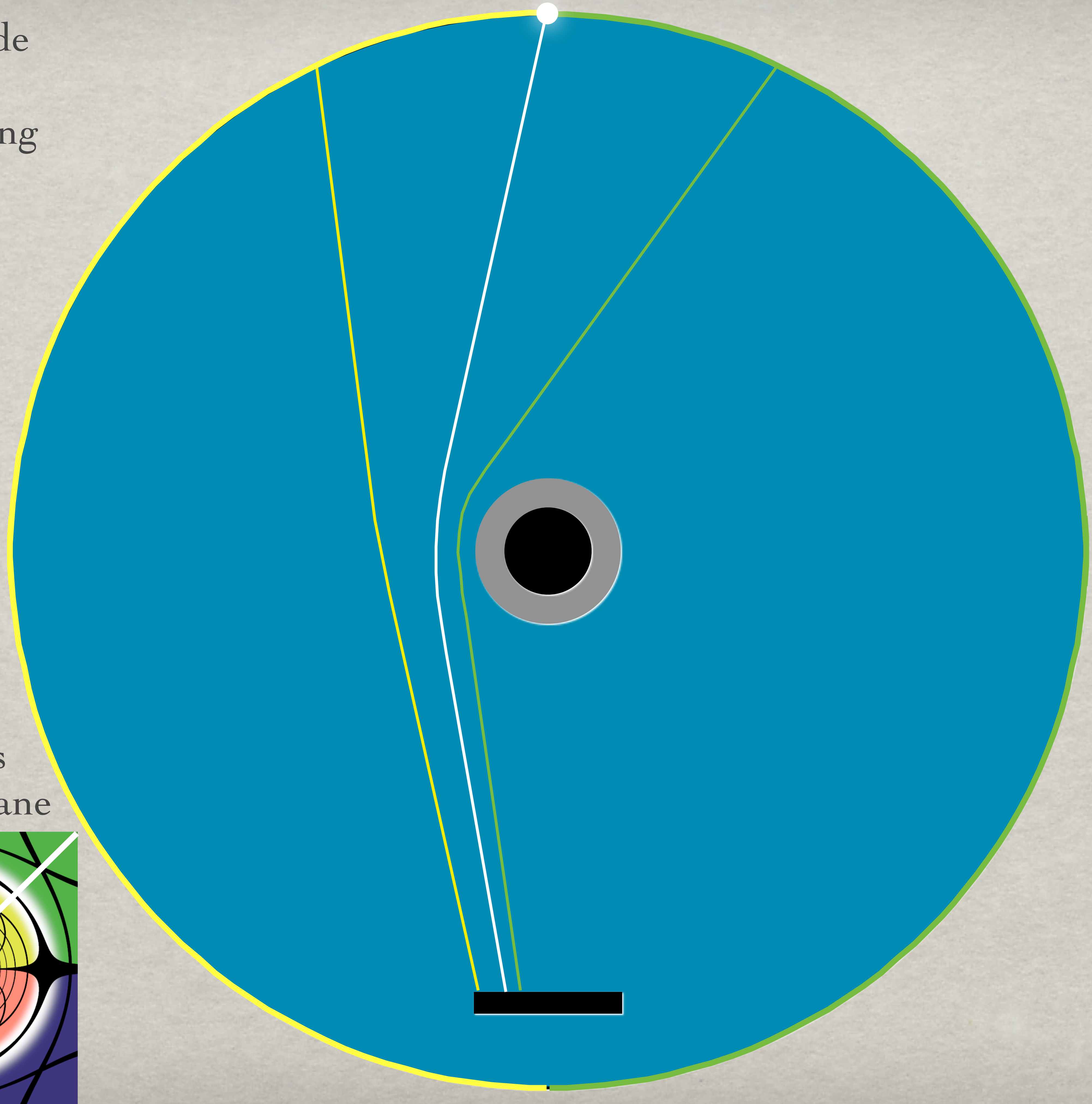
Point on the first
Einstein ring



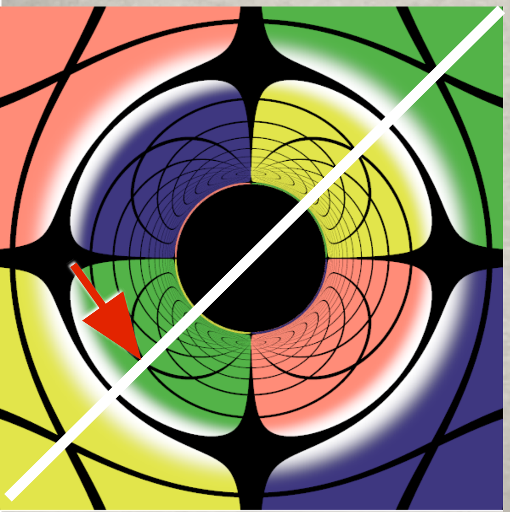
... on this
geodesic plane



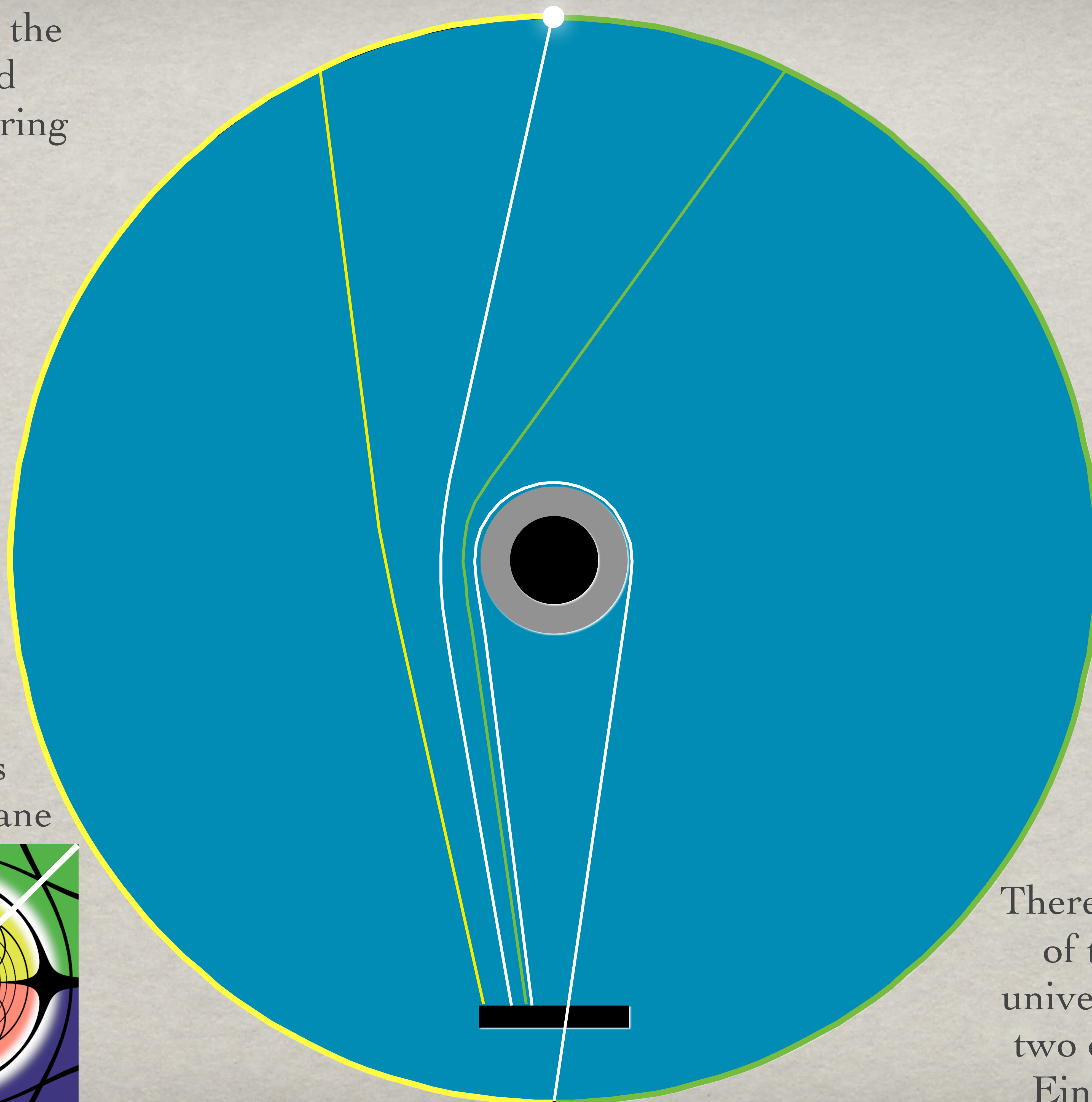
Point inside
the first
Einstein ring



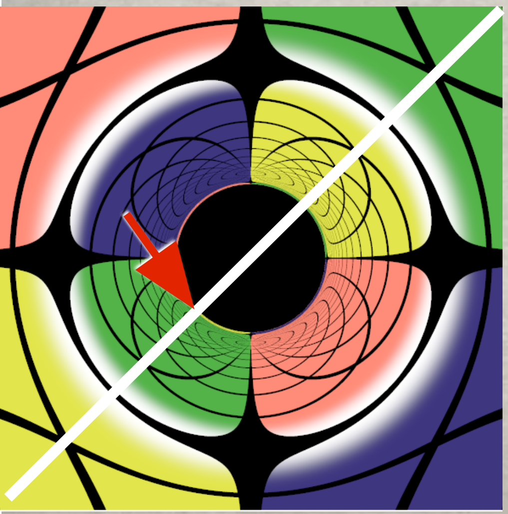
