Black holes and exotic compact objects

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Compact Objects, Gravitational Waves & Deep Learning

University of Aveiro, 21-25 September, 2020
Plan of the lectures:

Lecture 1
Black holes: astrophysical evidence and a theory (brief) timeline

Lecture 2
Spherical black holes: the Schwarzschild solution

Lecture 3
Spinning black holes: the Kerr solution

Lecture 4
Exotic compact objects: the example of bosonic stars

Lecture 5
Non-Kerr black holes
The field of GW detection is theory driven, in the sense one needs theoretical templates to guide searches.

Efforts to produce templates in modified gravity and with ECOs are underway.

Here we shall focus on the case of bosonic stars.
Lecture 4

I) Spherical boson (scalar/vector) star solutions

II) Lensing by spherical bosonic stars

III) Rotating bosonic (scalar/vector) star solutions

IV) Lensing by rotating bosonic stars
Stationary scalar solitons in field theory

The asymptotically flat black holes we have been discussing so far have an event horizon which cloaks a curvature singularity. An interesting question in any gravitational model is whether stationary particle-like solutions exist, i.e. gravitating solitons: everywhere regular configurations, without horizons, corresponding to localized lumps of (time-independent) gravitational+matter field energy.

In field theory (without gravity) there is a long history and a large literature on finding soliton-type solutions and solitary waves, starting with the Korteweg-de-Vries equation D. J. Korteweg and G. de Vries, Philosophical Magazine 39 (1895) 422-443.
Stationary scalar solitons in field theory

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These solutions occur for non-linear field theories and the constancy of their “shape” is interpreted as a cancellation between non-linear and dispersive effects.

There is, however, a generic argument, known as Derrick’s theorem, against the existence of stable, time-independent solutions of finite energy in a wide class of non-linear wave equations, in three or higher (spatial ) dimensions G. H. Derrick, J. Math. Phys. 5 (1964) 1252 (see also R.H. Hobart, Proc. Phys. Soc. 82 (1963)201).
Stationary scalar solitons in field theory

Derrick’s theorem (original scaling argument)

Consider the (possibly) non-linear Klein-Gordon equation in Minkowski spacetime:

\[ \Box \Phi = \frac{1}{2} \frac{dV}{d\Phi} \]

This can be derived from the action (using a variational principle):

\[ S = \int d^4x \left[ -\partial_\mu \Phi \partial^\mu \Phi - V(\Phi) \right] = \int dt d^3r \left[ \left( \frac{\partial \Phi}{\partial t} \right)^2 - (\nabla \Phi)^2 - V(\Phi) \right] \]

We will prove that no stable, time-independent, localized solutions exist, for any potential energy. By “localized” we mean that the integrals:

\[ I_1 = \int d^3r (\nabla \Phi)^2 \quad I_2 = \int d^3r V(\Phi) \]

converge, when taken over the whole 3-space.

We assume the field depends only on the spatial coordinates. Then \( \delta S = 0 \iff \delta E = 0 \)

\[ E = \int d^3r \left[ (\nabla \Phi)^2 + V(\Phi) \right] \]
Stationary scalar solitons in field theory

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\[ E = \int d^3r \left[ (\nabla \Phi)^2 + V(\Phi) \right] \]

Assume the existence of a localized solution of: \( \delta E = 0 \) Let the solution be: \( \Phi(r) \)

The solution is stable if: \( \delta^2 E \geq 0 \)

Define a 1-parameter family of configurations: \( \Phi_\lambda(r) = \Phi(\lambda r) \quad \lambda = \text{arbitrary constant} \)

Let:

\[ E_\lambda = \int d^3r \left[ (\nabla \Phi_\lambda)^2 + V(\Phi_\lambda) \right] = \frac{I_1}{\lambda} + \frac{I_2}{\lambda^3} \]

It follows that:

\[ \left( \frac{dE_\lambda}{d\lambda} \right)_{\lambda=1} = -I_1 - 3I_2 \quad \left( \frac{d^2E_\lambda}{d\lambda^2} \right)_{\lambda=1} = 2I_1 + 12I_2 \]

Thus:

\[ \left( \frac{dE_\lambda}{d\lambda} \right)_{\lambda=1} = 0 \iff I_2 = -\frac{I_1}{3} \quad \text{and} \quad \left( \frac{d^2E_\lambda}{d\lambda^2} \right)_{\lambda=1} = -2I_1 < 0 \]

Then, \( \delta^2 E < 0 \) for a uniform “stretching” of the hypothetical soliton and it is unstable (q.e.d.).
Stationary scalar solitons in field theory

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Thus:

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Then, \( \delta^2E < 0 \) for a uniform “stretching” of the hypothetical soliton and it is unstable (q.e.d.).

