PyR@TE 3

Renormalization group equations for general gauge theories

Lohan Sartore

[LS, I. Schienbein, arXiv:2007.12700; https://github.com/LSartore/pyrate]

Tools 2020

PyR@TE 3: What for ?

→ Computation of RGEs for general (4D, non-SUSY) gauge theories

→ Essential tool for perturbative studies of a wide range of phenomena:

- Study of RG flows (asymptotic behavior, critical exponents, IR/UV fixed points, ...)
- Gauge coupling unification, general GUT studies
- Study and stability of (effective) scalar potentials
- High scale boundary conditions
- ...

General (non-SUSY) RGEs: a brief history

Two-loop results (80's):

- M.E. Machacek, M.T. Vaughn, Nucl. Phys. B222, 83 (**1983**)
- M.E. Machacek, M.T. Vaughn, Nucl. Phys. B236, 221 (**1984**)
- M.E. Machacek, M.T. Vaughn, Nucl. Phys. B249, 709 (**1985**)

Some corrections + extension to dimensionful couplings:

- M.-x. Luo, H.-w. Wang and Y. Xiao, Phys. Rev. D67 (**2003**)

Kinetic mixing:

- M. Luo, Y. Xiao, Phys. Lett. B 555, 279 (**2003**)
- R. M. Fonseca, M. Malinsky, F. Staub, Phys. Lett. B 726, 882 (**2013**)

} Implemented in SARAH and PyR@TE 2

Some corrections + correct treatment for off-diagonal W.F. renormalization:

- I. Schienbein, F. Staub, T. Steudtner, K. Svirina, Nucl. Phys. B939 (**2019**)

3-loop gauge coupling RGEs for theories based on a simple group:

- A. Pickering, J. Gracey, D. Jones, Phys. Lett. B 510 (**2001**)

Beyond 2-loops: recent developments

In a recent paper [C. Poole, A. E. Thomsen, arXiv:1906.04625]:

Constraints on 3- and 4-loop β-functions in a general four-dimensional Quantum Field Theory

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Abstract

The β-functions of marginal couplings are known to be closely related to the A-function through Osborn's equation, derived using the local renormalization group. It is possible to derive strong constraints on the β-functions by parametrizing the terms in Osborn's equation as polynomials in the couplings, then eliminating unknown $\tilde{A}$ and T_{IJ} coefficients. In this paper we extend this program to completely general gauge theories with arbitrarily many Abelian and non-Abelian factors. We detail the computational strategy used to extract consistency conditions on β-functions, and discuss our automation of the procedure. Finally, we implement the procedure up to 4-, 3-, and 2-loops for the gauge, Yukawa and quartic couplings respectively, corresponding to the present forefront of general β-function computations. We find an extensive collection of highly non-trivial constraints, and argue that they constitute an useful supplement to traditional perturbative computations; as a corollary, we present the complete 3-loop gauge β-function of a general QFT in the $\overline{\text{MS}}$ scheme, including kinetic mixing.

- 3-loop general gauge coupling RGEs
- Partial results at order 4-3-2 (+ full 4-loop gauge coupling RGEs in the SM)
- Fixed the γ_5 ambiguity at order 4-3-2

New formalism + Weyl consistency conditions

General gauge theory: Lagrangian density

- Gauge group $\mathcal{G} = \prod \mathcal{G}_u$

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General gauge theory: Lagrangian density

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- Real scalars ϕ_a

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$$\mathcal{L} = -\frac{1}{4}\left(G^{-2}\right)_{AB} F^A_{\mu\nu} F^{B\,\mu\nu} + \frac{1}{2}(D_\mu\phi)_a(D^\mu\phi)_a + \frac{i}{2}\Psi^{\mathrm{T}}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} D_\mu\Psi$$
$$-\frac{1}{2}y_{aij}\Psi_i\Psi_j\,\phi_a - \frac{1}{2}m_{ij}\Psi_i\Psi_j$$
$$-\frac{1}{2}\mu_{ab}\phi_a\phi_b - \frac{1}{3!}t_{abc}\phi_a\phi_b\phi_c - \frac{1}{4!}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d\,.$$

$$D_\mu\Psi_i = \partial_\mu\Psi_i - i\sum_A V^A_\mu\left(T^A_\Psi\right)_{ij}\Psi_j$$
$$D_\mu\phi_a = \partial_\mu\phi_a - i\sum_A V^A_\mu\left(T^A_\phi\right)_{ab}\phi_b$$