... on this
geodesic plane



Point on the second Einstein ring

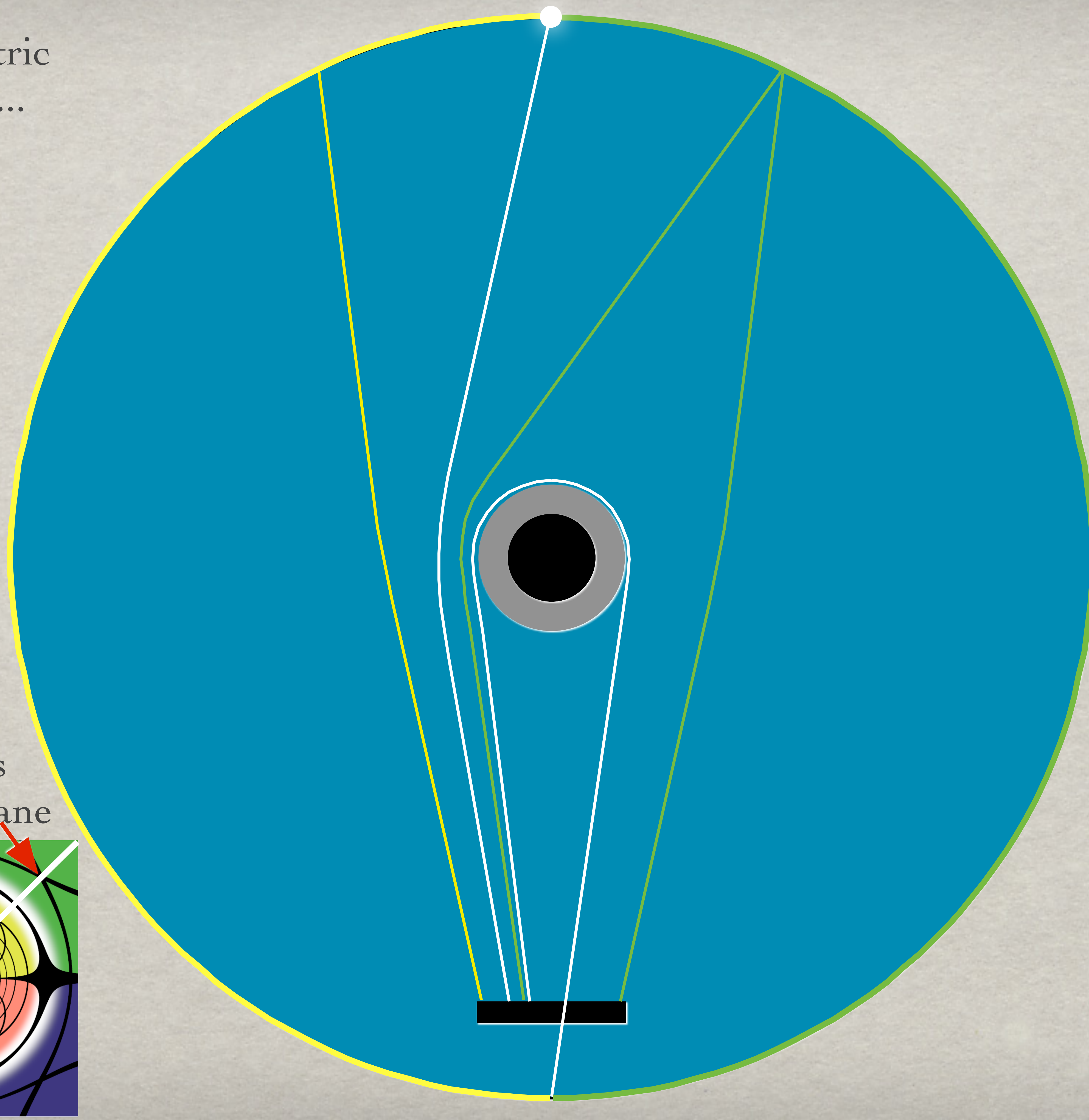


... on this geodesic plane

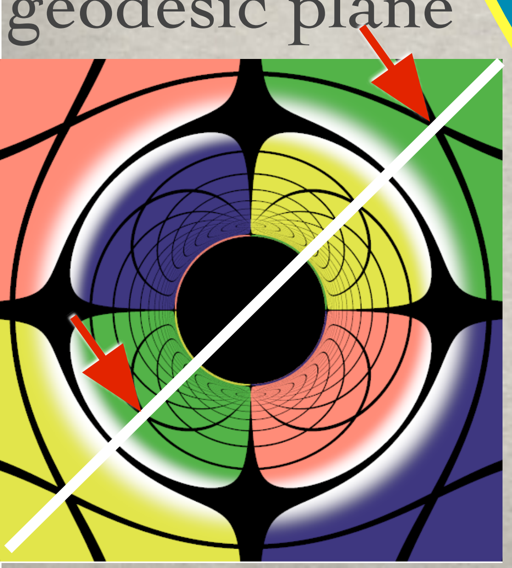


There is an image of the **whole** universe between two consecutive Einstein rings