Observe, moreover that if the potential is strictly positive (or zero if the field vanishes) there are no solutions at all, since

\[ I_1 + 3I_2 \geq 0 \]

and the equality only holds for the trivial solution.

**Exercise 4.1**

Does the argument apply in \( d=1,2 \) (spatial dims)?
Derrick observed that one way to circumvent the theorem would be to allow for localized solutions that are periodic in time, rather than time independent. Various authors, starting with Rosen, considered a complex field with a harmonic time dependence, which guarantees a time-independent energy momentum tensor:

\[ \Phi(t, \mathbf{r}) = e^{-i \omega t} \varphi(\mathbf{r}) \]

Moreover there is a global symmetry and a conserved scalar charge (typically called Q). Then, for some classes of potentials (yielding non-linear models), localized stable solutions exist, which are now known, following Coleman, as \textit{Q-balls}:


But in the presence of gravity, no scalar non-linear interactions are required. Effectively, such non-linearities are provided by the self-gravity of the field.
Gravitating scalar solitons: boson stars
Gravitating scalar solitons: boson stars

Discussion of particle-like solutions in Einstein’s gravity gained notoriety with the work of Wheeler, who searched for these configurations in electro-vacuum. He coined the term “Geon” as a contraction of “gravitational electromagnetic entity”, meaning an electromagnetic or gravitational wave which is held together in a confined region by the gravitational attraction of its own field energy J. A. Wheeler, Phys. Rev. 97 (1955) 511.

Although no explicit solutions were found in electro-vacuum a Klein-Gordon Geon was found in massive scalar-vacuum by Kaup D. J. Kaup, Phys. Rev. 172 (1968) 1331. Later these gravitating solitons became known as boson stars.
Gravitating scalar solitons: boson stars

The model (mini-boson stars):

\[ S = \int d^4 x \sqrt{-g} \left[ \frac{R}{16\pi} - \frac{1}{2} g^{\alpha\beta} \left( \Phi^*_\beta \Phi^\beta + \Phi^*_\beta \Phi^\beta \right) - \mu^2 \Phi^* \Phi \right], \]

The field equations:

\[ G_{\alpha\beta} = 8\pi \left\{ \Phi^*_\beta \Phi^\beta + \Phi^*_\beta \Phi^\beta - g_{\alpha\beta} \left[ \frac{1}{2} g^{\gamma\delta} (\Phi^*_\gamma \Phi^\delta + \Phi^*_\gamma \Phi^\delta) + \mu^2 \Phi^* \Phi \right] \right\} \]

\[ \Box \Phi = \mu^2 \Phi \]

The action is invariant under a U(1) global symmetry: \( \Phi \rightarrow e^{i\alpha} \Phi \)

This leads to a conserved current: \( j^\alpha = -i (\Phi^* \partial^\alpha \Phi - \Phi \partial^\alpha \Phi^*) \)

Integrating the temporal component of this 4-current on a timelike slice leads to a conserved charge - the Noether charge \( Q \):

\[ Q = \int \Sigma \ j^t \]

The Noether charge counts the number of scalar particles. Notice that this is conserved in the sense of a local continuity equation; there is no associated Gauss law!
Gravitating scalar solitons: boson stars

Spherically symmetric solutions ansatz (three unknown functions):

\[ ds^2 = -N(r)\sigma^2(r)dt^2 + \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) , \quad N(r) \equiv 1 - \frac{2m(r)}{r} , \quad \Phi = \phi(r)e^{-iwt} \]

The time dependence cancels at the level of the energy momentum tensor, being therefore compatible with a stationary metric. Thus \( k = \partial/\partial t \) is a Killing vector field, but it does not preserve the scalar field - the metric and the matter field do not share the same symmetries.

The above ansatz makes the Einstein equations simpler as compared to other choices (such as isotropic coordinates). The two “essential” Einstein equations read:

\[ m' = 4\pi r^2 \left( N\phi'^2 + \mu^2 \phi^2 + \frac{w^2 \phi^2}{N\sigma^2} \right) , \quad \sigma' = 8\pi \sigma r \left( \phi'^2 + \frac{w^2 \phi^2}{N^2\sigma^2} \right) \]

(one further constraint equation is found, but which is a differential consequence of these).