General gauge theory: β-functions

Perturbative expansion:

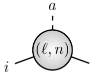
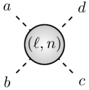
$$\text{Gauge:}\quad \beta_{AB} \equiv \frac{dG^2_{AB}}{dt} = \frac{1}{2}\sum_{\text{perm}}\sum_{\ell}\frac{1}{(4\pi)^{2\ell}}\, G^2_{AC}\,\beta^{(\ell)}_{CD}\,G^2_{DB}\,,$$

$$\text{Yukawa:}\quad \beta_{aij} \equiv \frac{dy_{aij}}{dt} = \frac{1}{2}\sum_{\text{perm}}\sum_{\ell}\frac{1}{(4\pi)^{2\ell}}\,\beta^{(\ell)}_{aij}\,,$$

$$\text{Quartic:}\quad \beta_{abcd} \equiv \frac{d\lambda_{abcd}}{dt} = \frac{1}{4!}\sum_{\text{perm}}\sum_{\ell}\frac{1}{(4\pi)^{2\ell}}\,\beta^{(\ell)}_{abcd}$$

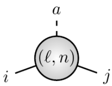
At fixed loop order, factorization of scheme dependence / model dependence:

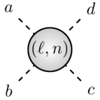
$$\beta^{(\ell)}_{AB} = \sum_n \mathfrak{g}^{(\ell)}_n$$  ,

$$\beta^{(\ell)}_{aij} = \sum_n \mathfrak{y}^{(\ell)}_n$$  , and $$\beta^{(\ell)}_{abcd} = \sum_n \mathfrak{q}^{(\ell)}_n$$ 

β-functions and Weyl consistency conditions

$\beta_{AB}^{(\ell)} = \sum_n \mathfrak{g}_n^{(\ell)}$ A  B ,

$\beta_{aij}^{(\ell)} = \sum_n \mathfrak{y}_n^{(\ell)}$ 

, and $\beta_{abcd}^{(\ell)} = \sum_n \mathfrak{q}_n^{(\ell)}$ 

$$\beta_{abcd}^{(1)} = \mathfrak{q}_1^{(1)}\,(T_\phi^A T_\phi^C)_{ab} G_{AB}^2 G_{CD}^2 (T_\phi^B T_\phi^D)_{cd} \quad + \mathfrak{q}_2^{(1)}\,[C_2(S)]_{ae}\lambda_{ebcd} \quad + \mathfrak{q}_3^{(1)}\,\lambda_{abef}\lambda_{efcd}$$
$$+\mathfrak{q}_4^{(1)}\,[Y_2(S)]_{ae}\lambda_{ebcd} \quad + \mathfrak{q}_5^{(1)}\,\mathrm{Tr}[y_a \tilde{y}_b y_c \tilde{y}_d]$$

Weyl consistency conditions

Constraints involving $\mathfrak{g}^{(\ell)}$, $\boldsymbol{\eta}^{(\ell-1)}$ and $\mathfrak{q}^{(\ell-2)}$

- Valid in any renormalization scheme
- Non-trivial check of existing results
- Derive missing coefficients at higher loop orders!
  e.g. 3-loop gauge from 2-loop Yukawa + 1-loop quartic

PyR@TE 3: Motivations

In summary, we've got a new, compact, easy-to-implement formalism...

- Gauge coupling RGEs up to 3-loop + partial 4-3-2 results
- Natural embedding of semi-simple groups & gauge kinetic mixing
- One missing piece though: RGEs for dimensionful couplings.
  $\rightarrow$ Computed in this new formalism in [LS, arXiv:2006.12307]

In addition ...

- PyR@TE 2 not maintained since 2017 (some bugs and implementation flaws)
- In PyR@TE 2 (and SARAH), execution time increases drastically with the complexity of the model
- Complex models can get quite cumbersome to implement

PyR@TE 3: What's new

- To a great extent rewritten from scratch
- Python 2.7 $\rightarrow$ Python 3.6+
- A number of new features (see next slide)

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- 3-loop gauge coupling RGEs
- New model file syntax
- Performance **drastically** improved [$\mathcal{O}(100)$ to $\mathcal{O}(10\,000)$ times faster]

PyR@TE 3: What's new

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Model	Loop order	PyR@TE 2	PyR@TE 3
SM B-L	1	114	1.5
	2	8823	11
	2 + 3 (gauge)	/	23
SM + complex triplet	1	385	1.0
	2	**59936**	**3.2**
	2 + 3 (gauge)	/	5.7
SM + scalar singlet	1	79	0.9
	2	5765	4.3
	2 + 3 (gauge)	/	5.6
SM + complex doublet	1	153	1.2
	2	**39666**	**6.2**
	2 + 3 (gauge)	/	9.4
SM + Majorana triplet + Vectorlike doublet	1	262	1.3
	2	15653	10.7
	2 + 3 (gauge)	/	13.2

Execution time (in seconds)

New features: quick overview

Available to the user...