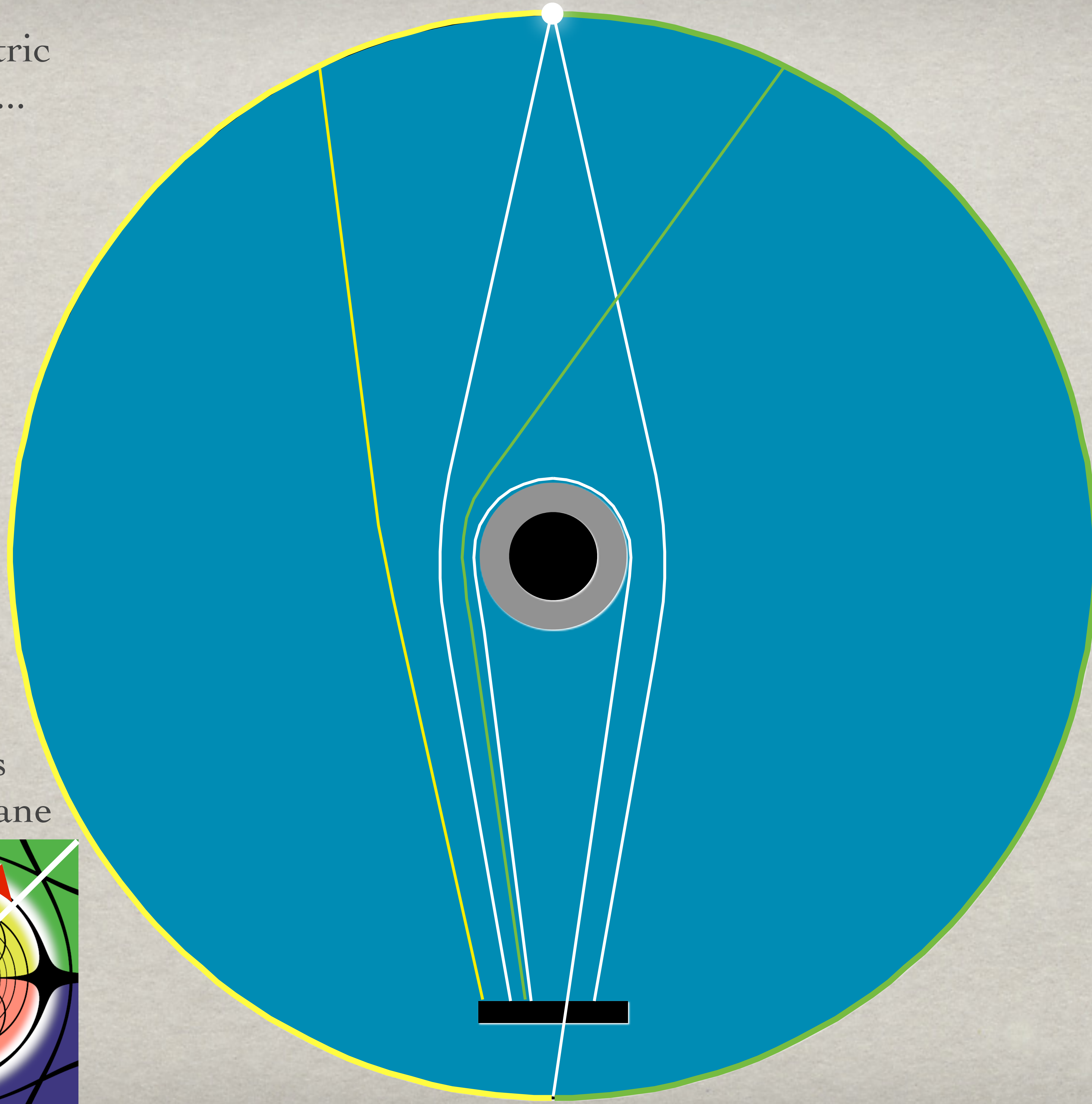
Symmetric points...



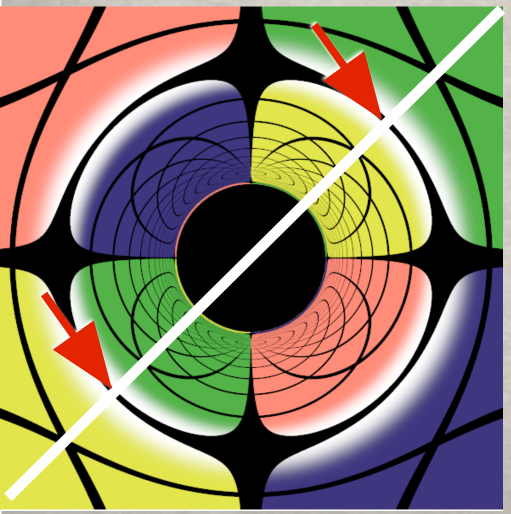
... on this geodesic plane



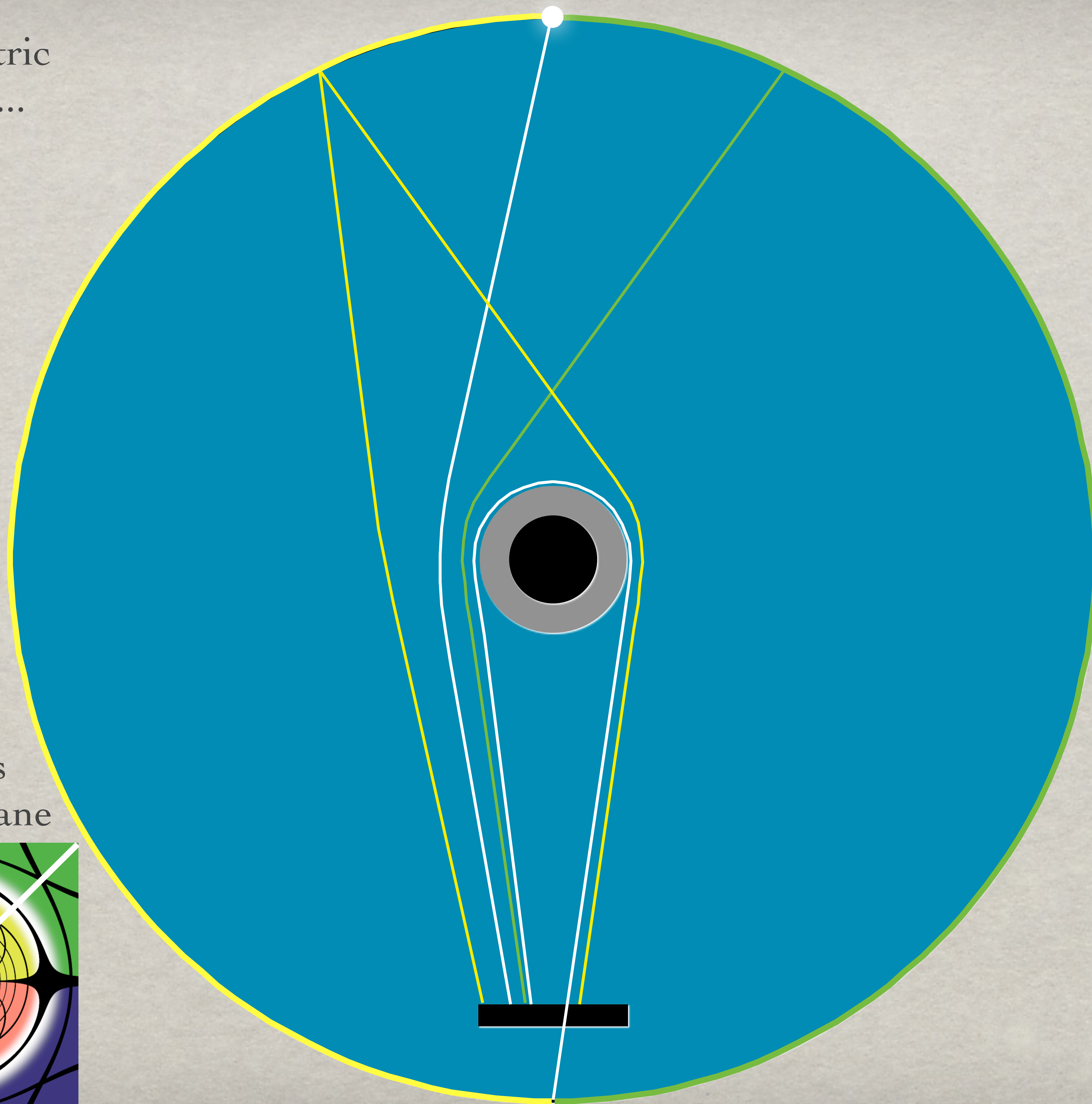
Symmetric points...