The Klein-Gordon equation gives (thus completing three equations):

\[ \phi'' + \frac{2\phi'}{r} + \frac{N'\phi'}{N} + \frac{\sigma'\phi'}{\sigma} - \frac{\mu^2\phi}{N} + \frac{w^2\phi}{N^2\sigma^2} = 0 \]

Exercise 4.2

Obtain these equations.
Gravitating scalar solitons: boson stars

These equations are solved numerically subject to the following boundary conditions:

\[ m(0) = 0, \quad \sigma(0) = s_0, \quad \phi(0) = b, \quad m(\infty) = M, \quad \sigma(\infty) = 1, \quad \phi(\infty) = 0 \]

To start integrating from the origin, one finds a power series solution of the field equations near the origin:

\[ m(r) = \frac{4\pi (\mu^2 s_0^2 + w^2)}{3s_0} b^2 r^3 + O(r^5), \quad \sigma(r) = s_0 + \frac{4\pi w^2 b^2}{s_0} r^2 + O(r^4), \quad \phi(r) = b + \frac{1}{6} \left( \mu^2 - \frac{w^2}{s_0^2} \right) b r^2 + O(r^4) \]

Then, one integrates the coupled ODEs by a standard Runge-Kutta solver, with input parameters \( b, w, s_0 \). The solution has 2-parameters (\( w, n \)); don’t define the solution uniquely.

A shooting strategy is implemented to assure that (at the level of machine precision), the scalar field value vanishes at some large \( r \). Then, one extracts the ADM mass \( M \) from the value of \( m \).

The shooting method is implemented in terms of the scalar field value at \( r=0 \), for fixed \( w, s_0 = 1 \). This is actually a 1-parameter shooting, with shooting parameter \( b \). At some large value of \( r \), \( \phi(r_{\text{max}}) = 0 \), to machine precision, but \( \sigma(r_{\text{max}}) = s_{\text{fin}} \neq 1 \). But there is a scaling symmetry of the field equations \( (\sigma \rightarrow \sigma/\lambda, \quad w \rightarrow w/\lambda) \) that maps the solution to another \( w \) and the \( \sigma(r_{\text{max}}) = 1 \).

One finds various (or no) solutions, corresponding to different ADM masses \( M \), for the same number of nodes and for a given frequency.

Solutions without/with nodes are the fundamental family/excited states.
Gravitating scalar solitons: boson stars

ADM mass $M$ (and Noether charge $Q$) vs. frequency $w$ diagram:

- Solutions only exist for a range of frequencies: $\frac{w_{\min}}{\mu} < \frac{w}{\mu} < 1$ \quad $w_{\min} \approx 0.767 \mu$

- There is a range of frequencies for which more than one solution exists. This defines the first, second, third, etc, branches.

- There is a maximum value for the ADM mass: $M_{\text{ADM}}^{\text{max}} \approx \alpha_{\text{BS}} \frac{M_{\text{Pl}}^2}{\mu} \approx \alpha_{\text{BS}} 10^{-19} M_\odot \left( \frac{\text{GeV}}{\mu} \right)$

\[ \alpha_{\text{BS}} = 0.633 \]
Gravitating scalar solitons: boson stars

ADM mass $M$ (and Noether charge $Q$) vs. frequency $w$ diagram:

- This spiral corresponds to nodeless solutions. These are regarded as fundamental modes. Excited solutions also exist.

- Along the spiral, $M$ becomes larger than $Q$. This signals instability, since there is excess, rather than binding, energy.

Simulations
Zilhão (2017)

$\mathcal{R}(\Phi)$

$\mathcal{I}(\Phi)$

$w=0.95$
first branch

$|\Phi|^2$
Scalar Boson Stars can form dynamically:

Initial data: a coherent complex scalar field: \[ \Phi = 0.0025 e^{-r^2/90^2}, \quad \dot{\Phi} = 0.9iw \]

Evolution:
\[ \text{time in units of } 1/\mu \]

Final state:
(perturbed) boson star with mass
\[ M_i \sim 0.56 \frac{M_{Pl}^2}{\mu} \]

\[ M_i \sim 0.644 \frac{M_{Pl}^2}{\mu} > 0.633 \frac{M_{Pl}^2}{\mu} \quad \text{(maximal mass for boson stars in this model; without cooling a BH would form)} \]

\[ \frac{M_i - M_f}{M_i} \sim 0.13 \]

Gravitational cooling:
ejected scalar field “radiation” carries excess kinetic energy
(analogous to violent relaxation in collisionless stellar systems)

FIG. 1. The evolution of \( r^2 \rho \) for a massive, self-gravitating complex scalar field is shown. Because of the self-gravitation, the field collapses quickly and a perturbed boson star is formed at the center. The star oscillates and begins to settle down as scalar material is radiated through the gravitational cooling process discussed in the text.
For **massless** field there is complete dispersion: collapse, rebounces and complete dispersion to infinity.