- Gauge invariance check of the Lagrangian
- RGEs for vacuum-expectation values
- Assumptions on the form of Yukawa matrices
- GUT normalizations, substitutions of couplings (e.g. $\alpha = \frac{g^2}{4\pi}$)
- Improved output
- Real time output & progress bars

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On the technical side...

- **Optimized tensor algebra: sparse tensors, efficient tensor contraction**
- Optimization of PyLie's algorithms
- A new interface between PyLie and PyR@TE: PyLieDB
- The program is overall more robust against bugs & implementation errors

PyR@TE 3: overview

Input

- Gauge group
- Particle content
- Lagrangian

↓

PyR@TE model file

►

Mapping on the general formalism

Fill ϕ_a and Ψ_i

Fill G_{AB}, $\underbrace{(T^A_\phi)_{ab} \text{ and } (T^A_\Psi)_{ij}}_{\text{PyLie}}$

Expand Lagrangian

↓

Fill $y_{aij}, \lambda_{abcd}, t_{abc}, \mu_{ab}, m_{ij}$

►

Compute the RGEs

Minimal set of linearly independent couplings

$$\tilde{\mathbf{g}} = M\mathbf{g}$$

Compute the $\beta(\tilde{g}_i)$'s

Infer

$$\beta(g_i) = (M^{-1})^{ij}\beta(\tilde{g}_j)$$

Output

- LaTeX
- Mathematica (expressions + RGE solver)
- Python (expressions + RGE solver)

The model file

- Gauge group, particle content:

```
Author: Lohan Sartore
Date: 08.06.2020
Name: SM
Groups: {U1Y: U1, SU2L: SU2, SU3c: SU3}


Fermions: {
    Q : {Gen: 3, Qnb: {U1Y: 1/6, SU2L: 2, SU3c: 3}},
    L : {Gen: 3, Qnb: {U1Y: -1/2, SU2L: 2}},
    uR : {Gen: 3, Qnb: {U1Y: 2/3, SU3c: 3}},
    dR : {Gen: 3, Qnb: {U1Y: -1/3, SU3c: 3}},
    eR : {Gen: 3, Qnb: {U1Y: -1}},
}

RealScalars: {
}

ComplexScalars: {
    H : {RealFields: [Pi, Sigma], Norm: 1/sqrt(2), Qnb: {U1Y: 1/2, SU2L: 2}},
}
```

The model file: new syntax

- Lagrangian (Yukawa couplings & Scalar potential): syntax *à la* FeynRules

```
Potential: {

    Definitions: {
        Htilde[i] : Eps[i,j]*Hbar[j]
    },


    Yukawas: {
        Yu : Qbar[i,a] Htilde[i] uR[a],
        Yd : Qbar[i,a] H[i] dR[a],
        Ye : Lbar[i] H[i] eR
    },



    QuarticTerms: {
        lambda : (Hbar[i] H[i])**2

        # or any other normalization of your choosing...
        lambda : 1/2 (Hbar[i] H[i])**2
    },

    ScalarMasses: {
        mu : -Hbar[i] H[i]
    }
}
```

$$\widetilde{H}^i = \varepsilon^{ij} H_j^*$$

$$\begin{aligned}\mathcal{L}_Y &= Y_u\, \bar{Q}_{i,a}\, \widetilde{H}^i\, u_R^a \\ &+ Y_d\, \bar{Q}_{i,a}\, H^i\, d_R^a \\ &+ Y_e\, \bar{L}_a\, H^i\, e_R\end{aligned}$$

$$\begin{aligned}\mathcal{V} &= \lambda \left(H_i^\dagger H^i\right)^2 \\ &- \mu\, H_i^\dagger H^i\end{aligned}$$

