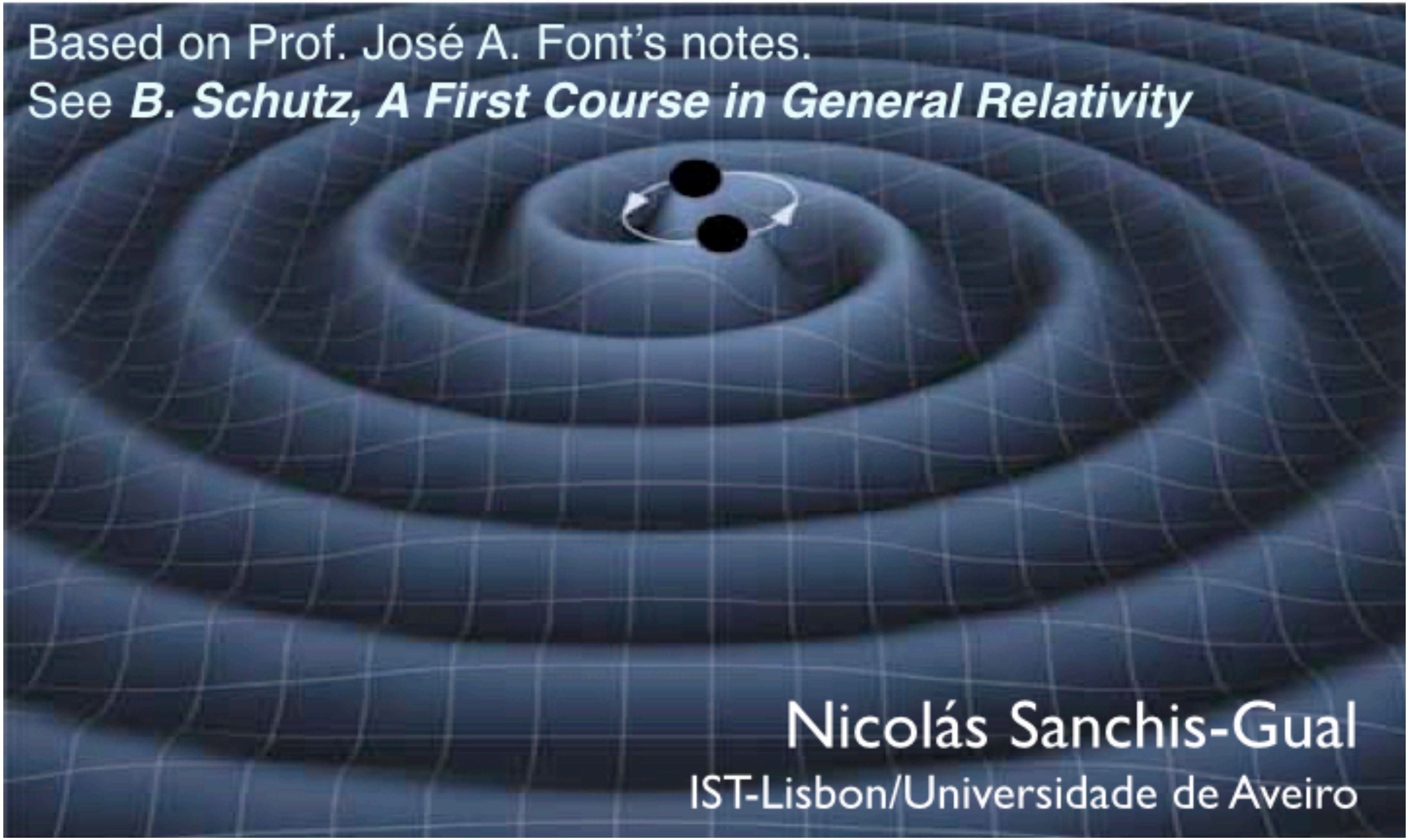
Introduction to gravitational waves: from theory to numerical simulations (2)



Linearized Einstein equations

- Physical principles: must relate the distribution of energy and momentum with the spacetime curvature, must be covariant equations valid in any coordinate system, must reduce to Newton's equations for weak gravitational fields and low velocities, and must satisfy the energy and momentum conservation law.

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

- The Einstein tensor is the only second-order tensor field that can be constructed with the components of the metric and their partial derivatives up to the second order.

$$\nabla_\nu G^{\mu\nu} = 0, \forall g_{\mu\nu}, \nabla_\nu T^{\mu\nu} = 0$$

- We want to obtain the equations of the gravitational field in the curved space-time framework. These equations should give us the geometry of space-time, that is, the metric.

- In Newton's theory the gravitational field is represented by a single function, the gravitational potential Φ. The equation that allows us to find the gravitational potential generated by a certain distribution of matter is the Poisson equation on

$$\nabla^2\Phi = 4\pi G\rho$$

- Since the source is a tensor, we look for a tensor equation in which the left-hand side depends on the metric and its derivatives up to the second order. We will then have the following equation

$$\hat{G} = \kappa \hat{T}$$

$$T^{\alpha\beta}{}_{;\beta} = \nabla_\beta T^{\alpha\beta} = 0$$

- We then look for a tensor $\hat{G}$ such that its divergence is zero.

$$\hat{G} = \hat{R}_{icc} - \frac{1}{2} R \hat{g} + \Lambda \hat{g}$$

- Einstein equations constitute a system of 10 second-order partial differential equations, coupled and nonlinear. Solving Einstein's equations consists in finding the elements of the metric as a function of their coordinates.

- Einstein's equations admit an evolution formalism in which arises a well-posed initial value problem, or Cauchy problem. This means that given some initial and boundary conditions, we can evolve in time the metric (and its derivatives). The initial instant of time allows to define a 3-spatial surface Σ_0 characterized by having a normal time vector and some coordinates(x_0, x_i), such that $x_0 = t = 0$ and x_i are the spatial coordinates over Σ_0. In this way, a family of 3-spatial surfaces (or hypersurfaces) will be represented by

$$\Sigma_\lambda \equiv \{(x^0, x^i)/x^0 = \lambda \in \mathbb{R}\}$$

- Einstein's equations can be seen as:

- $G_{ij} = \kappa T_{ij}$: 6 evolution equations containing time derivatives.

- $G_{0\alpha} = \kappa T_{0\alpha}$: 4 constraint equations on each hypersurface Σ. The implication of these equations is that the initial conditions cannot be arbitrary.

- The evolution of the stress-energy tensor - the right hand side of Einstein's equations - is obtained using the conservation equation

$$\nabla \cdot \hat{T} = 0 \implies \nabla_\beta T^{\alpha\beta} = 0.$$

$$\hat{T} = (\rho + p)\frac{\vec{u}}{c} \otimes \frac{\vec{u}}{c} + p\hat{g} \qquad p = p(\rho_0, \varepsilon)$$

Weak field approximation. Linearized Einstein's equations

- The gravitational field outside a spherical object or an object with slow rotation like the Earth or the Sun, can be approximated by the Schwarzschild metric:

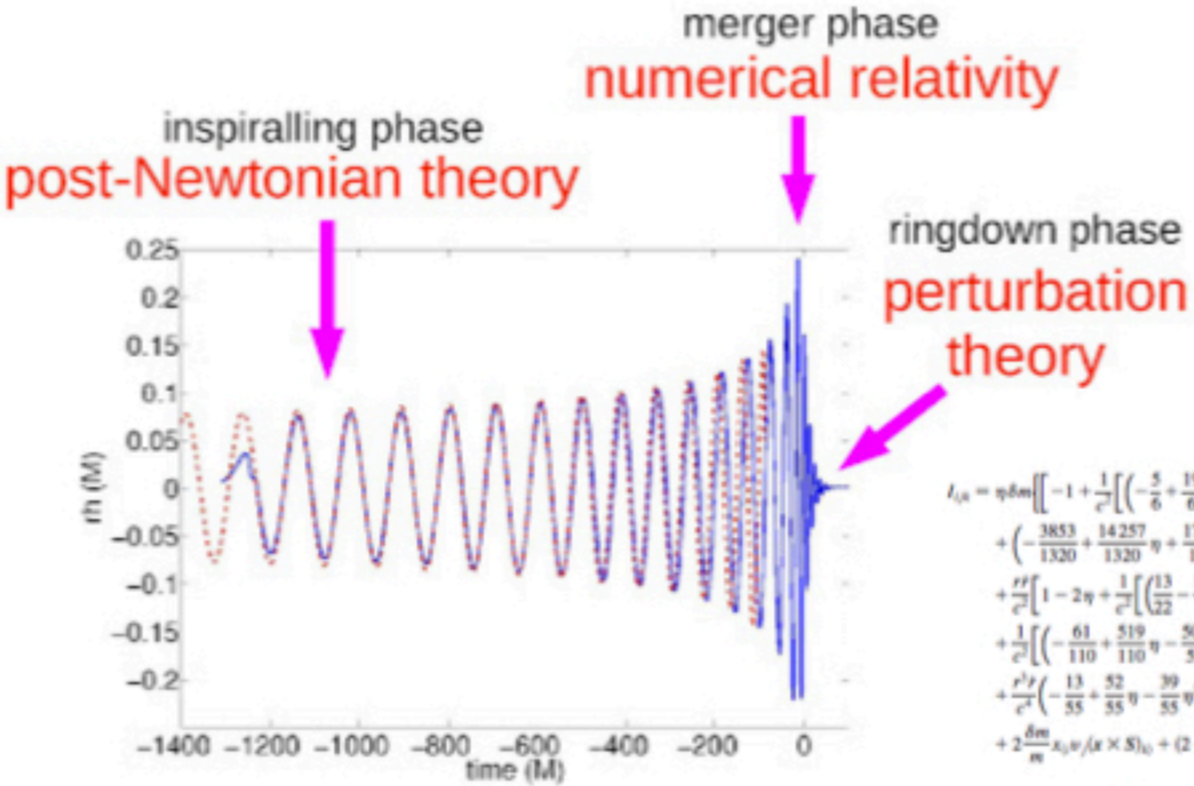
$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,.$$

- For such weak gravitational fields, we can use approximations that greatly simplify Einstein's equations:

- Vacuum, without including terms like $\frac{M}{R}$.