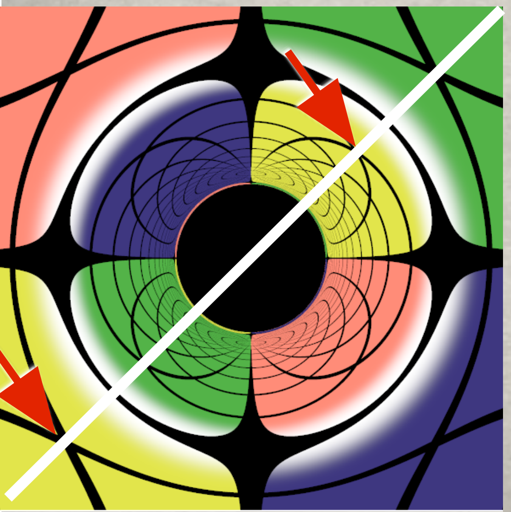
... on this geodesic plane



Symmetric points...

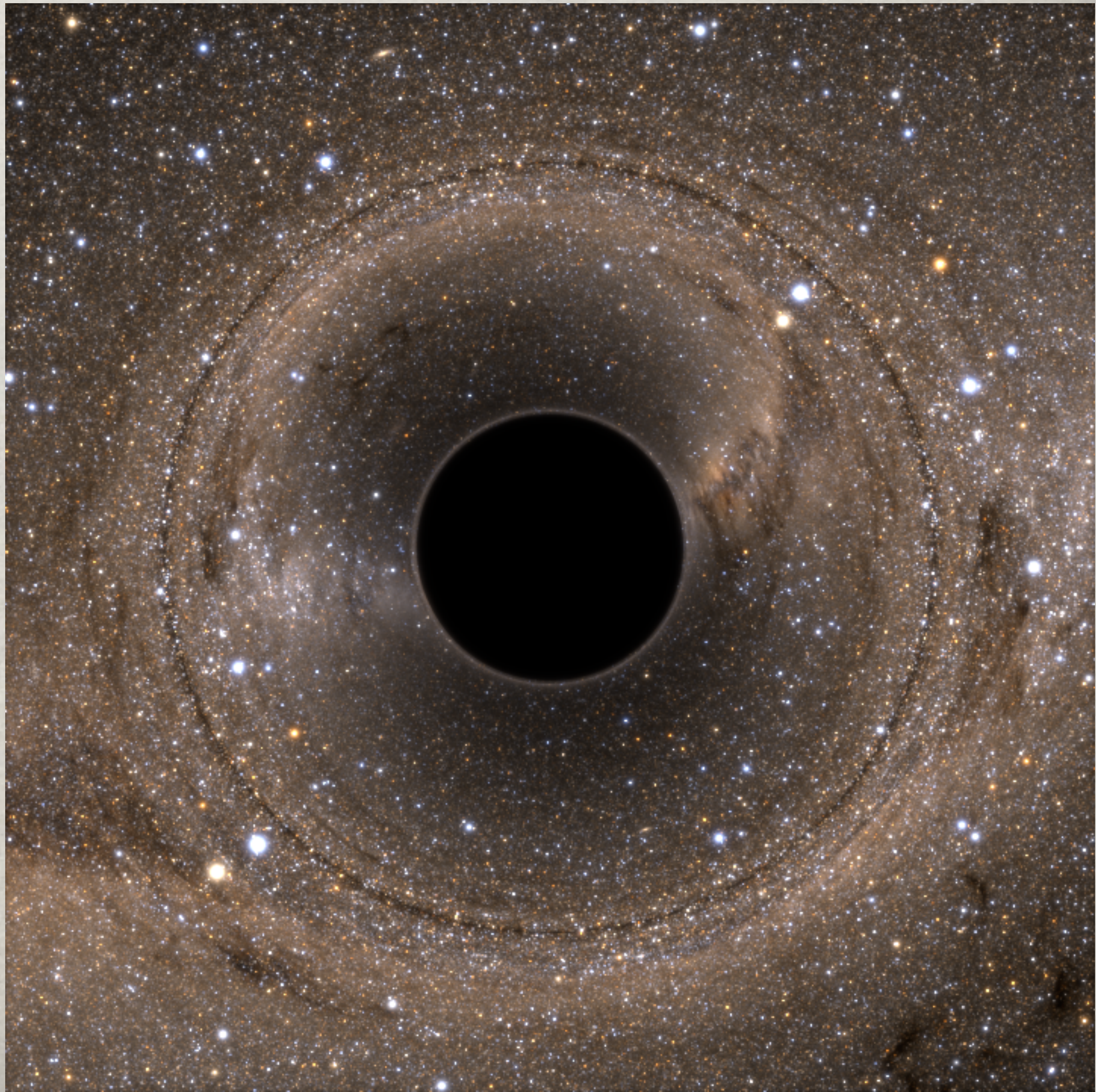


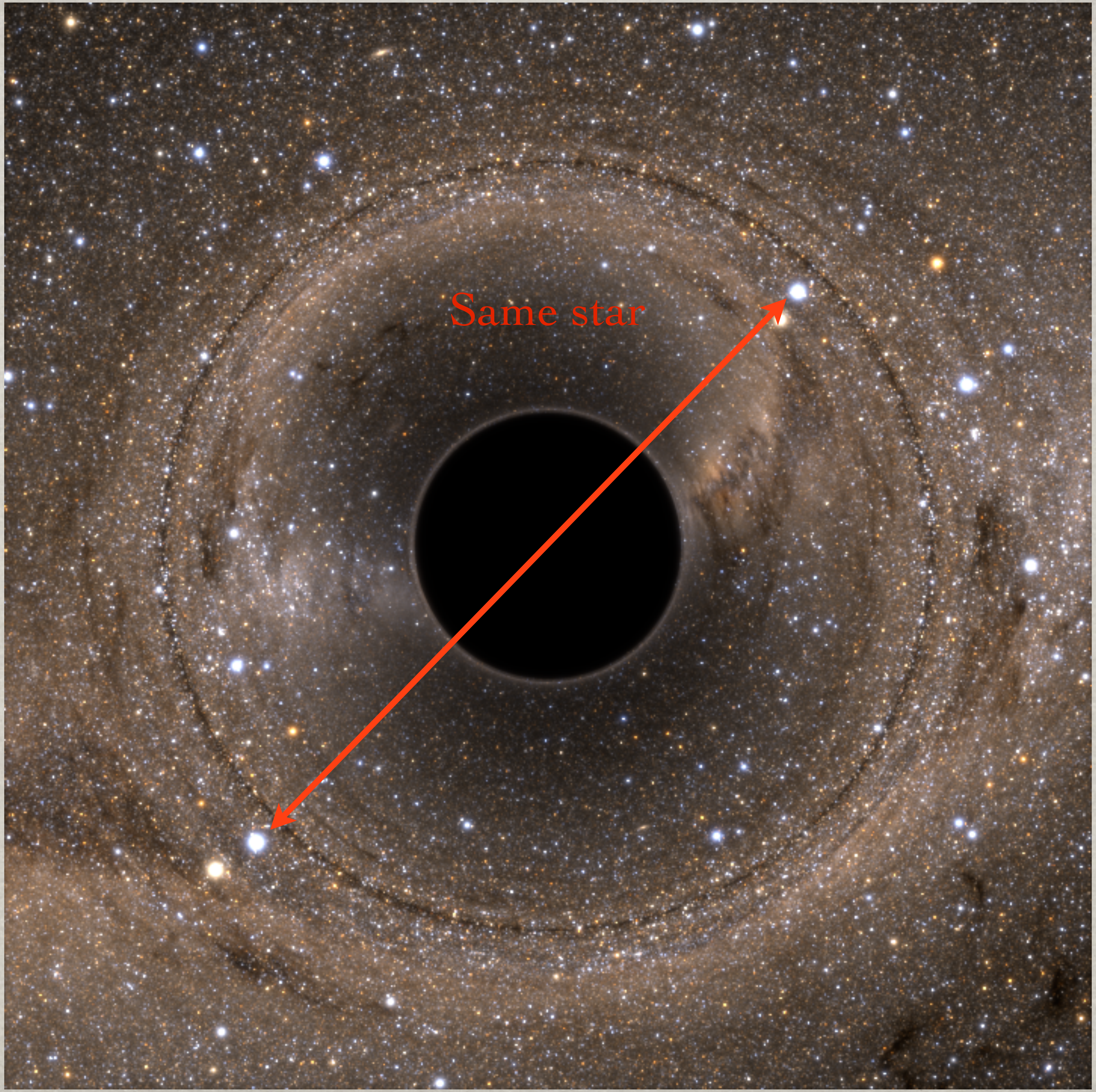
... on this geodesic plane

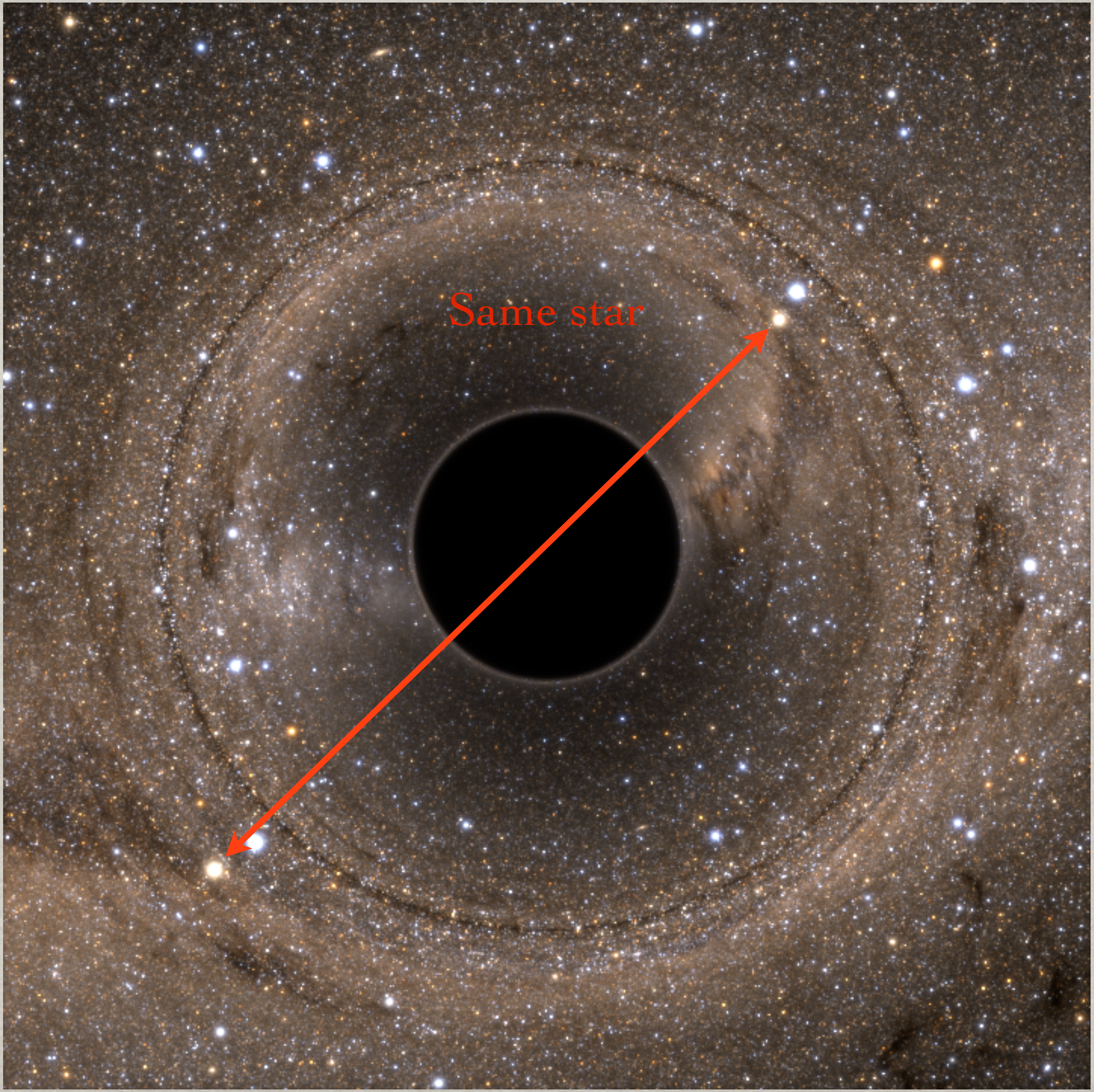