Similar results for a **real** (rather than complex) massive scalar field: an oscillation (rather than boson star) forms.

Result holds for generic initial data; main restriction spherical symmetry.
The vector cousin: spherical Proca stars


\[ S = \int d^4x \sqrt{-g} \left( \frac{1}{16\pi G} R - \frac{1}{4} \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta} - \frac{1}{2} \mu^2 A_\alpha \bar{A}^\alpha \right). \]

A similar construction holds yielding spherical solitonic objects: spherical Proca stars

Very similar domain of existence;
Energetic instability around minimal frequency;
Similar structure of fundamental family and excited states;
but in Proca case $A_0$ has at least one node.
Ultra-compact spherical Proca stars

Cunha, Font, Herdeiro, Radu, Sanchis-Gual, Zilhão, PRD 96 (2017) 104040

scalar

vector/Proca
Dynamics of spherical Proca stars

1) As in the scalar case, vector boson stars are perturbatively stable up to the maximal mass; then they share the same three possible fates: migration, collapse or dispersion

2) As in the scalar case, ultracompact vector boson stars are unstable against collapse
Cunha, Font, Herdeiro, Radu, Sanchis-Gual, Zilhão, PRD 96 (2017) 104040

3) As in the scalar case, vector boson stars can form dynamically via gravitational cooling
Di Giovanni, Sanchis-Gual, Herdeiro and Font, PRD 98 (2018) 064044

4) As in the scalar case, one can study binaries of spherical Proca stars and their gravitational wave emission

Stable model; apparent horizon forms at t~200
Lecture 4

I) Spherical boson (scalar/vector) star solutions

II) Lensing by spherical bosonic stars

III) Rotating bosonic (scalar/vector) star solutions

IV) Lensing by rotating bosonic stars
Lensing by spherically symmetric scalar boson stars. We assume only minimal interaction between light and the scalar field.

We take

\[ \mu = \text{constante} \]

so the solutions have varying mass \( M \)
We have performed ray tracing to compute lensing and shadows.  
Fix the observer’s distance.

The full celestial sphere  

The “camera” opening angle

Following A. Bohn et al. CQG32(2015)065002
Configuration 0 (no boson star)
Configuration 1, first branch, w=0.95

Lensing is visible; but no multiplication of images
An Einstein ring arises, corresponding to the lensing of the front point (F); two further images for a part of the celestial sphere.
Configuration 2, first branch, w=0.9

Region with two extra images

Lensing yields triple image
Configuration 3, first branch, $w=0.85$

Einstein rings (plus central point):

Central circle gets to maximal size in agreement with ADM mass
Configuration 4, first branch, $w=0.8$

Einstein rings (plus central point):
F F

Larger region is “duplicated”
Configuration 5, first branch, $w=0.7677$

Einstein rings (plus central point):

\[
\begin{array}{c}
F \\
F
\end{array}
\]
Configuration 6, second branch, w=0.8 (zoomed)

Full celestial sphere is “duplicated” just after the backbending; then, a new pair of Einstein rings appears.

Einstein rings (plus central point):
F B B F

Appearance of two inner Einstein rings means two extra images of the full celestial sphere and four extra images of a part of the celestial sphere. The two new Einstein rings are images of the point behind the observer (B).
Configuration 7, second branch, w=0.82

Einstein rings (plus central point):
F B B F
Configuration 8, second branch, $w=0.84$ (zoomed)

In between the latter pair of Einstein rings, new pairs of Einstein rings appear.

Einstein rings (plus central point):

F B F F B F

Two new Einstein rings appear, implying four extra images of the full celestial sphere and six extra images of a part of the celestial sphere.
Appearance of light ring - third branch
\[ w=0.842, M=0.382, Q=0.330, \Phi(0)=1.1375 \]

Lensing of a boson star beyond the light ring?
A self-similar structure?
Analysis of ultracompact spherical boson stars
Cunha, Font, Herdeiro, Radu, Sanchis-Gual, Zilhão, PRD 96 (2017) 104040

Configuration 2
w=0.8402, M=0.3767
Lensing and shadow of a Schwarzschild BH (left) and a comparable bosonic star (right, Proca star model), in similar observation conditions.