- Newtonian gravity, where in the dynamics of the system only appear terms like $\frac{M}{R}$.

- Post-Newtonian gravity of order n, where in the dynamics of the system only appear terms like $(\frac{M}{R})^n$.



$$
\begin{aligned}
I_{ijk} = {}& \eta\,\delta m \Big\{\Big[-1 + \frac{1}{c^2}\Big[\Big(-\frac{5}{6}+\frac{19}{6}\eta\Big)v^2 + \Big(\frac{5}{6}-\frac{13}{6}\eta\Big)\frac{m}{r}\Big] + \frac{1}{c^4}\Big[\Big(-\frac{257}{440}+\frac{7319}{1320}\eta-\frac{5501}{440}\eta^2\Big)v^4 \\
& + \Big(-\frac{3853}{1320}+\frac{14257}{1320}\eta+\frac{17371}{1320}\eta^2\Big)\frac{m}{r}v^2 + \Big(\frac{247}{1320}-\frac{531}{440}\eta+\frac{1347}{440}\eta^2\Big)\frac{m}{r}\dot r^2 + \Big(\frac{47}{33}+\frac{1591}{132}\eta-\frac{235}{66}\eta^2\Big)\frac{m^2}{r^2}\Big]\Big] x_{\langle ijk\rangle} \\
& + \frac{r\dot r}{c^2}\Big[1-2\eta+\frac{1}{c^2}\Big[\Big(\frac{13}{22}-\frac{107}{22}\eta+\frac{102}{11}\eta^2\Big)v^2+\Big(\frac{2461}{660}-\frac{8689}{660}\eta-\frac{1389}{220}\eta^2\Big)\frac{m}{r}\Big]\Big] x_{\langle ij}v_{k\rangle} + \frac{r^2}{c^2}\Big[-1+2\eta \\
& + \frac{1}{c^2}\Big[\Big(-\frac{61}{110}+\frac{519}{110}\eta-\frac{504}{55}\eta^2\Big)v^2+\Big(\frac{1}{11}-\frac{4}{11}\eta+\frac{3}{11}\eta^2\Big)\dot r^2+\Big(-\frac{1949}{330}-\frac{62}{165}\eta+\frac{483}{55}\eta^2\Big)\frac{m}{r}\Big]\Big] x_{\langle i}v_{jk\rangle} \\
& + \frac{r^3\dot r}{c^4}\Big(-\frac{13}{55}+\frac{52}{55}\eta-\frac{39}{55}\eta^2\Big)v_{\langle ijk\rangle}\Big\} + \frac{3\eta}{2}\frac{1}{c^3}\Big\{-3\frac{\delta m}{m}x_{\langle ij}(\mathbf{v}\times\mathbf{S})_{k\rangle} + (-3+11\eta)x_{\langle ij}(\mathbf{v}\times\boldsymbol{\Delta})_{k\rangle} \\
& + 2\frac{\delta m}{m}x_{\langle i}v_j(\mathbf{x}\times\mathbf{S})_{k\rangle} + (2-6\eta)x_{\langle i}v_j(\mathbf{x}\times\boldsymbol{\Delta})_{k\rangle}\Big\}.
\end{aligned}
\tag{A2}
$$

$$
\begin{aligned}
I_{ijkl} = {}& \eta m\Big\{\Big[1-3\eta+\frac{1}{c^2}\Big[\Big(\frac{103}{110}-\frac{147}{22}\eta+\frac{279}{22}\eta^2\Big)v^2+\Big(-\frac{10}{11}+\frac{61}{11}\eta-\frac{105}{11}\eta^2\Big)\frac{m}{r}\Big]\Big]x_{\langle ijkl\rangle} \\
& - \frac{72}{55}\frac{r\dot r}{c^2}(1-5\eta+5\eta^2)v_{\langle i}x_{jkl\rangle} + \frac{78}{55}\frac{r^2}{c^2}(1-5\eta+5\eta^2)v_{\langle ij}x_{kl\rangle}\Big\}.
\end{aligned}
\tag{A3}
$$

$$
I_{ijklm} = -\eta\,\delta m(1-2\eta)x_{\langle ijklm\rangle}
\tag{A4}
$$

$$
\begin{aligned}
J_{ij} = {}& \eta\,\delta m\Big\{\Big[-1+\frac{1}{c^2}\Big[\Big(-\frac{13}{28}+\frac{17}{7}\eta\Big)v^2+\Big(-\frac{27}{14}-\frac{15}{7}\eta\Big)\frac{m}{r}\Big]+\frac{1}{c^4}\Big[\Big(-\frac{29}{84}+\frac{11}{3}\eta-\frac{505}{56}\eta^2\Big)v^4 \\
& +\Big(-\frac{671}{252}+\frac{1297}{126}\eta+\frac{121}{12}\eta^2\Big)\frac{m}{r}v^2+\Big(\frac{5}{252}+\frac{241}{252}\eta+\frac{335}{84}\eta^2\Big)\frac{m}{r}\dot r^2+\Big(\frac{43}{252}+\frac{1543}{126}\eta-\frac{293}{84}\eta\Big)\frac{m^2}{r^2}\Big]\Big](\mathbf{x}\times\mathbf{v})_{\langle i}x_{j\rangle} \\
& +\frac{r\dot r}{c^2}\Big[-\frac{5}{28}+\frac{5}{14}\eta+\frac{1}{c^2}\Big[\Big(-\frac{25}{168}+\frac{25}{24}\eta-\frac{25}{14}\eta^2\Big)v^2+\Big(-\frac{103}{63}-\frac{337}{126}\eta+\frac{173}{84}\eta^2\Big)\frac{m}{r}\Big]\Big](\mathbf{x}\times\mathbf{v})_{\langle i}v_{j\rangle}\Big\} \\
& +\frac{\eta}{c}\Big\{-\frac{3}{2}x_{\langle i}\Delta_{j\rangle}+\frac{1}{c^2}\Big[r\dot r\Big(\frac{3}{7}-\frac{16}{7}\eta\Big)v_{\langle i}\Delta_{j\rangle}+r\dot r\frac{3}{7}\frac{\delta m}{m}v_{\langle i}S_{j\rangle}+\Big[\Big(\frac{27}{14}-\frac{109}{14}\eta\Big)(\mathbf{v}\cdot\boldsymbol{\Delta})+\frac{27}{14}\frac{\delta m}{m}(\mathbf{v}\cdot\mathbf{S})\Big]x_{\langle i}v_{j\rangle} \\
& +\Big[\Big(-\frac{11}{14}+\frac{47}{14}\eta\Big)(\mathbf{x}\cdot\boldsymbol{\Delta})-\frac{11}{14}\frac{\delta m}{m}(\mathbf{x}\cdot\mathbf{S})\Big]v_{\langle ij\rangle}+\Big[\Big(-\frac{29}{28}+\frac{143}{28}\eta\Big)v^2+\Big(\frac{19}{28}+\frac{13}{28}\eta\Big)\frac{m}{r}\Big]x_{\langle i}\Delta_{j\rangle} \\
& +\frac{1}{r^2}\frac{m}{r}\Big[\Big(-\frac{4}{7}+\frac{31}{14}\eta\Big)(\mathbf{x}\cdot\boldsymbol{\Delta})-\frac{29}{14}\frac{\delta m}{m}(\mathbf{x}\cdot\mathbf{S})\Big]x_{\langle ij\rangle}+\frac{\delta m}{m}\Big[-\frac{2}{7}v^2-\frac{1}{14}\frac{m}{r}\Big]x_{\langle i}S_{j\rangle}\Big]\Big\}
\end{aligned}
\tag{A5}
$$

Racine, E., Buonanno, A., & Kidder, L. (2009). Recoil velocity at second post-Newtonian order for spinning black hole binaries. *Physical Review D, 80*(4), 044010.

- In a weak gravitational field, we can linearize the field equations, making the hypothesis that the metric of space-time is a small perturbation of the Minkowski metric, choosing a system of coordinates in which the metric is written in the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

- where $\eta_{\alpha\beta}$ is the Minskowski metric and $h_{\alpha\beta}$ is a small correction. We will have then

$$|h_{\alpha\beta}| \ll 1,$$

- Next, we are going to obtain Einstein's equations for this metric, remaining in linear order in h.