3.4*10^8 stars from the Two Micron All Sky Survey (2MASS)

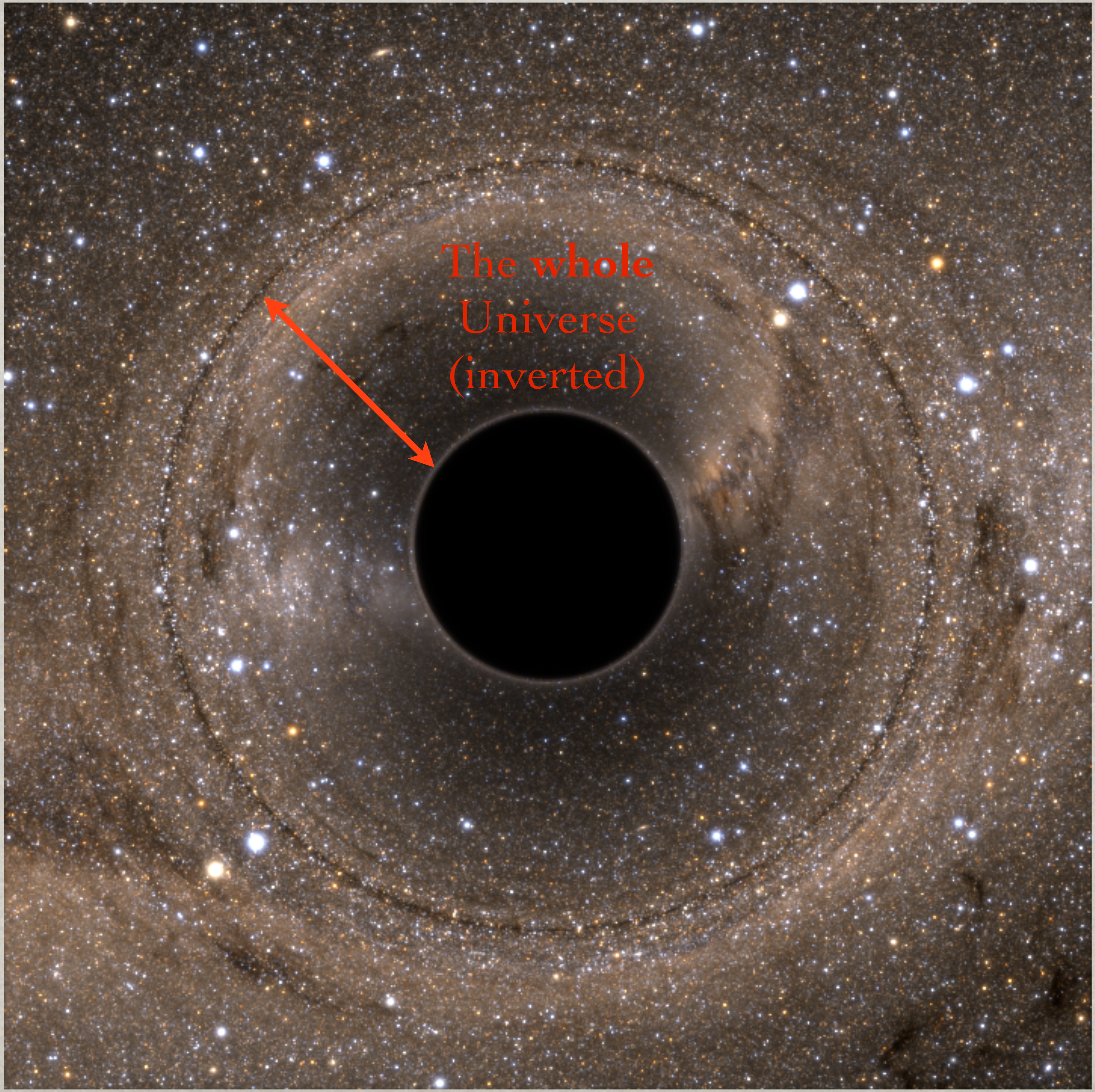
http://www.black-holes.org/the-science-numerical-relativity/numerical-relativity/gravitational-lensing