The Einstein ring has a similar dimension (white lensed region), but the strongly lensed region - shadow and near its edge for the BH vs. central rings for the star - is much smaller for the star.
All ultra compact bosonic stars in these models are perturbatively unstable and collapse into a Schwarzschild black hole in a short time scale.

Cunha, Font, Herdeiro, Radu, Sanchis-Gual, Zilhão, PRD 96 (2017) 104040

Scalar stars

Vector/Proca stars
Lecture 4

I) Spherical boson (scalar/vector) star solutions

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III) Rotating bosonic (scalar/vector) star solutions

IV) Lensing by rotating bosonic stars
Rotating boson stars


\[ ds^2 = -e^{2F_0(r,\theta)} dt^2 + e^{2F_1(r,\theta)} \left( dr^2 + r^2 d\theta^2 \right) + e^{2F_2(r,\theta)} r^2 \sin^2 \theta \left( d\varphi - W(r, \theta) dt \right)^2 \]
\[ \Phi = \phi(r, \theta) e^{i(m\varphi - wt)} \]

The solution has three parameters: \((w, m, n)\), but again these do not define solutions uniquely.

Solutions preserved by
a single helicoidal
Killing vector field:
\[ \frac{\partial}{\partial t} + \frac{w}{m} \frac{\partial}{\partial \varphi} \]

The Klein-Gordon plus Einstein equations yield now a system of five coupled PDEs (plus two “constraint” equations which are differential consequences of the others). To solve them:

- one performs an expansion of the unknown functions, both near the origin and asymptotically;
- the equations can be solved using a relaxation method (Newton-Raphson). For each fixed frequency \(w\) one can find various or no solutions, corresponding to different ADM masses \(M\).
Rotating boson stars

Klein Gordon equation:

\[ \phi_{,rr} + \frac{1}{r^2} \phi_{,\theta \theta} + \phi_{,r} (F_0, r + F_2, r) + \frac{1}{r^2} \phi, \theta (F_0, \theta + F_2, \theta) + \frac{2}{r} \phi_{,r} + \cot \frac{\theta}{r^2} \phi_{,\theta} \\
- \left( \frac{e^{-2F_2 m^2}}{r^2 \sin^2 \theta} - \frac{e^{-2F_0 (w - mW)^2} + \mu^2}{2} \right) e^{2F_1} \phi = 0 \]

The Einstein equations are combined to have second derivatives of a single function:

\[ F_{1,rr} + \frac{1}{r^2} F_{1,\theta \theta} - \left( F_{0,r} F_{2, r} + \frac{1}{r^2} F_{0, \theta} F_{2, \theta} \right) - \frac{e^{-2F_0 + 2F_2 r^2 \sin^2 \theta}}{4} \left( W_{r, r} + \frac{1}{r^2} W_{\theta} \right) - \frac{F_{0,r}}{r} \\
+ \frac{F_{1,r}}{r} - \frac{\cot \theta F_{0, \theta}}{r^2} + 8 \pi \left( \phi_{,r}^2 + \frac{1}{r^2} \phi_{,\theta}^2 + e^{2F_1} \left[ e^{-2F_0 (w - mW)^2} - \frac{e^{-2F_2 m^2}}{r^2 \sin^2 \theta} \right] \phi^2 \right) = 0 \]

\[ F_{2,rr} + \frac{1}{r^2} F_{2,\theta \theta} + F_{2, r} + \frac{1}{r^2} F_{2, \theta} + F_{0,r} F_{2, r} + \frac{1}{r^2} F_{0, \theta} F_{2, \theta} + \frac{e^{-2F_0 + 2F_2 r^2 \sin^2 \theta}}{2} \left( W_{r, r} + \frac{1}{r^2} W_{\theta} \right) \\
+ \frac{1}{r} \left( F_{0,r} + \frac{\cot \theta F_{0, \theta}}{r} \right) + \frac{3F_{2, r}}{r} + \frac{2 \cot \theta F_{2, \theta}}{r^2} + 8 \pi e^{2F_1} \left( \mu^2 + \frac{2e^{-2F_2 m^2}}{r^2 \sin^2 \theta} \right) \phi^2 = 0 \]