The Christoffel symbols:

$$
\begin{aligned}
\Gamma^{\alpha}_{\beta\gamma} &= \frac{1}{2} g^{\alpha\mu}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}), \\
&= \frac{1}{2} \eta^{\alpha\mu}(h_{\mu\beta,\gamma} + h_{\mu\gamma,\beta} - h_{\beta\gamma,\mu}).
\end{aligned}
$$

The Riemann tensor:

$$
R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\gamma,\delta} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\delta} = \Gamma^{\alpha}_{\beta\gamma,\delta} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \mathcal{O}(h^2).
$$

The Ricci tensor:

$$
\begin{aligned}
R_{\beta\delta} &= R^{\alpha}{}_{\beta\alpha\delta} = \Gamma^{\alpha}_{\beta\delta,\alpha} - \Gamma^{\alpha}_{\beta\alpha,\delta} \\
&= \frac{1}{2}\eta^{\alpha\mu}(h_{\mu\beta,\delta\alpha} + h_{\delta\mu,\beta\alpha} - h_{\beta\delta,\mu\alpha}) - \frac{1}{2}\eta^{\alpha\mu}(h_{\mu\beta,\alpha\delta} + h_{\alpha\mu,\beta\delta} - h_{\beta\alpha,\mu\delta}) \\
&= \frac{1}{2}\eta^{\alpha\mu}(h_{\delta\mu,\beta\alpha} - h_{\beta\delta,\mu\alpha} - h_{\alpha\mu,\beta\delta} + h_{\beta\alpha,\mu\delta}) + \mathcal{O}(h^2).
\end{aligned}
$$

The Ricci scalar:

$$
\begin{aligned}
R &= \eta^{\beta\delta}R_{\beta\delta} = \frac{1}{2}\eta^{\beta\delta}\eta^{\alpha\mu}h_{\delta\mu,\beta\alpha} - \frac{1}{2}\eta^{\beta\delta}\eta^{\alpha\mu}h_{\beta\delta,\mu\alpha} - \frac{1}{2}\eta^{\beta\delta}\eta^{\alpha\mu}h_{\alpha\mu,\beta\delta} + \frac{1}{2}\eta^{\beta\delta}\eta^{\alpha\mu}h_{\beta\alpha,\mu\delta} \\
&= \frac{1}{2}h^{\beta\alpha}{}_{,\beta\alpha} - \frac{1}{2}h^{\delta}{}_{\delta,\mu}{}^{,\mu} - \frac{1}{2}h^{\mu}{}_{\mu,\delta}{}^{,\delta} + \frac{1}{2}h^{\delta\mu}{}_{,\delta\mu} \\
&= h^{\mu\nu}{}_{,\mu\nu} - h^{\mu}{}_{\mu,\nu}{}^{,\nu} + \mathcal{O}(h^2).
\end{aligned}
$$

The Einstein tensor:

$$
\begin{aligned}
G_{\alpha\beta} &= R_{\alpha\beta} - \frac{1}{2}R\eta_{\alpha\beta} = \frac{1}{2}\eta^{\mu\nu}(h_{\beta\nu,\alpha\mu} - h_{\alpha\beta,\nu\mu} - h_{\mu\nu,\alpha\beta} + h_{\alpha\mu,\nu\beta}) - \frac{1}{2}(h^{\mu\nu}{}_{,\mu\nu} - h^{\mu}{}_{,\mu\nu}{}^{,\nu})\eta_{\alpha\beta} \\
&= -\frac{1}{2}(h_{\alpha\beta,\mu}{}^{,\mu} + h^{\mu}{}_{\mu,\alpha\beta} - h_{\beta}{}^{\mu}{}_{,\alpha\mu} - h_{\alpha}{}^{\mu}{}_{,\beta\mu} + h^{\mu\nu}{}_{,\mu\nu}\eta_{\alpha\beta} - h^{\mu}{}_{\mu,\nu}{}^{,\nu}\eta_{\alpha\beta}) + \mathcal{O}(h^2).
\end{aligned}
$$

- The Einstein equations are:

$$-\frac{1}{2}(h_{\alpha\beta,\mu}^{\ \ \ ,\mu} + h^{\mu}{}_{\mu,\alpha\beta} - h_{\beta}{}^{\mu}{}_{,\alpha\mu} - h_{\alpha}{}^{\mu}{}_{,\beta\mu} + h^{\mu\nu}{}_{,\mu\nu}\eta_{\alpha\beta} - h^{\mu}{}_{\mu,\nu}{}^{,\nu}\eta_{\alpha\beta}) = \kappa T_{\alpha\beta}.$$

- We can simplify this equation defining:

$$\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta}$$

- Where $h = h^{\mu}_{\mu}$. Moreover

$$\begin{aligned}\bar{h} &= \bar{h}^{\alpha}{}_{\alpha} = \eta^{\alpha\beta}\bar{h}_{\alpha\beta} = \eta^{\alpha\beta}\left[h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta}\right] = h^{\alpha}{}_{\alpha} - \frac{1}{2}h\eta^{\alpha\beta}\eta_{\alpha\beta} \\ &= h - \frac{1}{2}4h = h - 2h = -h \implies \bar{h} = -h\end{aligned}$$

- Therefore $$h_{\alpha\beta} = \bar{h}_{\alpha\beta} + \frac{1}{2}h\,\eta_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2}\bar{h}\,\eta_{\alpha\beta}$$

- With these definitions, we can write the different terms that appear in the left-hand side

$$
\begin{aligned}
h_{\alpha\beta,\mu}{}^{,\mu} &= \bar{h}_{\alpha\beta,\mu}{}^{,\mu} - \frac{1}{2}\bar{h}_{,\mu}{}^{,\mu}\eta_{\alpha\beta}, \\
h^{\mu}{}_{\mu,\alpha\beta} &= -\bar{h}_{,\alpha\beta}, \\
h_{\beta}{}^{\mu}{}_{,\alpha\mu} &= \bar{h}_{\beta}{}^{\mu}{}_{,\alpha\mu} - \frac{1}{2}\bar{h}_{,\alpha\mu}\eta_{\beta}{}^{\mu} = \bar{h}_{\beta}{}^{\mu}{}_{,\alpha\mu} - \frac{1}{2}\bar{h}_{,\alpha\beta}, \\
h_{\alpha}{}^{\mu}{}_{,\beta\mu} &= \bar{h}_{\alpha}{}^{\mu}{}_{,\beta\mu} - \frac{1}{2}\bar{h}_{,\beta\alpha}, \\
h^{\mu\nu}{}_{,\mu\nu} &= \bar{h}^{\mu\nu}{}_{,\mu\nu} - \frac{1}{2}\bar{h}_{,\mu\nu}\,\eta^{\mu\nu} = \bar{h}^{\mu\nu}{}_{,\mu\nu} - \frac{1}{2}\bar{h}_{,\mu}{}^{,\mu}, \\
h^{\mu}{}_{\mu,\nu}{}^{,\nu} &= h_{,\nu}{}^{,\nu} = \bar{h}_{,\nu}{}^{,\nu}.
\end{aligned}
$$

- We have

$$
\begin{aligned}
- &\frac{1}{2}\Big(\bar{h}_{\alpha\beta,\mu}{}^{,\mu} - \frac{1}{2}\bar{h}_{,\mu}{}^{,\mu}\,\eta_{\alpha\beta} - \bar{h}_{,\alpha\beta} - \bar{h}_{\beta}{}^{\mu}{}_{,\alpha\mu} + \frac{1}{2}\bar{h}_{,\alpha\beta} - \bar{h}_{\alpha}{}^{\mu}{}_{,\beta\mu} + \frac{1}{2}\bar{h}_{,\beta\alpha} \\
+ &\bar{h}^{\mu\nu}{}_{,\mu\nu}\,\eta_{\alpha\beta} - \frac{1}{2}\bar{h}_{,\mu}{}^{,\mu}\,\eta_{\alpha\beta} + \bar{h}_{,\nu}{}^{,\nu}\,\eta_{\alpha\beta}\Big) \\
= &-\frac{1}{2}\big(\bar{h}_{\alpha\beta,\mu}{}^{,\mu} - \bar{h}_{\beta}{}^{\mu}{}_{,\alpha\mu} - \bar{h}_{\alpha}{}^{\mu}{}_{,\beta\mu} + \bar{h}^{\mu\nu}{}_{,\mu\nu}\,\eta_{\alpha\beta}\big).
\end{aligned}
$$

- The Einstein's equations in linear order in h are written

$$-\frac{1}{2}\left(\bar{h}_{\alpha\beta,\mu}{}^{,\mu} - \bar{h}_{\beta}{}^{\mu}{}_{,\alpha\mu} - \bar{h}_{\alpha}{}^{\mu}{}_{,\beta\mu} + \bar{h}^{\mu\nu}{}_{,\mu\nu}\,\eta_{\alpha\beta}\right) = \kappa T_{\alpha\beta}.$$

- This last expression can be considerably simplified if we carry out a coordinate transformation compatible with the linearized approximation, that is, a transformation that does not change equations

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \rightarrow g_{\alpha\beta}^{\mathrm{new}} = \eta_{\alpha\beta} + h_{\alpha\beta}^{\mathrm{new}}$$

- Gauge transformation.

- Coordinate change:

$$x^\alpha \to \bar{x}^\alpha = x^\alpha + \xi^\alpha(x),$$

- where ξ^α represents a very small (infinitesimal) change in the x^α coordinates, in the sense that $|\xi^\alpha_\beta| \ll 1$. Under this transformation, the metric is transformed as follows

$$\begin{aligned} g_{\alpha\beta} &= \bar{g}_{\mu\nu}\frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial \bar{x}^\nu}{\partial x^\beta} = \bar{g}_{\mu\nu}(\delta^\mu{}_\alpha + \xi^\mu{}_{,\alpha})(\delta^\nu{}_\beta + \xi^\nu{}_{,\beta}) \\ &= \bar{g}_{\alpha\beta} + \bar{g}_{\mu\nu}\,\xi^\mu{}_{,\alpha}\delta^\nu{}_\beta + \bar{g}_{\mu\nu}\,\delta^\mu{}_\alpha\,\xi^\nu{}_{,\beta} + \mathcal{O}(\xi^2) \\ &= \bar{g}_{\alpha\beta} + \bar{g}_{\mu\nu}\,\xi^\mu{}_{,\alpha} + \bar{g}_{\alpha\nu}\,\xi^\nu{}_{,\beta} + \mathcal{O}(\xi^2) \\ &= \bar{g}_{\alpha\beta} + \xi_{\beta,\alpha} + \xi_{\alpha,\beta} + \mathcal{O}(\xi^2). \end{aligned}$$