Same star

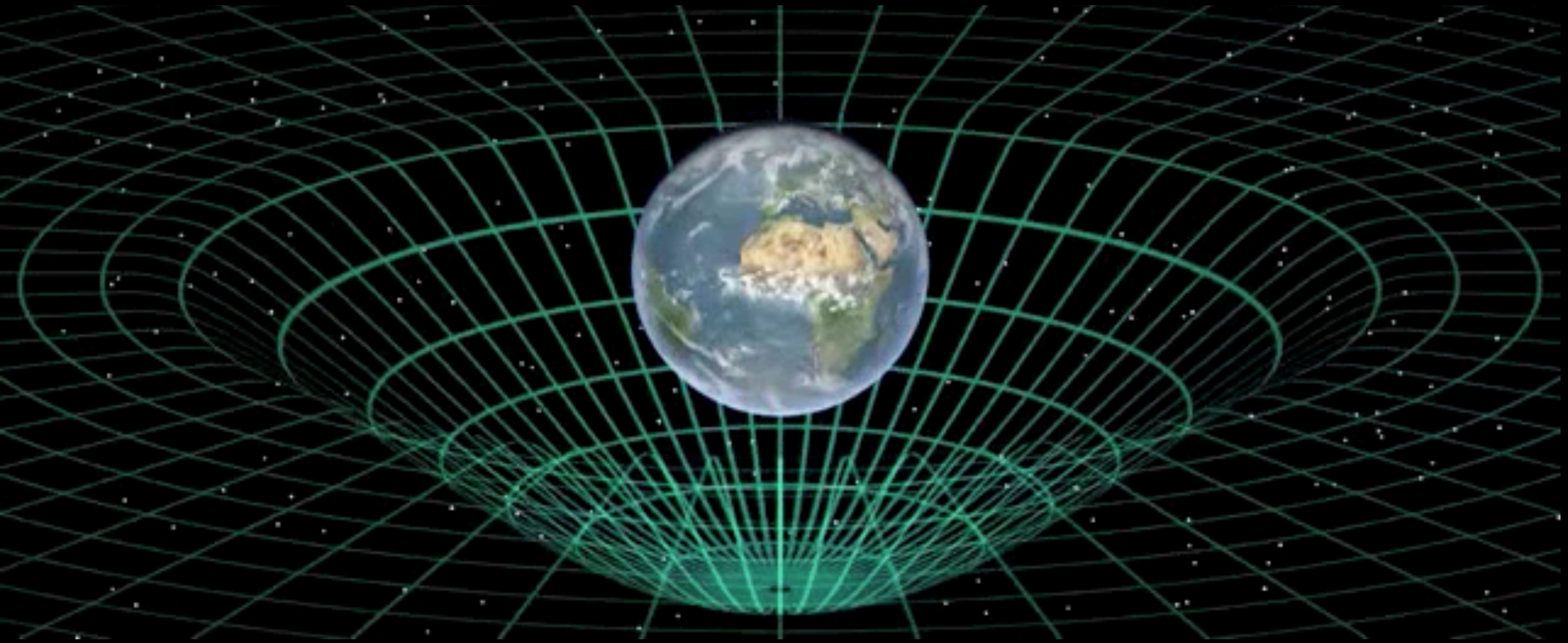

Same star

Between the 2nd and 3rd Einstein rings there is again the whole Universe (upright)



The lensing structure of a BH exhibits self-similarity.