\[ F_{0,rr} + \frac{1}{r^2} F_{0,\theta \theta} + F_{0, r} + \frac{1}{r^2} F_{0, \theta} + F_{0,r} F_{2, r} + \frac{1}{r^2} F_{0, \theta} F_{2, \theta} - \frac{e^{-2F_0 + 2F_2 r^2 \sin^2 \theta}}{2} \left( W_{r, r} + \frac{1}{r^2} W_{\theta} \right) \\
+ \frac{2F_{0,r}}{r} + \frac{\cot \theta F_{0, \theta}}{r^2} + \frac{N' F_{2, r}}{2} - 8 \pi e^{2F_1} \left( 2e^{-2F_0 (w - mW)^2} - \mu^2 \right) \phi^2 = 0 \]

\[ W_{rr} + \frac{1}{r^2} W_{,\theta \theta} + (3F_{2, r} - F_{0, r}) W_{r} + \frac{1}{r^2} (3F_{2, \theta} - F_{0, \theta}) W_{\theta} \\
+ \frac{4}{r} \left( W_{r} + \frac{3 \cot \theta W_{, \theta}}{4r} \right) + 32 \pi \frac{e^{2F_1 - 2F_2 m (w - mW)} \phi^2}{r^2 \sin^2 \theta} = 0 \]
The maximum value for the ADM mass increases with $m$:

$$M_{\text{ADM}}^\text{max} \simeq \alpha_{\text{BS}} \frac{M_{\text{Pl}}^2}{\mu} \simeq \alpha_{\text{BS}} 10^{-19} M_\odot \left( \frac{\text{GeV}}{\mu} \right)$$

For $m=0$: $\alpha_{\text{BS}} = 0.633$

For $m=1$: $\alpha_{\text{BS}} = 1.315$

For $m=2$: $\alpha_{\text{BS}} = 2.216$


For rotating boson stars:


$$J = mQ$$

**Exercise 4.3**
Show this.
Rotating boson stars

ADM mass $M$ vs. frequency $w$ diagram and $M$ vs. $J$ diagram for $m=1,2,3$:

Rotating boson stars

Scalar field profile (left) and for a typical rotating boson star, \( m=1 \), first branch, \( w=0.85 \):
Rotating boson stars

Surfaces of constant scalar energy density:


Rotating boson stars are rotating “mass” tori in GR
How `compact' are these BSs?

BSs have no surface - the scalar field decays exponentially - thus there is no BS's `radius'. One estimate is the following:
1) use the `perimetal' radius, a radial coordinate $R$ such that a circumference along the equatorial plane has perimeter $2\pi R$.
2) compute $R_{99}$, the perimetal radius containing 99% of the BS mass, $M_{99}$.
3) define the inverse compactness by comparing $R_{99}$ with the Schwarzschild radius associated to 99% of the BS's mass, $R_{Schw}=2M_{99}$.

\[ \text{Compactness}^{-1} \equiv \frac{R_{99}}{2M_{99}} \]

With this measure, the inverse compactness is always greater than unity; in other words, BSs are less compact than BHs, as one would expect.
Simulations
Zilhão (2017)
Longer evolutions show the appearance of a non-axisymmetric instability
http://gravitation.web.ua.pt/node/1740
Instability may be associated to toroidal structure and is absent in cousin Proca model
http://gravitation.web.ua.pt/node/1740

Rotating boson stars
Brito, Cardoso, Herdeiro and Radu, PLB 752 (2016) 291
Herdeiro, Radu and Rúnarsson, CQG 33 (2016) 154001
Herdeiro, Perapechka, Radu and Shnir, PLB 797 (2019) 134845

Rotating Proca stars
Evolution of a perturbed spinning Proca star
http://gravitation.web.ua.pt/node/1740
Evolution of an excited spinning Proca star
http://gravitation.web.ua.pt/node/1740
The (ultra) light in the dark: A potential vector boson of $8.7 \times 10^{-13}$ eV from GW190521

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Advanced LIGO-Virgo reported a short gravitational-wave signal (GW190521) interpreted as a quasi-circular merger of black holes, one populating the pair-instability supernova gap, forming a remnant black hole of $M_f \sim 142M_\odot$ at a luminosity distance of $d_L \sim 5.3$ Gpc. With barely visible pre-merger emission, however, GW190521 merits further investigation of the pre-merger dynamics and even of the very nature of the colliding objects. We show that GW190521 is consistent with numerically simulated signals from head-on collisions of two (equal mass and spin) horizonless vector boson stars (aka Proca stars), forming a final black hole with $M_f = 231^{+13}_{-17} M_\odot$, located at a distance of $d_L = 571^{+348}_{-181}$ Mpc. The favoured mass for the ultra-light vector boson constituent of the Proca stars is $\mu_V = 8.72^{+0.73}_{-0.82} \times 10^{-13}$ eV. This provides the first demonstration of close degeneracy between these two theoretical models, for a real gravitational-wave event. Confirmation of the Proca star interpretation, which we find statistically slightly preferred, would provide the first evidence for a long sought dark matter particle.
FIG. 1. Time-series and spectrum of GW190521. Left: Whitened strain data of the LIGO Livingston detector at the time of GW190521, together with the best fitting waveforms for a head-on merger of two BHs (green), two equal/unequal mass PSs (red and blue) and for a quasi-circular BH merger (black). The time axis is expressed so that the GPS time is equal to \( t_{\text{GPS}} = t + 1242442965.6069 \) s.