- Therefore

$$\bar{g}_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - (\xi_{\alpha,\beta} + \xi_{\beta,\alpha}).$$

- We have redefined $h_{\alpha\beta}$

$$h_{\alpha\beta} \to h_{\alpha\beta}^{\text{new}} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}.$$

- and the metric $g_{\alpha\beta}$

$$g_{\alpha\beta} \to g_{\alpha\beta}^{\text{new}} = \eta_{\alpha\beta} + h_{\alpha\beta}^{\text{new}}.$$

- We define the following tensor

$$\bar{h}_{\alpha\beta}^{\text{new}} = h_{\alpha\beta}^{\text{new}} - \frac{1}{2} h^{\text{new}}\, \eta_{\alpha\beta},$$

- With

$$h^{\text{new}} = h^{\mu}{}_{\mu}{}^{\text{new}} = h - \xi^{\mu}{}_{,\mu} - \xi_{\mu}{}^{,\mu} = h - 2\xi^{\mu}{}_{,\mu}$$

$$
\begin{aligned}
\bar{h}^{\text{new}}_{\alpha\beta} &= h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} - \frac{1}{2}(h - 2\xi^{\mu}{}_{,\mu})\eta_{\alpha\beta} \\
&= h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} - \frac{1}{2}h\eta_{\alpha\beta} + \xi^{\mu}{}_{,\mu}\eta_{\alpha\beta} \\
&= \bar{h}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \xi^{\mu}{}_{,\mu}\eta_{\alpha\beta}.
\end{aligned}
$$

- If we impose the following condition

$$
\bar{h}^{\text{new}\,\mu\nu}{}_{,\nu} = 0 \quad \text{(Lorenz gauge)}
$$

- We have

$$
\bar{h}^{\mu\nu}{}_{,\nu} - \xi^{\mu,\nu}_{\ \ ,\nu} - \xi^{\nu,\mu}_{\ \ ,\nu} + \xi^{\nu}{}_{,\nu}{}^{,\mu} = 0 \implies \xi^{\mu,\nu}_{\ \ ,\nu} = \bar{h}^{\mu\nu}{}_{,\nu} \implies \Box\xi^{\mu} = \bar{h}^{\mu\nu}{}_{,\nu},
$$

$$-\frac{1}{2}\left(\bar{h}_{\alpha\beta,\mu}{}^{,\mu} - \bar{h}_{\beta}{}^{\mu}{}_{,\alpha\mu} - \bar{h}_{\alpha}{}^{\mu}{}_{,\beta\mu} + \bar{h}^{\mu\nu}{}_{,\mu\nu}\,\eta_{\alpha\beta}\right) = \kappa T_{\alpha\beta}.$$

- We can use the previous condition (Lorenz gauge) to simplify the Einstein equations to linear order in h

$$G_{\alpha\beta} = -\frac{1}{2}\bar{h}_{\alpha\beta,\mu}{}^{,\mu} \implies \Box\bar{h}_{\alpha\beta} = -2\kappa T_{\alpha\beta}.$$

- These equations are the linearized Einstein equations and represent a wave equation for the perturbations, $\bar{h}_{\alpha\beta}$, whose solution are the so-called gravitational waves.

Newtonian limit of the equations

- Let us now study the Newtonian limit of linearized Einstein equations. At this limit, the gravitational field is so weak that it can only produce speeds that are small compared to the speed of light. The energy-momentum tensor of matter in this case will satisfy that

$$|T^{00}| \gg |T^{0i}| \gg |T^{ij}|,$$

- These inequalities imply that

$$|\bar{h}^{00}| \gg |\bar{h}^{0i}| \gg |\bar{h}^{ij}|.$$

- Therefore the dominant equation will be

$$\Box \bar{h}^{00} = -2\kappa T^{00}.$$

$$\Box\bar{h}^{00} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\bar{h}^{00} + \nabla^2\bar{h}^{00} = \nabla^2\bar{h}^{00} + \mathcal{O}(v^2\nabla^2),$$

- We have

$$\nabla^2\bar{h}^{00} = -2\kappa\rho,$$

- Since

$$T^{00} = \rho + \mathcal{O}(\rho v^2),$$

- (non-relativistic matter, small rest speeds and energy density and furthermore, $\rho = \rho_0 c^2$).

- From Poisson's equation $\nabla^2\Phi = 4\pi G\rho_0$,

$$\bar{h}^{00} = -\frac{c^2\kappa}{2\pi G}\,\Phi.$$

- From

$$h^{\alpha\beta} = \bar{h}^{\alpha\beta} - \frac{1}{2}\bar{h}\eta^{\alpha\beta} \quad \text{y} \quad \bar{h} = \bar{h}^{\alpha}{}_{\alpha} \approx \bar{h}^{0}{}_{0} = \bar{h}^{00}\eta_{00} = -\bar{h}^{00}.$$

- We get

$$\begin{aligned}
h^{00} &= \bar{h}^{00} - \frac{1}{2}(-\bar{h}^{00})\eta^{00} = \frac{1}{2}\bar{h}^{00} = -\frac{c^2\kappa}{4\pi G}\,\Phi, \\
h^{xx} &= \bar{h}^{xx} - \frac{1}{2}(-\bar{h}^{00})\eta^{xx} = -\frac{c^2\kappa}{4\pi G}\,\Phi, \\
h^{yy} &= h^{zz} = -\frac{c^2\kappa}{4\pi G}\,\Phi.
\end{aligned}$$

- The metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu$

$$ds^2 = -\left(1 + \frac{c^2\kappa}{4\pi G}\,\Phi\right)c^2dt^2 + \left(1 - \frac{c^2\kappa}{4\pi G}\,\Phi\right)(dx^2 + dy^2 + dz^2).$$

- From the geodesic equation

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma = 0, \qquad \frac{d^2x^i}{dt^2} + \Gamma^i_{00}c^2 = 0.$$

$$\Gamma^i_{00} = \frac{1}{2}g^{i\mu}(g_{\mu0,0} + g_{0\mu,0} - g_{00,\mu}) \approx -\frac{1}{2}g^{ij}g_{00,j} = \frac{c^2\kappa}{8\pi G}\delta^{ij}\frac{\partial\Phi}{\partial x^j},$$

$$\frac{d^2x^i}{dt^2} = -\frac{c^4\kappa}{8\pi G}\delta^{ij}\frac{\partial\Phi}{\partial x^j}.$$

$$\frac{c^4\kappa}{8\pi G} = 1 \implies \kappa = \frac{8\pi G}{c^4}.$$

Gravitational radiation

- Gravitational waves are an example of weak gravitational fields. Such waves are generated in the strong field regime - intense gravity - by self-gravitating sources such as binary black hole systems. When they propagate far from the sources that generate them, in the so-called wave zone, they represent weak disturbances in a Minkowski space-time and can be described using the weak field formalism.

- Gravitational waves represent ripples in the fabric of space-time that propagate at the speed of light and that induce, as we shall see, variations in the length of the objects they pass through.

- The detection of gravitational waves opens a new window to the detailed exploration of the universe and they will surely be the key to answering many puzzles related to gravitation, astrophysics and fundamental physics.

Properties of gravitational waves

- Consider the linearized Einstein equations obtained in the previous section, in vacuum,

$$\Box \bar{h}^{\alpha\beta} = 0,$$

- Which is a wave equation. This equation admits a plane wave solution:

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} e^{ik_\mu x^\mu},$$

- Which can be written as

$$\begin{aligned}
\Box \bar{h}^{\alpha\beta} \equiv \eta^{\nu\lambda}\partial_\nu\partial_\lambda \bar{h}^{\alpha\beta} &= \eta^{\mu\lambda}\bar{h}^{\alpha\beta}{}_{,\nu\lambda} \\
&= \eta^{\nu\lambda} A^{\alpha\beta}\left[e^{ik_\mu x^\mu} ik_\sigma \eta^\sigma{}_\nu\right]_{,\lambda} \\
&= \eta^{\nu\lambda} A^{\alpha\beta} e^{ik_\mu x^\mu}\, ik_\rho \eta^\rho{}_\lambda ik_\sigma \eta^\sigma{}_\nu \\
&= \eta^{\nu\lambda} A^{\alpha\beta} e^{ik_\mu x^\mu} (-1) k_\lambda k_\nu \\
&= -A^{\alpha\beta}\, e^{ik_\mu x^\mu}\, k^\nu k_\nu = 0.
\end{aligned}$$

- $\overrightarrow{k}$ must be an isotropic (null) vector

$$\vec{k} \cdot \vec{k} = k_\alpha k^\alpha = 0.$$

- Furthermore, in the gauge that we have chosen to write the wave equation we have

$$\bar{h}^{\alpha\beta}{}_{,\beta} = 0 \implies A^{\alpha\beta}\, e^{ik_\mu x_\mu}\, ik_\lambda \eta^\lambda{}_\beta = 0 \implies A^{\alpha\beta}\, k_{\alpha\beta} = 0,$$

- $A^{\alpha\beta}$ is orthogonal to $\overrightarrow{k}$.