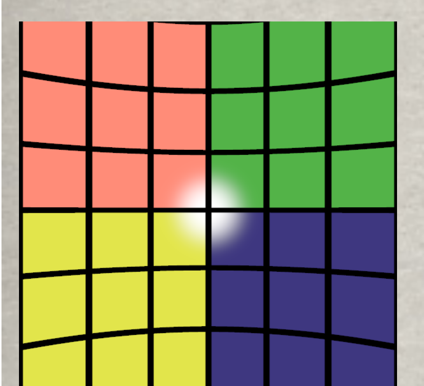
Rotation introduces frame dragging

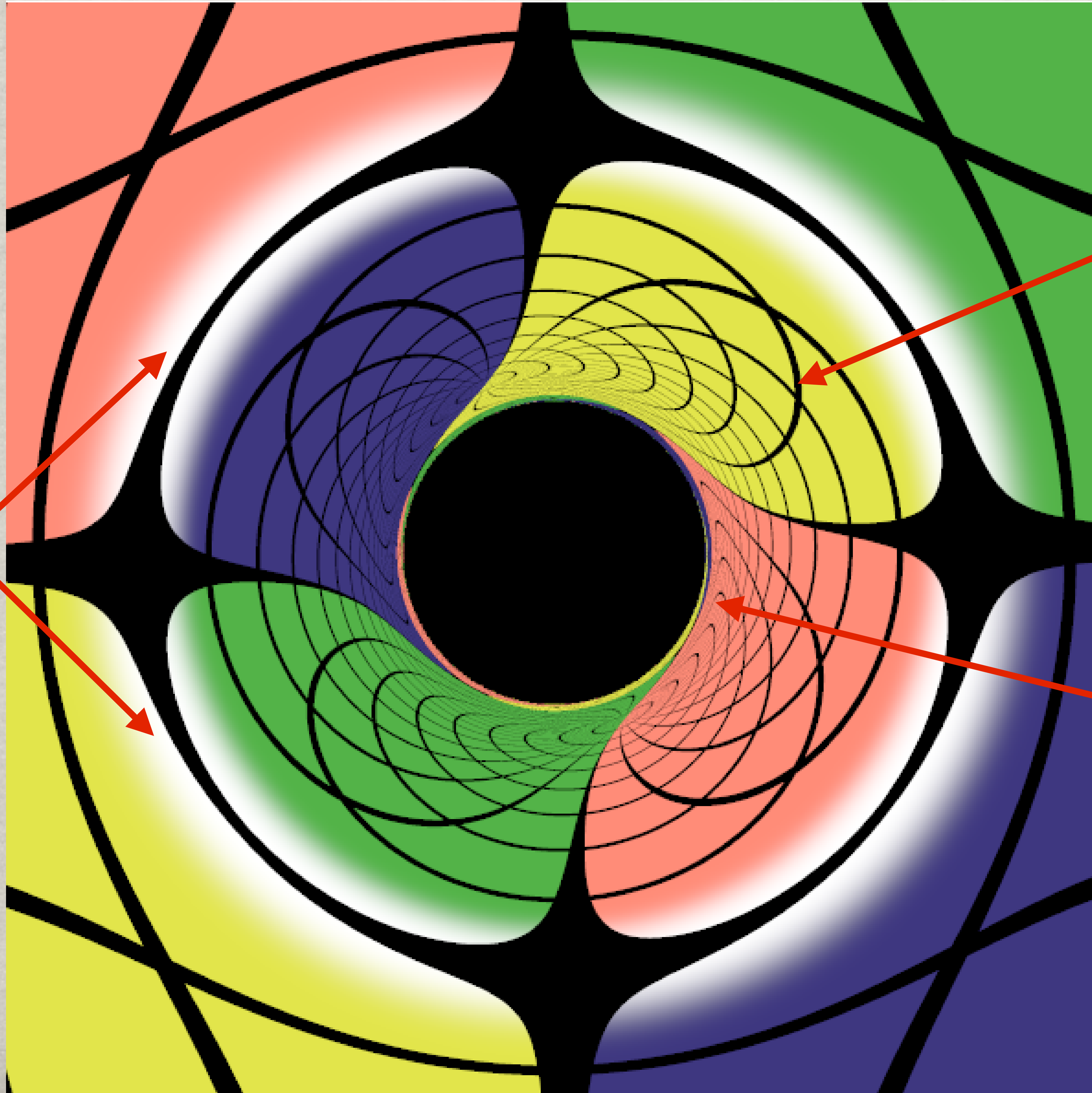


Animation: Gravity Probe B Team

Visualization from camera: Kerr (a=0.95M), spin axis along line of sight

White dot on grid at "infinity" has been lensed into an Einstein ring





The spin of the BH causes "frame dragging"; the effect on the photon trajectories produces an apparent dragging of the grid itself

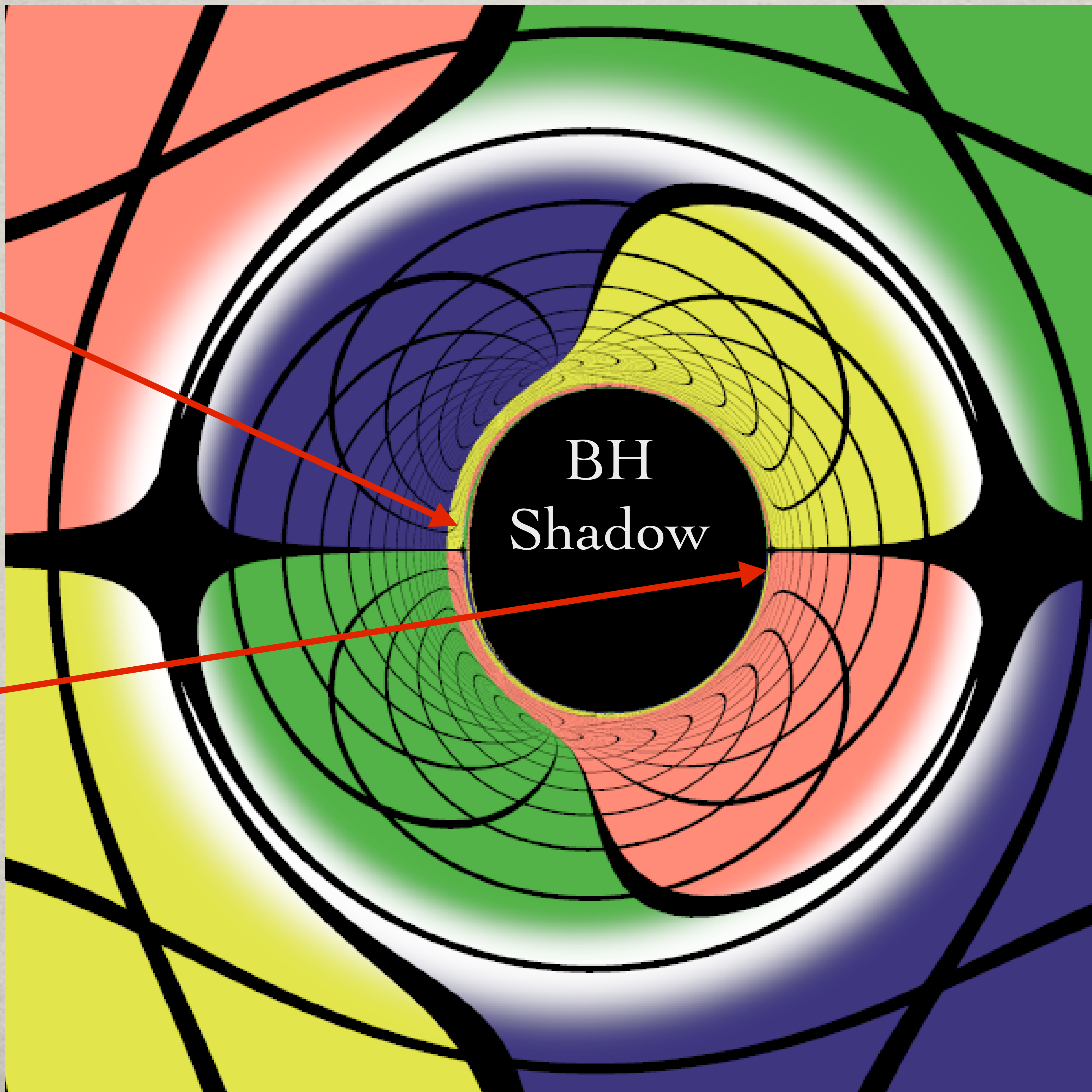
The strength of the frame dragging increases closer to the BH

Visualization from camera: Kerr (a=0.95M), spin perpendicular to the line of sight

A photon traveling in the direction of the frame dragging can orbit closer to the BH without being captured…

than a photon traveling opposite to the frame dragging…

This results in an offset, asymmetric shadow



How about a truly dynamical black hole binary?