Right: corresponding waveforms shown in the Fourier domain. Solid lines denote raw waveforms (scaled by a suitable, common factor) while dashed lines show the whitened versions. The vertical line denotes the 20 Hz limit, below which the detector noise increases dramatically. Due to this, a putative inspiral signal from a quasi-circular BBH merger (solid black) would be almost invisible to the detector (see dashed grey) and barely distinguishable from PHOC signals (dashed red and blue).

Scalar boson stars and their vector analogues, Proca stars [18, 19] (PSs), are self-gravitating stationary solutions of the Einstein-(complex, massive) Klein-Gordon [20] and of the Einstein-(complex) Proca [18] systems, respectively. These consist on complex bosonic fields oscillating at a well-defined frequency \( \omega \), which determines the mass and compactness of the star. Unlike other ECOs, bosonic stars can dynamically form without any fine-tuned condition through the gravitational cooling mechanism [21, 22]. While spinning solutions have been obtained for both scalar and vector bosons, the former are unstable against non-axisymmetric perturbations [23]. Hence, we will focus on the vector case in this work. For non-self-interacting bosonic fields, the maximum possible mass of the corresponding stars is determined by the boson particle mass \( \mu \). In particular, ultra-light bosons within \( 10^{13} \mu \lesssim 10^{10} \) eV, can form stars with maximal masses ranging between \( \sim 1000 \) and 1 solar masses, respectively.

We perform Bayesian parameter estimation and model selection on 4 seconds of publicly available data [24] from the two Advanced LIGO and Virgo detectors around the time of GW190521 (for full details, see the Parameter Estimation section within the Methods). We compare GW190521 to numerical simulations of HOCs, to simulations of equal-mass and equal-spin head-on PS mergers (PHOCs), and to the surrogate model for generically spinning BBH mergers NRSur7dq4 [25].

Our simulations include the gravitational-wave modes \( (\ell, m) = (2, 0), (2, \pm 2), (3, \pm 2) \) while the BBH model contains all modes with \( \ell \lesssim 4 \). The PHOC cases we consider form a Kerr BH with a feeble Proca remnant that does not impact on the GW emission [26]. Finally, to check the robustness of our results, we perform an exploratory study comparing GW190521 to a very limited family of simulations for unequal-mass (\( q_6 = 1 \)) head-on PS mergers.

Results. Figure 1 shows the whitened strain time series from the LIGO Livingston detector and the best fitting waveforms returned by our analyses for HOCs, PHOCs and BBH mergers. While the latter two show a similar morphology with slight pre-peak power, the HOC signal is noticeably shorter and has a slightly larger ringdown frequency. These features are more evident in the right panel, where we show the corresponding Fourier transforms (dashed) together the corresponding raw, non-whitened versions (solid). The HOC waveform displays a rapid power decrease at frequencies below its peak due to the absence of an inspiral. In contrast, PHOCs show a low-frequency tail due to the pre-collapse emission that mimics the typical inspiral signal present in the BBH case down to \( f ' \approx 20 \) Hz. Below this limit, the putative inspiral signal from a BBH disappears behind the detector noise (dashed grey) making the signal barely distinguishable from that of a PHOC.

Fig. 2 shows the two-dimensional 90% credible intervals for the redshifted final mass and the final spin obtained by the LVC using BBH models covering inspiral, merger and ringdown (IMR, in black) and solely from the ringdown emission; starting at the signal peak (grey).
Accordingly, the BBH scenario yields considerably more evidence for the PHOC model. In Table I we report the Bayesian evidence for both models, in terms of the final spin and redshifted mass. We observe that the PHOC model is favored over the BBH one if an uniform distribution of distance is considered. However, the inclusion of a source-frame final mass of $8.0 \pm 0.5 M_{\odot}$ obtained by PHOC further strengthens this evidence, slightly favouring the PHOC model.