- The wave vector k_μ gives information about the frequency of the wave ω and its direction of propagation n_i

$$\omega \equiv k^0 = \sqrt{k_i k^i} \quad ; \quad n^i \equiv \frac{k^i}{k_0}.$$

- We can impose more restrictions on the amplitude $A^{\alpha\beta}$ using a gauge transformation

$$x_\alpha \to x_\alpha + \xi_\alpha,$$

- Where $\Box \xi_\alpha = 0$ (Lorenz gauge). Let's take

$$\xi_\alpha = B_\alpha e^{ik_\mu x^\mu},$$

- Under this change of coordinates we have

$$h_{\alpha\beta} \to h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha},$$

- And

$$\bar{h}_{\alpha\beta} \to \bar{h}_{\alpha\beta} - \xi_{\alpha\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta}\xi^{\mu}{}_{,\mu},$$

- It is easy to obtain the expression of the amplitudes of the wave, $A_{\alpha\beta}$, under the change of coordinates. We have

$$\bar{h}_{\alpha\beta} = A_{\alpha\beta}e^{ik_\mu x^\mu} \to \bar{h}_{\alpha\beta,\beta} = ik_\beta e^{ik_\mu x^\mu}\, A_{\alpha\beta}.$$

- Therefore

$$\begin{aligned}
\bar{h}^{\rm new}_{\alpha\beta} &= \bar{h}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \xi^{\mu}{}_{,\mu}\eta_{\alpha\beta} \\
\bar{h}^{\rm new}_{\alpha\beta,\beta} &= \bar{h}_{\alpha\beta,\beta} - \xi_{\alpha\beta,\beta} - \xi_{\beta,\alpha\beta} + \xi^{\mu}{}_{,\mu\beta}\eta_{\alpha\beta} \\
&= ik_\beta A_{\alpha\beta} e^{ik_\mu x^\mu} - ik_\beta ik_\beta B_\alpha e^{ik_\mu x^\mu} - ik_\beta ik_\alpha B_\beta e^{ik_\mu x^\mu} + ik_\beta ik_\mu B^\mu e^{ik_\mu x^\mu}\eta_{\alpha\beta} \\
&= ik_\beta e^{k_\mu x^\mu}[A_{\alpha\beta} - ik_\beta B_\alpha - ik_\alpha B_\beta + ik_\mu B^\mu \eta_{\alpha\beta}].
\end{aligned}$$

- The transformation of the wave amplitudes under the gauge transformation is

$$A_{\alpha\beta} \to A^{\rm new}_{\alpha\beta} = A_{\alpha\beta} - iB_\alpha k_\beta - iB_\beta k_\alpha + i\eta_{\alpha\beta} B^\mu k_\mu.$$

- On the other hand, it is possible to choose $\overrightarrow{B}$ so two additional constraints can be imposed on $A^{\text{new}}_{\alpha\beta}$:
  $A^{\alpha}{}_{\alpha} = 0$ and $A_{\alpha\beta}u^{\beta} = 0$,
  with $\overrightarrow{u}$ being any unitary time vector.

- The first condition leads to

$$A_{\alpha\beta} \rightarrow A^{\text{new}}_{\alpha\beta} = A_{\alpha\beta} - iB_{\alpha}k_{\beta} - iB_{\beta}k_{\alpha} + i\eta_{\alpha\beta}B^{\mu}k_{\mu}.$$

- While the second

$$A^{\alpha}{}_{\alpha} - iB^{\alpha}k_{\alpha} - iB^{\alpha}k_{\alpha} + i\eta^{\alpha}{}_{\alpha}B^{\mu}k_{\mu} = A^{\alpha} + 2iB^{\alpha}k_{\alpha} = 0,$$

- Therefore, we have the equations for $\overrightarrow{B}$ that will allow us, when doing the gauge transformation, to obtain a new $A^{\alpha\beta}$ such that $A^{\alpha}{}_{\alpha} = 0$ and $A_{\alpha\beta}u^{\beta} = 0$.

- We thus guarantee that with the gauge transformation $\Box \xi^\alpha = 0$, the components of the wave amplitude $A^{\alpha\beta}$ satisfy

$$A^{\alpha\beta} k_\beta = 0, \quad A_{\alpha\beta} u^\beta = 0, \quad A^\alpha{}_\alpha = 0.$$

- The above conditions are called the traceless, transverse gauge, or **``TT" gauge** conditions. We will denote $\bar{h}^{\mathrm{TT}}_{\mu\nu}$ to $\bar{h}_{\mu\nu}$ in the TT gauge.

- Let us now analyze the components of $A_{\alpha\beta}$. For simplicity, consider an observer with 4-velocity $u^\mu = \delta^\mu{}_0 = (1{,}0{,}0{,}0)$. For this observer it is true that

$$A_{\alpha\beta} u^\beta = A_{\alpha\beta} \delta^\beta{}_0 = A_{\alpha 0} = 0 \rightarrow \bar{h}^{\mathrm{TT}}_{\alpha 0} = 0.$$

- If we orient the axes of the spatial coordinates so that the wave is travelling in the direction of the z axis

$$\vec{k} = (\omega, 0, 0, \omega)$$

- We get

$$A_{\alpha\beta}k^{\beta} = 0 \to A_{\alpha 0} + A_{\alpha z} = 0 \to A_{\alpha z} = 0 \to \bar{h}^{\mathrm{TT}}_{\alpha z} = 0.$$

- This result explains the origin of the term "transverse" in the name of the gauge: $A_{\alpha\beta}$ is transversal to the direction of propagation. Thus, the name TT comes from the fact that the resulting tensor is transversal (with respect to the observer u^{μ}) and the spatial part has no trace. Additionally, since we are in the Lorenz gauge, the spatial part of $\bar{h}^{\mathrm{TT}}_{\mu\nu}$ also has zero divergence.

- Note also that since the trace of $\bar{h}_{\mu\nu}^{\mathrm{TT}}$ vanishes

$$\bar{h}^{\mathrm{TT}} \equiv \eta^{\mu\nu}\bar{h}_{\mu\nu}^{\mathrm{TT}} = \eta^{00}\bar{h}_{00}^{\mathrm{TT}} + \eta^{ij}\bar{h}_{ij}^{\mathrm{TT}} = 0,$$

- we get that $\bar{h}_{\mu\nu}^{\mathrm{TT}} = h_{\mu\nu}^{\mathrm{TT}}$. This allows to eliminate the notation with the bar since it is no longer necessary. Einstein's equations linearized in vacuum are thus

$$\Box h_{\mu\nu}^{\mathrm{TT}} = 0.$$

- The plane wave solution for the observer $u^\mu = \delta^\mu_0$ representing a wave traveling in the z direction is

$$h^{\rm TT}_{\mu\nu} = A_{\mu\nu} e^{ik_\alpha x^\alpha} = A_{\mu\nu} e^{i(k_0 t + k_z z)} = A_{\mu\nu} e^{-i\omega(t-z)}.$$

- Where we have used

$$\begin{aligned} k_0 &= \eta_{\mu 0} k^\mu = -k^0 = -\omega, \\ k_z &= \eta_{\mu z} k^\mu = k^z \implies \omega^2 = k_z k^z = (k^z)^2 \implies k_z = k^z = \omega. \end{aligned}$$

- In the TT gauge we have that $A_{\alpha 0} = A_{\alpha z} = 0$. Therefore, there are only 4 independent amplitude components different from zero

$$A^{\mathrm{TT}}_{\alpha\beta} = \begin{pmatrix} A_{00} & A_{0x} & A_{0y} & A_{0z} \\ A_{x0} & A_{xx} & A_{xy} & A_{xz} \\ A_{y0} & A_{yx} & A_{yy} & A_{yz} \\ A_{z0} & A_{zx} & A_{zy} & A_{zz} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{yx} & A_{yy} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- On the other hand, $A_{yx} = A_{xy}$, being symmetric and since $A^{\alpha}{}_{\alpha} = 0$, we have

$$A_{xx} + A_{yy} = 0 \implies A_{yy} = -A_{xx}.$$

- Therefore

$$A_{\alpha\beta}^{\mathrm{TT}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

- that is, there are only two independent constants. Therefore, in the plane wave solution only two degrees of freedom remain, corresponding to the two possible polarizations:

$$\begin{aligned} h_{xx}^{\mathrm{TT}} &= h_{yz}^{\mathrm{TT}} = h_{zz}^{\mathrm{TT}} = 0 \\ h_{xx}^{\mathrm{TT}} &= -h_{yy}^{\mathrm{TT}} = A_{+} e^{-i\omega(t-z)} \quad (+\ \text{polarization}) \\ h_{xy}^{\mathrm{TT}} &= h_{yx}^{\mathrm{TT}} = A_{\times} e^{-i\omega(t-z)} \quad (\times\ \text{polarization}), \end{aligned}$$

- where $A_{+} \equiv A_{xx}$ and $A_{\times} = A_{xy}$.

$$h_{\alpha\beta}^{\mathrm{TT}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx}^{\mathrm{TT}} & h_{xy}^{\mathrm{TT}} & 0 \\ 0 & h_{xy}^{\mathrm{TT}} & -h_{xx}^{\mathrm{TT}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left[A_+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + A_\times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] e^{-i\omega(t-z)}.$$

- In these coordinates (TT gauge), the metric of spacetime can be expressed as follows:

$$\begin{aligned} ds^2 = (\eta_{\alpha\beta} + h_{\alpha\beta}^{\mathrm{TT}}) dx^\alpha \, dx^\beta = -c^2 dt^2 \quad &+ \quad (1 + |A_\times| \cos(\omega z - \omega t + \varphi_+)) dx^2 \\ &+ \quad 2|A_\times| \cos(\omega z - \omega t + \varphi_\times) \, dx \, dy \\ &+ \quad (1 - |A_+| \cos(\omega z - \omega t + \varphi_+)) \, dy^2 + dz^2. \end{aligned}$$

Effect of a gravitational wave on a particle

- Let us consider a particle that is initially in a region free of gravitational waves and at rest. A free particle obeys the geodesic equation

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = 0.$$

- Since initially the particle is at rest, the initial value of the acceleration will be

$$\left(\frac{du^\alpha}{d\tau}\right)_0 = -\Gamma^\alpha_{00} u^0 u^0 = -\frac{1}{2}\eta^{\alpha\beta}(h_{\beta 0,0} + h_{0\beta,0} - h_{00,\beta} = 0,$$

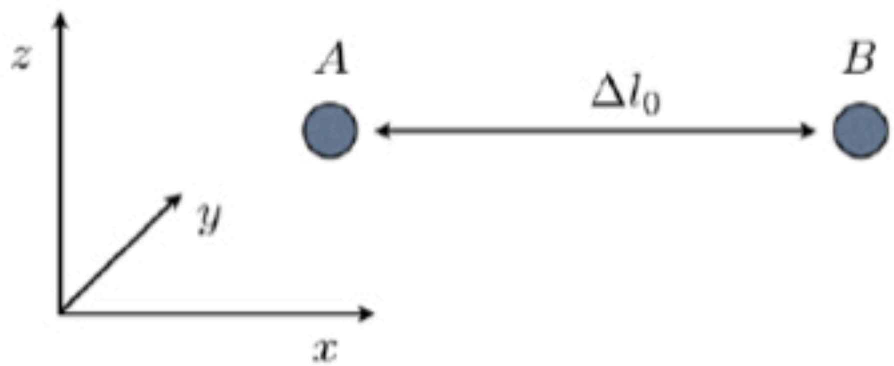
- since in the TT gauge, $h_{\alpha 0} = 0$. Therefore, the particle will remain at rest later and will not change its position. This does not mean that the particle does not move but only that the chosen coordinates follow the particle.