A. Bohn et al. CQG32(2015)065002

Animation:

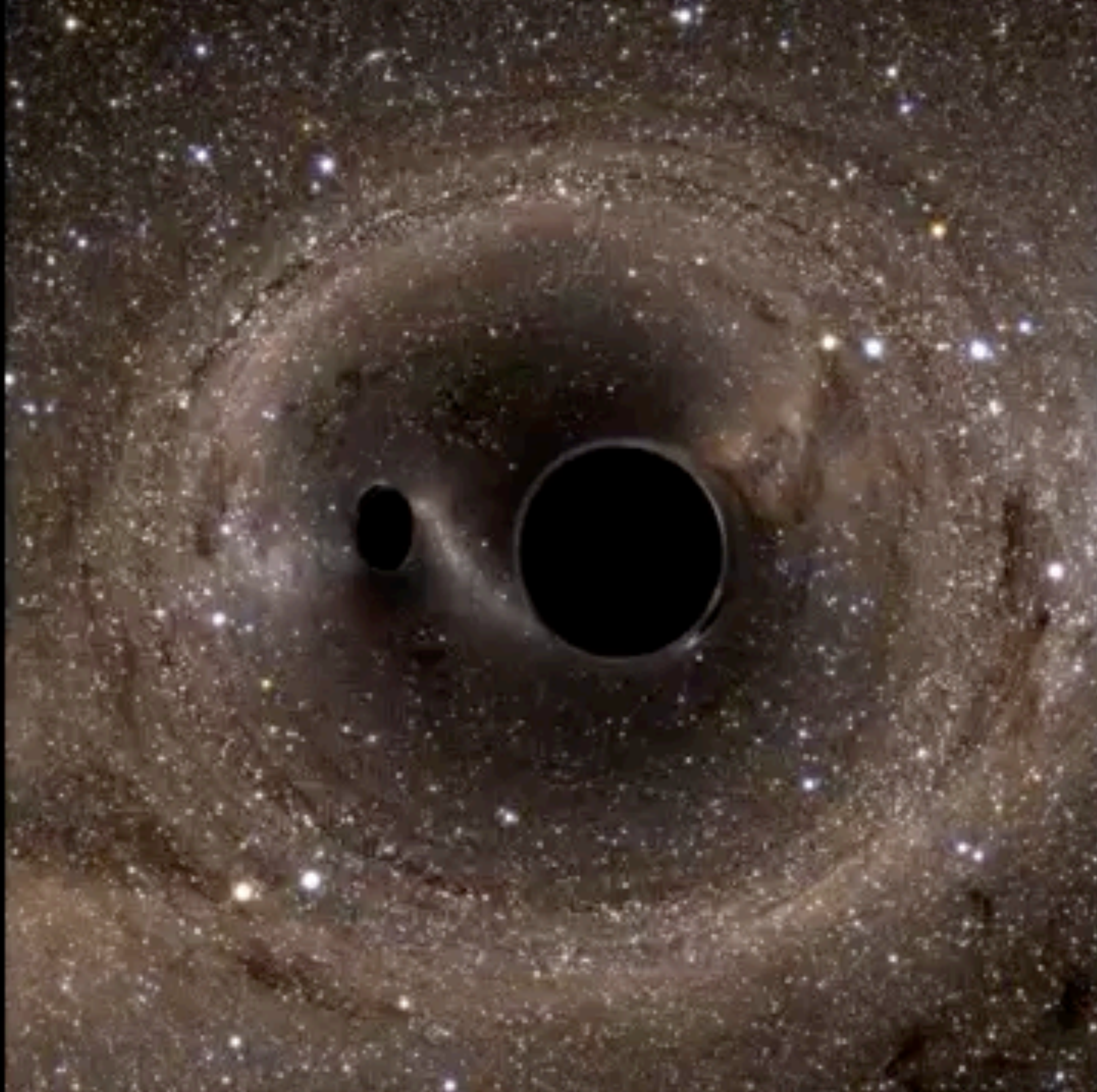
Last three orbits of a 3:1 mass ratio binary with arbitrarily chosen spins on both black holes.

The details of this merger can be found in *Taylor et al.* (Phys. Rev. D 88 (2013) 124010).

The stars used are the same as the ones used in the single black hole image shown before.

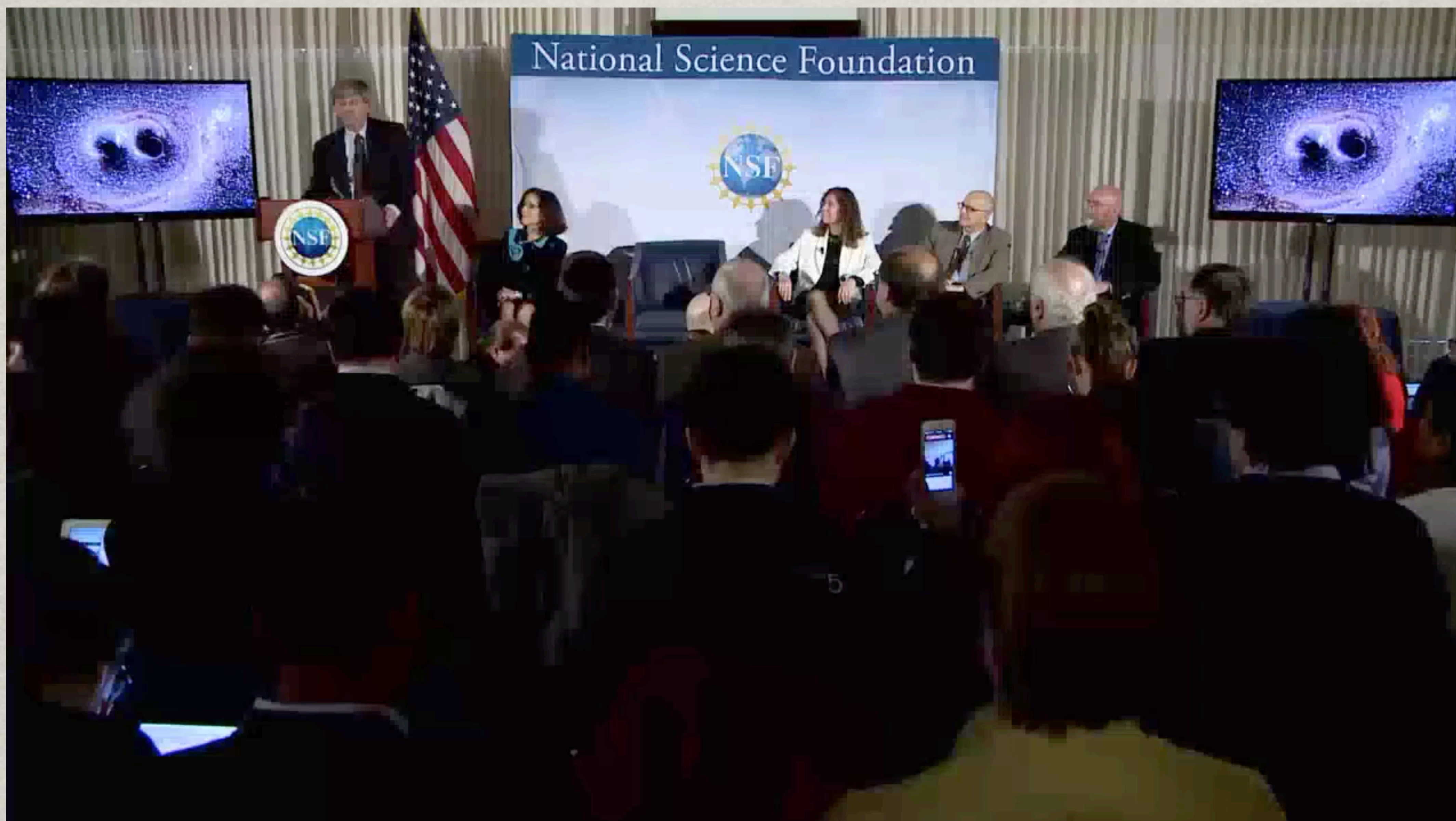
The camera is located above the orbital plane of the binary looking down.

Source:
http://www.black-holes.org/the-science-numerical-relativity/numerical-relativity/gravitational-lensing


SXS

11 February 2016

"Ladies and gentlemen, we have detected gravitational waves. We did it!"



David Reitze
Executive Director
Laser Interferometer Gravitational-Wave Observatory (LIGO)

Black holes and exotic compact objects

C. Herdeiro
Departamento de Matemática and CIDMA, Universidade de Aveiro, Portugal

Thank you for your attention!
Obrigado pela vossa atenção!

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Grav The Gravity group @ Aveiro University, Portugal
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