In Fig. 3 we show the posterior distribution for the values of the parameters $\mu_{V}$ and $\omega/\mu_{V}$, for the $q = 1$ and $q \neq 1$ models. We can see that the PHOC model favors lower values of $\mu_{V}$ and $\omega/\mu_{V}$ compared to the BBH model. The maximal boson star mass is $173_{-14}^{+19} M_{\odot}$ for the PHOC model, and $175_{-11}^{+13} M_{\odot}$ for the BBH model, indicating that the PHOC model can explain more compact objects.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$q = 1$ model</th>
<th>$q \neq 1$ model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primary mass</td>
<td>$115_{-8}^{+7} M_{\odot}$</td>
<td>$115_{-8}^{+7} M_{\odot}$</td>
</tr>
<tr>
<td>Secondary mass</td>
<td>$115_{-8}^{+7} M_{\odot}$</td>
<td>$111_{-15}^{+7} M_{\odot}$</td>
</tr>
<tr>
<td>Total / Final mass</td>
<td>$231_{-17}^{+13} M_{\odot}$</td>
<td>$228_{-15}^{+17} M_{\odot}$</td>
</tr>
<tr>
<td>Final spin</td>
<td>$0.75_{-0.04}^{+0.08}$</td>
<td>$0.75_{-0.04}^{+0.08}$</td>
</tr>
<tr>
<td>Inclination $\pi/2 -</td>
<td>\iota - \pi/2</td>
<td>$</td>
</tr>
<tr>
<td>Azimuth</td>
<td>$0.65_{-0.54}^{+0.86}$ rad</td>
<td>$0.78_{-1.20}^{+1.23}$ rad</td>
</tr>
<tr>
<td>Luminosity distance</td>
<td>$571_{-181}^{+348}$ Mpc</td>
<td>$700_{-279}^{+292}$ Mpc</td>
</tr>
<tr>
<td>Redshift</td>
<td>$0.12_{-0.04}^{+0.05}$</td>
<td>$0.14_{-0.05}^{+0.06}$</td>
</tr>
<tr>
<td>Total / Final redshifted mass</td>
<td>$258_{-9}^{+9} M_{\odot}$</td>
<td>$261_{-11}^{+10} M_{\odot}$</td>
</tr>
<tr>
<td>Bosonic field frequency $\omega/\mu_{V}$</td>
<td>$0.893_{-0.015}^{+0.015}$</td>
<td>($*)0.905_{-0.012}^{+0.015}$</td>
</tr>
<tr>
<td>Boson mass $\mu_{V}$ [×10^{-13}]</td>
<td>$8.72_{-0.82}^{+0.73}$ eV</td>
<td>$8.59_{-0.57}^{+0.58}$ eV</td>
</tr>
<tr>
<td>Maximal boson star mass</td>
<td>$173_{-14}^{+19} M_{\odot}$</td>
<td>$175_{-11}^{+13} M_{\odot}$</td>
</tr>
</tbody>
</table>
Lecture 4

I) Spherical boson (scalar/vector) star solutions

II) Lensing by spherical bosonic stars

III) Rotating bosonic (scalar/vector) star solutions

IV) Lensing by rotating bosonic stars
Configuration 0 (no boson star)

Again take

\[ \mu = \text{constant} \]

so the solutions have varying mass \( M \)
Rotating boson stars

Asymmetric lensing

Non-compact RBS
Rotating boson stars

Einstein ring appears at:

\[ w_{ER1}^{(b1)} \approx 0.92 \]

Compact RBS
Rotating boson stars

Larger region is “duplicated”

Compact RBS
Rotating boson stars

More Einstein rings but squashed D-shaped (not O-shaped)

Compact RBS
Rotating boson stars

(Presumably)
Infinitely many copies of the celestial sphere

Ultra-compact RBS
Rotating boson stars

Ulra-compact RBS
Ergo-region appears between configuration 10 and 11
Rotating boson stars

Clear chaotic behaviour

Ultra-compact RBS
There are images with a clear chaotic behaviour in some regions, e.g.:

The emergence of stable light rings turns out to allow the formation of pockets in the effective potentials. This induces chaotic scattering.

Cunha, Grover, Herdeiro, Radu, Runarsson and Wittig, PRD 94 (2016) 104023
Black holes and exotic compact objects

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Thank you for your attention!
Obrigado pela vossa atenção!

University of Aveiro, 21-25 September, 2020