- We must obtain some magnitude that is invariant with the coordinates. To do this, let us consider two test particles (A and B) located along the x-axis, initially separated by a distance Δl_0. The metric in the TT gauge is

$$ds^2 = -dt^2 + (\eta_{ij} + h_{ij}^{\mathrm{TT}})dx^i dx^j \, .$$

- Since A and B lie on the x axis, i.e. $dy = dz = 0$, for a given time ($dt = 0$), the line element is

$$ds^2 = dl^2 = (\eta_{xx} + h_{xx}^{\mathrm{TT}})dx^2 ,$$



- The proper distance between points A and B can be calculated as

$$\Delta l \equiv \int_A^B \sqrt{ds^2} = \int_A^B \sqrt{1 + h_{xx}^{\mathrm{TT}}}\, dx \simeq \int_A^B \left(1 + \frac{1}{2} h_{xx}^{\mathrm{TT}}\right) dx \simeq \Delta l_0 \left(1 + \frac{1}{2} h_{xx}^{\mathrm{TT}}\right),$$

- where $\Delta l_0 = \displaystyle\int_A^B dx$. To make this calculation we have used that $|\, h_{\mu\nu} | \ll 1$ (linear perturbations).

- The measurement of the relative variation of the proper distance between test particles is the basis for the design of modern gravitational wave detectors. The proper distance varies with time and it is the proper distance that we measure with light rays, for example using a Michelson-Morley interferometer.

- Another way to analyze the effect caused by the wave on the particles is studying the relative acceleration between two neighboring particles that follow geodesics. This is done from the geodetic deviation equation. Let ξ^α be the vector joining the two particles. This vector obeys the equation

$$\frac{d^2}{d\tau^2}\xi^\alpha = R^\alpha{}_{\mu\nu\beta}u^\mu u^\nu \xi^\beta,$$

- where $\overrightarrow{u} = d\overrightarrow{x}/d\tau$ is the 4-velocity of the particles. In first order in $h_{\mu\nu}$ we have

$$u^\mu = (1, 0, 0, 0) \qquad \xi^\alpha = (0, \epsilon, 0, 0),$$

- where we are considering that the first particle is at the origin of coordinates, $x = y = z = 0$, and the second is at $x = \epsilon$ and $y = z = 0$. Therefore, in first order in $h_{\mu\nu}$, the previous equation reduces to

$$\frac{d^2}{d\tau^2}\xi^\alpha = \frac{\partial^2}{\partial t^2}\xi^\alpha = R^\alpha{}_{00x}\epsilon = -\epsilon R^\alpha_{0x0}.$$

- Taking into account that the only non-zero components of the Riemann tensor are

$$\begin{aligned} R^{x}{}_{0x0} &= R_{x0x0} = -\frac{1}{2} h^{\mathrm{TT}}_{xx,00}, \\ R^{y}{}_{0x0} &= R_{y0x0} = -\frac{1}{2} h^{\mathrm{TT}}_{xy,00}, \\ R^{z}{}_{0y0} &= R_{y0y0} = -\frac{1}{2} h^{\mathrm{TT}}_{yy,00}, \end{aligned}$$

- We arrive to

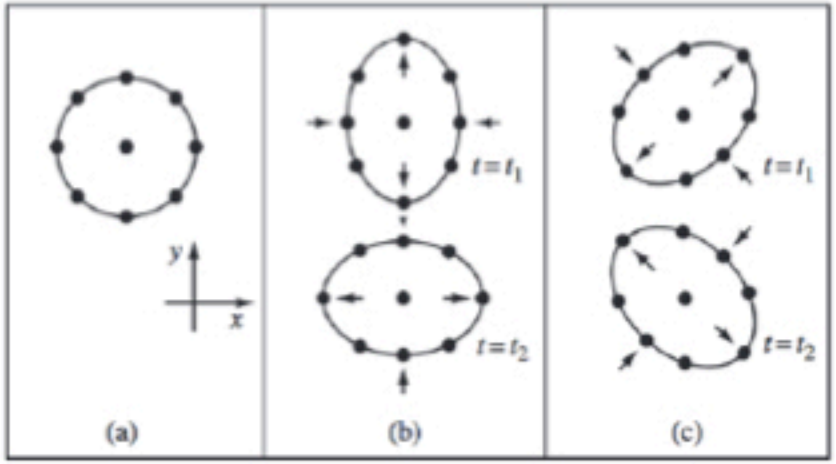
$$\begin{aligned} \frac{\partial^2}{\partial t^2}\xi^x &= \frac{1}{2}\epsilon \frac{\partial^2}{\partial t^2} h^{\mathrm{TT}}_{xx}, \\ \frac{\partial^2}{\partial t^2}\xi^y &= \frac{1}{2}\epsilon \frac{\partial^2}{\partial t^2} h^{\mathrm{TT}}_{xy}, \end{aligned}$$

- Similarly, if the particles were initially separated by a distance ϵ in the y direction, $\xi^\alpha = (0,0,\epsilon,0)$, we have

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\xi^y &= \frac{1}{2}\epsilon \frac{\partial^2}{\partial t^2} h^{\mathrm{TT}}_{yy} = -\frac{1}{2}\epsilon \frac{\partial^2}{\partial t^2} h^{\mathrm{TT}}_{xx}, \\ \frac{\partial^2}{\partial t^2}\xi^x &= \frac{1}{2}\epsilon \frac{\partial^2}{\partial t^2} h^{\mathrm{TT}}_{xy}. \end{aligned}$$

- These equations help us describe the polarization of a gravitational wave. Consider a test particle in a circle initially at rest, as shown in panel (a). Suppose that the gravitational wave has $h_{xx}^{\mathrm{TT}} \neq 0$ and $h_{xy}^{\mathrm{TT}} = 0$. In this case, the test particles of the circle will move, in terms of their proper distance relative to the central particle , as shown in panel (b). If, on the contrary, the gravitational wave has $h_{xy}^{\mathrm{TT}} \neq 0$ and $h_{xx}^{\mathrm{TT}} = h_{yy}^{\mathrm{TT}} = 0$, the perturbations on the circle of free particles would follow the pattern shown in panel (c).

- Since h_{xx}^{TT} and h_{xy}^{TT} are independent, the oscillations shown in panels (b) and (c) of Figure 2 provide a graphical representation of the two linear polarizations of a gravitational wave, and justify the notation. on used in its name: "+" for case (b) and "×" for case (c). Besides, the particles oscillate around their original position with the frequency ω of the incident gravitational wave.



The generation of gravitational waves

Quadrupolar formula

- The Einstein equations for weak field are

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}.$$

- We will assume the simplifying (but realistic) hypothesis that the time dependence of the source $T_{\mu\nu}$ is harmonic, that is, it is a sinusoidal oscillation with frequency Ω, the real part of

$$T_{\mu\nu} = S_{\mu\nu} e^{-i\Omega t}.$$

- The tensor $S_{\mu\nu}$ only depends on the spatial coordinates, $S_{\mu\nu} = S_{\mu\nu}(x^i)$. We will also assume that the region of space that generates the waves, in which $S_{\mu\nu} \neq 0$, is small compared to the wavelength of the gravitational wave of frequency Ω, $2\pi/\Omega$. This second hypothesis is called the small velocity hypothesis, since it implies that the typical velocities reached within the source region, which are given by the size of that region multiplied by the frequency Ω, should be much less than unity. Most sources of gravitational waves satisfy this condition.

- We will therefore look for a solution for $\bar{h}_{\mu\nu}$ with the same time dependence, that is, of the form

$$\bar{h}_{\mu\nu} = B_{\mu\nu} e^{-i\Omega t},$$

- where, again, $B_{\mu\nu} = B_{\mu\nu}(x^i)$. Substituting this ansatz in the wave equation, we arrive at the equation

$$(\nabla^2 + \Omega^2) B_{\mu\nu} = -16\pi S_{\mu\nu}.$$

- Outside the source, that is, where $S_{\mu\nu} = 0$, we want a solution $B_{\mu\nu}$ that represents outgoing radiation away from the source. Let r be the radial coordinate originating at the source. The solution sought is of the form

$$B_{\mu\nu} = \frac{A_{\mu\nu}}{r} e^{i\Omega r} + \frac{Z_{\mu\nu}}{r} e^{-i\Omega r},$$

- Since we are looking for waves emitted by the source, that is, outgoing, we can set $Z_{\mu\nu} = 0$ and, therefore, the solution will be of the form

$$B_{\mu\nu} = \frac{A_{\mu\nu}}{r} e^{i\Omega r}.$$

- To determine $A_{\mu\nu}$ we will have to integrate the equation in the spatial region that defines the source. For this we will make the approximation that the source is not zero ($S_{\mu\nu} \neq 0$) only inside a sphere of radius $\epsilon \ll 2\pi/\Omega$. We have

$$\int \nabla^2 B_{\mu\nu} d^3x + \int \Omega^2 B_{\mu\nu} d^3x = -16\pi \int S_{\mu\nu} d^3x.$$

- The second term of the left-hand-side is

$$\int \Omega^2 B_{\mu\nu} d^3x \leq \Omega^2 |B_{\mu\nu}|_{\max} \frac{4\pi}{3} \epsilon^3,$$

- where $|B_{\mu\nu}|_{\max}$ is the maximum value that $B_{\mu\nu}$ reaches within the source.

