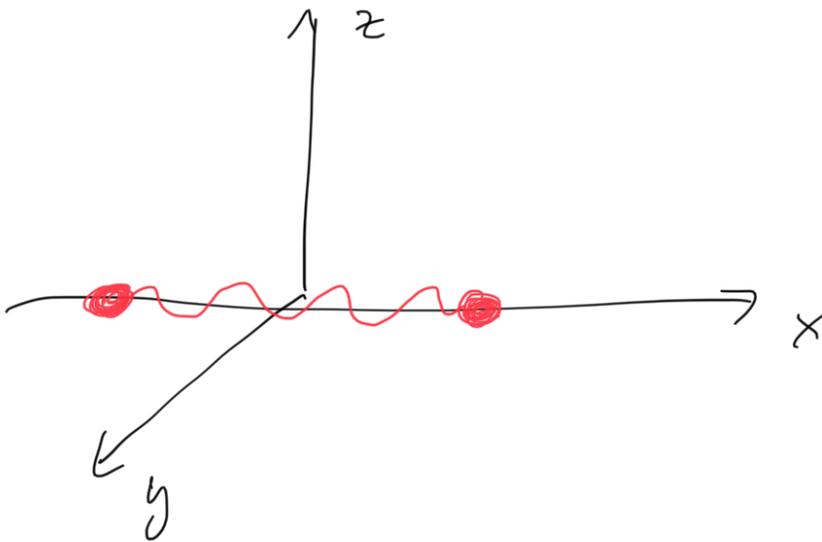

WORKSHOP ON COMPACT OBJECTS, GRAVITATIONAL WAVES AND DEEP LEARNING

Example 1: Gravitational wave emitted by an harmonic oscillator

We will estimate the GWs emitted during the oscillations of an harmonic oscillator composed by two masses

$$m_1 = m_2 = m$$

joined by a massless spring and separated a distance l_0 at rest.



The positions of the two masses are given by

$$x_1 = -\frac{1}{2} l_0 - A \cos(\omega t)$$

$$x_2 = \frac{1}{2} l_0 + A \cos(\omega t)$$

... and this

Let's compute the quadrupolar moment of this system

$$I^{lm} = \int T^{00} x^l x^m d^3x$$

For this discrete distribution of masses, the stress energy tensor is given by

$$T^{00} = \sum_{i=1}^2 m_i \delta(x - x_i) \delta(y) \delta(z)$$

where we have assumed low velocities,
 $v \ll c$

Therefore

$$I^{xx} = I_{xx} = \int m_1 \delta(x - x_1) x^2 \delta(y) \delta(z) d^3x + \int m_2 \delta(x - x_2) x^2 \delta(y) \delta(z) d^3x$$

$$= m(x_1^2 + x_2^2)$$

$$= m \left(\frac{1}{2} l_0^2 + 2A^2 \cos^2(\omega t) + 2Al_0 \cos(\omega t) \right)$$

$$= \text{const.} + mA^2 \cos(2\omega t) + 2mA l_0 \cos(\omega t)$$

where we have used $\cos(2\alpha) = 2\cos^2\alpha - 1$. The rest of the components of the quadrupole moment tensor are zero.

$$\begin{aligned} \int y^2 &= \int m_1 \delta(x-x_1) y^2 \delta(y) \delta(z) d^3x + \\ &\int m_2 \delta(x-x_2) y^2 \delta(y) \delta(z) d^3x \\ &= 0 \end{aligned}$$

since $\int \delta(y) y^2 dy = 0$.

Let's compute the gravitational wave that propagates in the z direction. In this case the normal unit vector the wave front is

$$\vec{n} = \frac{\vec{x}}{r} = (0, 0, 1)$$

and

$$P_{jk} = \delta_{jk} - n_j n_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The gravitational wave is given by

$$\bar{h}_{\mu 0}^{\tau\tau} = 0, \quad \mu = 0, \dots, 3$$

$$\bar{h}_{jk}^{\tau\tau} = \frac{2}{z} e^{i\Omega z} \frac{d^2}{dt^2} I_{jk}^{\tau\tau}$$

where

$$I_{jk}^{\tau\tau} = \mathcal{P}_{jk\ell m} I_{jk} = \left(P_{jl} P_{km} - \frac{1}{2} P_{jk} P_{\ell m} \right) I_{\ell m}$$

Since the only non-zero component of I_{ij} is I_{xx} , we have

$$\bar{I}_{jk}^{\tau\tau} = \left(P_{jx} P_{kx} - \frac{1}{2} P_{jk} P_{xx} \right) I_{xx} = \left(P_{jx} P_{kx} - \frac{1}{2} P_{jk} \right) I_{xx}$$

Therefore

$$\bar{I}_{xx}^{\tau\tau} = \left(P_{xx} P_{xx} - \frac{1}{2} P_{xx}^2 \right) I_{xx} = \frac{1}{2} I_{xx}$$

$$\bar{I}_{yy}^{\tau\tau} = \left(P_{yx} P_{xy} - \frac{1}{2} P_{yy} \right) I_{xx} = -\frac{1}{2} I_{xx}$$

$$\bar{I}_{xy}^{\tau\tau} = \left(P_{xy} P_{yx} - \frac{1}{2} P_{xy} \right) I_{xx} = 0$$

$$\bar{I}_{zz}^{\tau\tau} = \bar{I}_{zx}^{\tau\tau} = \bar{I}_{zy}^{\tau\tau} = 0$$

The gravitational wave that travels in the z direction is given by

$$\bar{h}_{\mu 0}^{\tau\tau} = 0, \quad \bar{h}_{xy}^{\tau\tau} = 0, \quad \bar{h}_{zi}^{\tau\tau} = 0$$

$$\bar{h}_{xx}^{\tau\tau} = -\bar{h}_{yy}^{\tau\tau} = \frac{e^{i\Omega z}}{z} \frac{d^2}{dt^2} I_{xx}$$

Since

$$\begin{aligned} I_{xx} &= \text{const.} + mA^2 \cos(2\omega t) + 2mA l_0 \cos(\omega t) \\ &= \text{const.} + \text{Re} [2mA l_0 e^{-i\omega t}] \\ &\quad + \text{Re} [mA^2 e^{-i2\omega t}] \end{aligned}$$

we have two frequencies to take into account for Ω , ω and 2ω .

$$\bar{h}_{xx}^{\tau\tau} = \frac{e^{i\omega z}}{z} \frac{d^2}{dt^2} (2mA l_0 e^{-i\omega t})$$

$$\begin{aligned}
 \ddot{x}_x &= \frac{1}{z} \frac{d^2}{dt^2} \left(m A^2 e^{-i2\omega t} \right) \\
 &+ \frac{e^{i2\omega z}}{z} \frac{d^2}{dt^2} \left(m A^2 e^{-i2\omega t} \right) \\
 &= -2m A l_0 \omega^2 \frac{e^{i\omega(z-t)}}{z} - 4m A^2 \omega^2 \frac{e^{i2\omega(z-t)}}{z}
 \end{aligned}$$

We can estimate the gravitational radiation generated by this kind of system in the laboratory, with the following (extreme) numbers: $m = 10^3 \text{ kg}$, $l_0 = 1 \text{ m}$, $A = 10^{-4} \text{ m}$ and $\omega = 10^4 \text{ Hz}$. We have then:

$$\overline{h_{xx}} \approx -\frac{2Gm}{c^4 z} \omega^2 A l_0 \cos(\omega(t-z)) \approx \frac{10^{-35}}{z}$$