The model file: new syntax

- The `Definitions` section, example of SM + complex triplet

```
ComplexScalars: {
    H : {RealFields: [Pi, Sigma], Norm: 1/sqrt(2), Qnb: {U1Y: 1/2, SU2L: 2}},
    delta : {RealFields: [dR, dI], Norm: 1/sqrt(2), Qnb: {U1Y: 1, SU2L: 3}},
}

Potential: {

    Definitions: {
        # Define the generators of the fundamental SU(2) rep
        tFund : t(SU2, 2),

        # Define the matrix Delta and its adjoint
        Delta[i,j] : tFund[a,i,j] delta[a],
        DeltaDag[i,j] : tFund[a,i,j] deltabar[a]
    }


    QuarticTerms: {
        lambda1 : (Hbar[i] H[i])**2,

        lambda2 : Hbar[i] H[i] * Delta[j,k] DeltaDag[k,j],

        lambda3 : Hbar[i] Delta[i,j] DeltaDag[j,k] H[k],

        lambda4 : (Delta[i,j] DeltaDag[j,i])**2,

        lambda5 : (Delta[i,j] DeltaDag[j,k] Delta[k,l] DeltaDag[l,i])
    }
}
```

$$H \sim (\frac{1}{2}, \mathbf{2}, \mathbf{1}), \quad \delta \sim (1, \mathbf{3}, \mathbf{1})$$

$SU(2)$ generator: $(t^a)^i_{\ j}$

$$\Delta^i_{\ j} = \delta_a (t^a)^i_{\ j}$$

$$(\Delta^\dagger)^i_{\ j} = \delta^*_a (t^a)^i_{\ j}$$

$$\begin{aligned}\mathcal{V} = &\lambda_1 (H^\dagger H)^2 \\ &+ \lambda_2 (H^\dagger H) \mathrm{Tr}(\Delta^\dagger \Delta) \\ &+ \lambda_3\, H^\dagger \Delta \Delta^\dagger H \\ &+ \lambda_4\, \mathrm{Tr}(\Delta^\dagger \Delta)^2 \\ &+ \lambda_5\, \mathrm{Tr}(\Delta^\dagger \Delta \Delta^\dagger \Delta)\end{aligned}$$

In the future...

Higher order results (most likely) coming soon

- Partially implemented, tests ongoing
- “Factorially" increasing number of contributions
  $\rightarrow$ Keep reasonable (and competitive !) execution times
- Parallelization showed promising results (≈ 2 times faster on 4 cores)

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Useful for parameter scans where a running step is involved

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Implementation of tensor representations ?

e.g. in $SU(3)$:

- $\mathbf{6} = (\mathbf{3} \otimes \mathbf{3})_S$

e.g. in $SO(10)$:

- $\mathbf{45} = (\mathbf{10} \otimes \mathbf{10})_A$
- $\mathbf{54} = (\mathbf{10} \otimes \mathbf{10})_S$

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User feedback $\rightarrow$ improvements & new features !

Thank you for your attention

General gauge theory: Gauge coupling matrix

The gauge coupling matrix G^2_{AB}:

SM

$$G^2_{AB} = \begin{pmatrix} g_1^2 & & & & & & \\ & g_2^2 & & & & & \\ & & g_2^2 & & & & \\ & & & g_2^2 & & & \\ & & & & g_3^2 & & \\ & & & & & \ddots & \\ & & & & & & g_3^2 \end{pmatrix}_{AB}$$

SM $\times\, U(1)_{B-L}$

$$G^2_{AB} = \begin{pmatrix} G_1 \cdot G_1^{\mathrm{T}} & & & & & & \\ & g_2^2 & & & & & \\ & & g_2^2 & & & & \\ & & & g_2^2 & & & \\ & & & & g_3^2 & & \\ & & & & & \ddots & \\ & & & & & & g_3^2 \end{pmatrix}_{AB}$$

$$G_1 = \begin{pmatrix} g_Y & g_{\text{mix}} \\ 0 & g_{B-L} \end{pmatrix}$$

Gauge indices A and B run over all the gauge bosons of the theory.

SM: $A, B \in \{1_{U(1)}, 1_{SU(2)}, 2_{SU(2)}, 3_{SU(2)}, 1_{SU(3)}, \ldots, 8_{SU(3)}\}$

SM $\times\, U(1)_{B-L}$: $A, B \in \{1_{U(1)_Y}, 1_{U(1)_{B-L}}, 1_{SU(2)}, 2_{SU(2)}, 3_{SU(2)}, 1_{SU(3)}, \ldots, 8_{SU(3)}\}$