- We can compute the first term on the left side by applying Gauss's theorem

$$\begin{aligned}\int \nabla^2 B_{\mu\nu}\, d^3x &= \int \vec{\nabla}\cdot(\vec{\nabla} B_{\mu\nu})\, d^3x = \oint \vec{n}\cdot\vec{\nabla} B_{\mu\nu} dS = \oint \vec{n}\cdot A_{\mu\nu}\vec{\nabla}\left(\frac{e^{i\Omega r}}{r}\right) dS \\ &= A_{\mu\nu}\frac{d}{dr}\left(\frac{e^{i\Omega r}}{r}\right)_{r=\epsilon} 4\pi\epsilon^2.\end{aligned}$$

- On the other hand

$$\frac{d}{dr}\left(\frac{e^{i\Omega r}}{r}\right)_{r=\epsilon} 4\pi\epsilon^2 = \left(\frac{i\Omega r e^{i\Omega r} - e^{i\Omega r}}{r^2}\right)_{r=\epsilon} 4\pi\epsilon^2 = (i\Omega\epsilon - 1)e^{i\Omega\epsilon} 4\pi \simeq -4\pi,$$

- since $\epsilon\dfrac{\Omega}{2\pi} \ll 1 \to \lambda \gg \epsilon$. Therefore, we have

$$\int \nabla^2 B_{\mu\nu} d^3x \simeq -4\pi A_{\mu\nu}.$$

- Moreover, if we denote

$$J_{\mu\nu} \equiv \int S_{\mu\nu} d^3x,$$

- we have that, in the limit $\epsilon \to 0$, the integral can be written as

$$-4\pi A_{\mu\nu} + \Omega^2 |B_{\mu\nu}|_{\max} \frac{4\pi}{3} \epsilon^3 = -16\pi J_{\mu\nu} \to A_{\mu\nu} = 4J_{\mu\nu}.$$

- Therefore, we arrive at the following solution for the gravitational wave:

$$\bar{h}_{\mu\nu} = 4J_{\mu\nu} \frac{e^{i\Omega(r-t)}}{r}.$$

- We can simplify this equation using the relation between $J_{\mu\nu}$ and the tensor $T_{\mu\nu}$. Integrating in the 3-volume

$$T_{\mu\nu} = S_{\mu\nu} e^{-i\Omega t} \implies \int T_{\mu\nu} \, d^3x = e^{-i\Omega t} \int S_{\mu\nu} \, d^3x = e^{-i\Omega t} J_{\mu\nu}.$$

- from where we get $e^{-i\Omega t} J^{\mu 0} = \int T^{\mu 0} \, d^3x.$

- Deriving this last expression with respect to time

$$-i\Omega e^{-i\Omega t} J^{\mu 0} = \int T^{\mu 0}{}_{,0} \, d^3x.$$

- From the conservation law of the stress-energy tensor, we have

$$T^{\mu\nu}{}_{,\nu} = 0 \implies T^{\mu 0}{}_{,0} + T^{\mu k}{}_{,k} = 0,$$

- and, therefore, $i\Omega e^{-i\Omega t} J^{\mu 0} = \int T^{\mu k}{}_{,k} d^3x = \oint T^{\mu k} n_k dS,$

- In the last step we have applied Gauss's theorem on a volume that completely contains the source. This means that $T^{\mu\nu} = 0$ on the surface (of integration) that encloses said volume, so the right hand side of the previous expression is identically null. Therefore, if $\Omega \neq 0$, we conclude that

$$J^{\mu 0} = 0 \implies \bar{h}^{\mu 0} = 0.$$

- On the other hand, we can obtain the expression of J_{lm} using the so-called tensor virial theorem:

$$\frac{d^2}{dt^2} \int T^{00} x^l x^m d^3x = 2 \int T^{lm} d^3x.$$

- Therefore, we have

$$J^{lm} = \int S^{lm} d^3x = e^{i\Omega t} \int T^{lm} d^3x \implies J^{lm} = \frac{e^{i\Omega t}}{2} \frac{d^2}{dt^2} \int T^{00} x^l x^m d^3x.$$

- For a source with small velocities (not relativistic), $T^{00} \simeq \rho$, where ρ is the mass density. The integral is the so-called quadrupole moment tensor of the mass distribution

$$I^{lm} \equiv \int T^{00} x^l x^m d^3x = e^{-i\Omega t} \int S^{00} x^l x^m d^3x = e^{-i\Omega t} D^{lm}.$$

- Therefore, we have

$$J^{lm} = \frac{e^{i\Omega t}}{2} \frac{d^2}{dt^2} I^{lm} = \frac{e^{i\Omega t}}{2} \frac{d^2}{dt^2} \left(e^{-i\Omega t} D^{lm}\right) = -\frac{\Omega^2}{2} D^{lm}.$$

- and

$$\bar{h}_{jk} = 4 J_{jk} \frac{e^{i\Omega(r-t)}}{r} = -2\Omega^2 D_{jk} \frac{e^{i\Omega(r-t)}}{r} = \frac{2}{r} e^{i\Omega r} \frac{d^2}{dt^2} I_{jk}.$$

- The solution that this equation provides is known as the quadrupole approximation to gravitational radiation, or simply the quadrupole formula. It should be noted that if we include the appropriate factors G and c, the last equation is written

$$\bar{h}_{jk} = \frac{2G}{rc^4} e^{i\Omega r} \frac{d^2}{dt^2} I_{jk},$$

- which indicates that gravitational waves are extraordinarily weak, since the factor $G/c^4 \sim 8 \times 10^{-50}$ s g^{-1} cm^{-1}.

- The quadrupole formula gives us the gravitational radiation emitted by a mass-energy distribution that evolves in time. The result depends only on the movement of the source and not on the forces acting on it. Furthermore, unlike electromagnetic radiation, gravitational radiation has a quadrupole nature. In the case of electromagnetism, for a system of accelerated charged particles, the associated dipole moment

$$\vec{d}_{\mathrm{EM}} = \sum_i q_i \vec{r}_i,$$

- can vary in time, giving rise to dipole radiation, whose flux depends on the second time derivative of $\vec{d}_{\mathrm{EM}}$. For an isolated system of masses, we can define a gravitational dipole moment in an analogous way

$$\vec{d}_{\mathrm{G}} = \sum_i m_i \vec{r}_i,$$

- which satisfies the law of conservation of the total momentum of an isolated system, $\frac{d}{dt}\vec{d}_{\mathrm{G}} = 0$. For this reason, gravitational radiation does not have a dipole contribution. It should also be emphasized that a distribution of matter with spherical or axial symmetry has a constant quadrupole moment, even though the system is rotating. Therefore, a spherical or axisymmetric star does not emit gravitational waves. To produce gravitational waves, a certain degree of asymmetry is necessary, as occurs, for example, in non-radial pulsations of stars, in a non-spherical gravitational collapse, or in collisions of massive objects in binary systems.

- Next, let us obtain the quadrupole formula on gauge TT. In said gauge we obtained that

$$\begin{aligned} \bar{h}^{\mathrm{TT}}_{\mu 0} &= 0, \\ n^j \bar{h}^{\mathrm{TT}}_{jk} &= 0, \end{aligned}$$

- where n_j is the unit vector in the direction of propagation of the wave, that is, normal to the wave front,

$$\vec{n} = \frac{\vec{x}}{r}.$$

- To move to the TT gauge, we must define the operator that projects a vector onto the plane perpendicular to the direction of $\vec{n}$. This operator is

$$P_{jk} \equiv \delta_{jk} - n_j n_k.$$

- It can be simply verified that P_{jk} is symmetric, it is a projector since $P_{jk} P_{kl} = P_{jl}$ and it is transverse, $n^j P_{jk} = 0$.

- Next, we define the transverse traceless projector:

$$\mathcal{P}_{jkmn} \equiv P_{jm}P_{kn} - \frac{1}{2}P_{jk}P_{mn},$$

- that ``extracts" the tranverse and traceless part of tensor of type $\binom{0}{2}$.

- We want to compute

$$\bar{h}_{jk}^{\mathrm{TT}} = (\mathcal{P}\bar{h})_{jk} = \mathcal{P}_{jklm}\bar{h}_{lm}.$$

- We have

$$\bar{h}_{jk}^{\mathrm{TT}} = \mathcal{P}_{jklm}\bar{h}_{lm} = \frac{2G}{rc^4}e^{i\Omega r}\frac{d^2}{dt^2}\mathcal{P}_{jklm}I_{lm} = \frac{2G}{rc^4}e^{i\Omega r}\frac{d^2}{dt^2}I_{jk}^{\mathrm{TT}}.$$

- Introducing the reduced quadrupolar moment tensor:

$$\hat{I}_{ij} \equiv I_{ij} - \frac{1}{3}\delta_{ij} I,$$

- with $I = I^i_i = \mathrm{tr}(I_{ij})$, we can rewrite the expression above as

$$\bar{h}^{\mathrm{TT}}_{jk} = \frac{2G}{rc^4} e^{i\Omega r} \frac{d^2}{dt^2} \hat{I}^{\mathrm{TT}}_{ij},$$

- It is straightforward to show that

$$\bar{I}^{\mathrm{TT}}_{ij} = \mathcal{P}_{jklm} \bar{I}_{ij} = I^{\mathrm{TT}}_{jk},$$

- as the reduced quadrupole moment tensor is a traceless tensor, by definition.

- Introducing the reduced quadrupolar moment tensor:

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$$\bar{h}^{\mathrm{TT}}_{jk} = \frac{2G}{rc^4} e^{i\Omega r} \frac{d^2}{dt^2} \hat{I}^{\mathrm{TT}}_{ij},$$

- It is straightforward to show that

$$\bar{I}^{\mathrm{TT}}_{ij} = \mathcal{P}_{jklm} \bar{I}_{ij} = I^{\mathrm{TT}}_{jk},$$

- as the reduced quadrupole moment tensor is a traceless tensor, by definition.

- As a particular case, let us consider a gravitational wave propagating in the z direction, so that $\overrightarrow{n} = \frac{\overrightarrow{x}}{r} = (0,0,1)$. The projector operator is

$$P_{jk} = \delta_{jk} - n_j n_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- We have

$$I_{jk}^{\mathrm{TT}} = \mathcal{P}_{jklm} I_{lm} = \left(P_{jk} P_{km} - \frac{1}{2} P_{jk} P_{lm} \right) I_{lm},$$

- whose components turn out to be

$$
\begin{aligned}
I_{xx}^{\mathrm{TT}} &= \left(P_{xl}P_{xm} - \frac{1}{2}P_{xx}P_{lm}\right) I_{lm} = P_{xx}P_{xx}I_{xx} - \frac{1}{2}P_{xx}P_{lm}I_{lm} = I_{xx} - \frac{1}{2}(I_{xx} + I_{yy}) \\
&= \frac{1}{2}(I_{xx} - I_{yy}), \\
I_{xy}^{\mathrm{TT}} &= \left(P_{xl}P_{ym} - \frac{1}{2}P_{xy}P_{lm}\right) I_{lm} = P_{xx}P_{yy}I_{xy} = I_{xy}, \\
I_{xz}^{\mathrm{TT}} &= \left(P_{xl}P_{zm} - \frac{1}{2}P_{xz}P_{lm}\right) I_{lm} = 0, \\
I_{xy}^{\mathrm{TT}} &= \left(P_{yl}P_{ym} - \frac{1}{2}P_{yy}P_{lm}\right) I_{lm} = I_{yy} - \frac{1}{2}(I_{xx} + I_{yy}) = \frac{1}{2}(I_{yy} - I_{xx}), \\
I_{yz}^{\mathrm{TT}} &= \left(P_{yl}P_{zm} - \frac{1}{2}P_{yz}P_{lm}\right) I_{lm} = 0, \\
I_{zz}^{\mathrm{TT}} &= \left(P_{zl}P_{zm} - \frac{1}{2}P_{zz}P_{lm}\right) I_{lm} = 0.
\end{aligned}
$$

- and

$$
\begin{aligned}
\bar{h}_{xx}^{\mathrm{TT}} &= \frac{2G}{rc^4} e^{i\Omega r} \frac{1}{2} \frac{d^2}{dt^2} (I_{xx} - I_{yy}), \\
\bar{h}_{xy}^{\mathrm{TT}} &= \frac{2G}{rc^4} e^{i\Omega r} \frac{1}{2} \frac{d^2}{dt^2} I_{xy}, \\
\bar{h}_{xz}^{\mathrm{TT}} &= 0, \\
\bar{h}_{yy}^{\mathrm{TT}} &= \frac{2G}{rc^4} e^{i\Omega r} \frac{1}{2} \frac{d^2}{dt^2} (I_{yy} - I_{xx}) = -\bar{h}_{xx}^{\mathrm{TT}}, \\
\bar{h}_{yz}^{\mathrm{TT}} &= 0, \\
\bar{h}_{zz}^{\mathrm{TT}} &= 0.
\end{aligned}
$$

- Finally, from $\dfrac{d^2 I_{jk}}{dt^2} = -\Omega^2 D_{jk} e^{-i\Omega t} = -\Omega^2 I_{jk},$
- The previous expressions reduce to

$$
\begin{aligned}
\bar{h}^{\mathrm{TT}}_{xx} &= -\frac{G}{rc^4}\Omega^2 e^{i\Omega r}(I_{xx} - I_{yy}), \\
\bar{h}^{\mathrm{TT}}_{xy} &= -\frac{2G}{rc^4}\Omega^2 e^{i\Omega r} I_{xy} = \bar{h}^{\mathrm{TT}}_{yx}, \\
\bar{h}^{\mathrm{TT}}_{xz} &= 0, \\
\bar{h}^{\mathrm{TT}}_{yy} &= -\bar{h}^{\mathrm{TT}}_{xx}, \\
\bar{h}^{\mathrm{TT}}_{yz} &= 0, \\
\bar{h}^{\mathrm{TT}}_{zz} &= 0.
\end{aligned}
$$

Detection of gravitational waves

- One of the most convenient ways to measure the distance to a distant object is by means of a radar: a pulse of electromagnetic radiation is sent towards the object and it is measured how long it takes to return after being reflected from the distant object. Dividing this time by two and multiplying it by c, we obtain the distance to the object. This method also provides an excellent way to measure distances even in curved spacetime and is the basis for laser interferometric gravitational wave detectors.

- Let's see how to use light to measure the distance between two freely falling objects. We will use the TT coordinate system. We will start, first, by considering for simplicity a wave traveling in the z direction with a single polarization, h_+, so that the metric is given by

$$ds^2 = -dt^2 + [1 + h_+(z - t)]dx^2 + [1 - h_+(t - z)]dy^2 + dz^2,$$

- Suppose, again for simplicity, that the two objects lie on the x-axis, one of them at the origin $x = 0$ and the other at the position $x = L$. In the TT coordinates, the objects remain in those coordinate positions. all the time. To make our measurement, the object at the origin sends a photon along the x-axis to the other object, which reflects it back. The first object measures the amount of proper time that has elapsed since the photon was emitted. Since a photon traveling along the x-axis moves along a null universe line ($ds^2 = 0$) with $dy = dz = 0$, we can calculate its effective velocity

$$\left(\frac{dx}{dt}\right)^2 = \frac{1}{1 + h_+}.$$

- Although this speed is not equal to one, relativity is not contradicted, since it is a coordinate speed. A photon emitted at time t_{ini} from the origin reaches a point of coordinate x at time $t(x)$. Integrating the effective speed of light from the previous equation, we obtain the coordinate time in which the photon reaches the object located at the far end, in the position $x = L$,

$$t_{\text{far}} = t_{\text{ini}} + \int_0^L [1 + h_+(t(x))]^{1/2} dx.$$

- This is an implicit equation, since the function we are looking for, $t(x)$, is inside the integral. We can use the fact that h_+ is small to integrate it, making $t(x) = t_{\rm ini} + x$ inside the integral and expanding the square root. The result is the following explicit equation

$$t_{\rm far} = t_{\rm ini} + L + \frac{1}{2}\int_0^L h_+(t_{\rm ini} + x)dx.$$

- After being reflected, the light returns to $x = 0$. Using the same argument, we can obtain the total round trip coordinate time:

$$t_{\rm final} = t_{\rm ini} + 2L + \frac{1}{2}\int_0^L h_+(t_{\rm ini} + x)dx + \frac{1}{2}\int_0^L h_+(t_{\rm ini} + L + x)dx.$$

- Since in the TT coordinates the coordinate time is the proper time, the above equation gives a quantity that can be measured.

- We are interested in using this equation in some way to measure the metric of the wave. The simplest thing is to derive $t_{\rm final}$ with respect to $t_{\rm ini}$, that is, to monitor the rate at which the lap time changes as the gravitational wave passes. The integral with respect to x will be an integral of the derivative of h_+ with respect to its argument, which simply produces h_+ again. Therefore, it has to

$$\frac{dt_{\rm final}}{dt_{\rm ini}} = 1 + \frac{1}{2}[h_+(t_{\rm ini} + 2L) - h_+(t_{\rm ini})].$$

- This result indicates that the rate of change of the return time depends only on the metric of the wave at the moment the photon was emitted and the moment it was received again at the origin. In particular, the amplitude of the wave when the photon is reflected off the farthest object plays no role.

- If instead of a single photon, a continuous electromagnetic wave of frequency ν is emitted from the origin, each crest of the wave can be interpreted as another ray of light or photon being sent and collected back. The derivative of the time it takes for these rays to return to the origin is not more than the change in the frequency of the electromagnetic wave,

$$\frac{dt_{\mathrm{final}}}{dt_{\mathrm{ini}}} = \frac{\nu_{\mathrm{final}}}{\nu_{\mathrm{ini}}}.$$

- Thus, changes in the redshift of the return wave, which we can monitor, are directly related to changes in the amplitude of the gravitational waves.

- The above discussion has assumed a special distribution of the objects and the wave: the wave traveled in the direction perpendicular to the separation of the two objects. If the wave were to travel at an arbitrary angle θ with the z axis in the xz plane, the derivative of the return time would involve the amplitude of the wave at the moment of reflection on the farthest object,

$$\frac{dt_{\rm final}}{dt_{\rm ini}} = 1+\frac{1}{2}[(1-\sin\theta)h_+(t_{\rm ini}+2L)-(1+\sin\theta)h_+(t_{\rm ini})+2\sin\theta(t_{\rm ini}+(1-\sin\theta)L)].$$