Leading hadronic contribution to the muon magnetic moment from lattice QCD

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Budapest–Marseille–Wuppertal-collaboration

[2002.12347v2]

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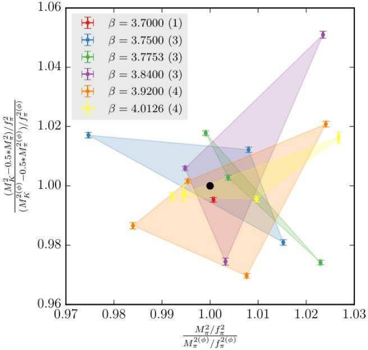
Outline

1. Simulation setup
2. Improvements
3. Continuum limit
4. Window

Simulations

- Tree-level Symanzyk gauge action
- $N_f = 2+1+1$ staggered fermions
- stout smearing 4 steps, $\varrho = 0.125$
- $L \sim 6\,\text{fm}$, $T \sim 9\,\text{fm}$
- M_π and M_K are around physical point

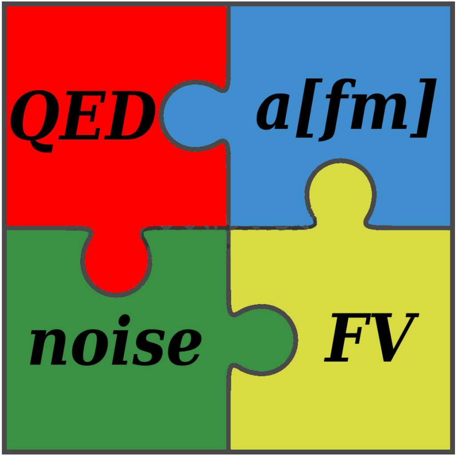


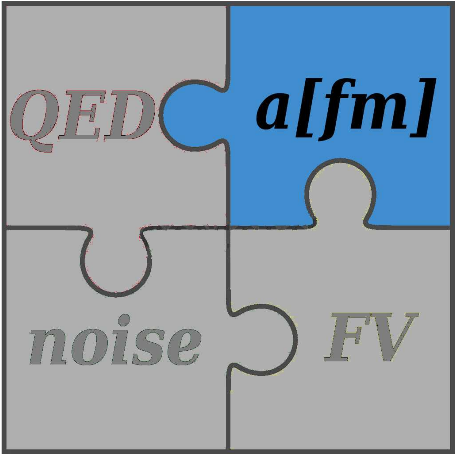
β	a[fm]	$L \times T$	#conf
3.7000	0.1315	48×64	904
3.7500	0.1191	56×96	2072
3.7753	0.1116	56×84	1907
3.8400	0.0952	64×96	3139
3.9200	0.0787	80×128	4296
4.0126	0.0640	96×144	6980

- Ensembles for dynamical QED

β	a[fm]	$L \times T$	#conf
3.7000	0.1315	24×48	716
		48×64	300
3.7753	0.1116	28×56	887
3.8400	0.0952	32×64	4253

Improvements since 1711.04980




QED
a[fm]
noise
FV

Scale determination

Scale enters into a_μ determination:

- physical value of m_μ
- physical values of m_π, m_K

$\longrightarrow$ $\Delta_{\text{scale}} a_\mu \sim 2 \cdot \Delta(\text{scale})$

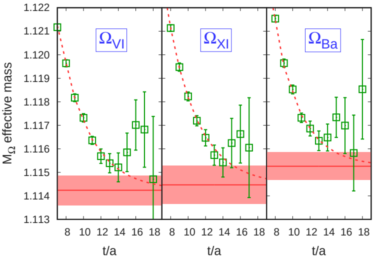
1. For final results: M_Ω scale setting
   - Experimentally well known: 1672.45(29) MeV [PDG 2018]
   - Moderate m_q dependence
   - Can be precisely determined on the lattice
2. For separation of isospin breaking effects: w_0 scale setting
   - Moderate m_q dependence
   - Can be precisely determined on the lattice
   - No experimental value

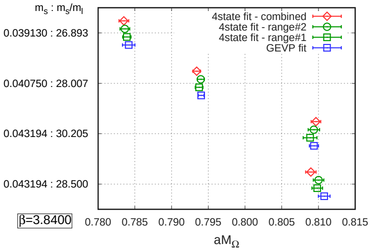
     $\longrightarrow$ Determine value of w_0 from $M_\Omega \cdot w_0$

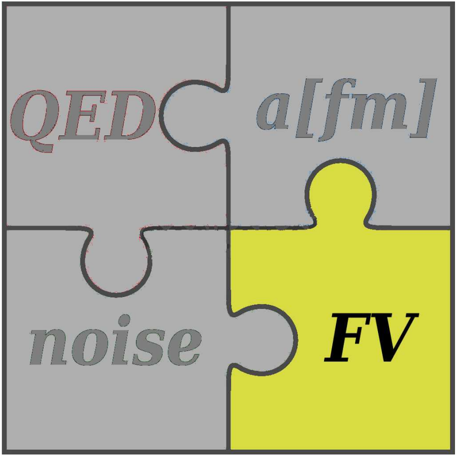
$$w_0 = 0.17236(29)(63)[70] \text{ fm}$$

M_Ω determination

- Staggered baryon operators [Golterman & Smit 1985] [Bailey 2007]
- 2 fit ranges with 4-state fits
- mass extraction using GEVP [Aubin & Orginos 2011] [DeTar & Lee 2015]
- include $\mathcal{O}(e^2)$ dynamical QED effects
- $\approx 0.1\%$ precision on each ensemble



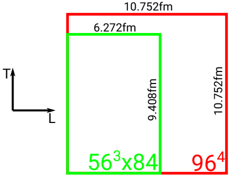



QED
a[fm]
noise
FV

FV: lattice

$L_{\mathrm{ref}} = 6.272\,\mathrm{fm} \qquad T_{\mathrm{ref}} = \frac{3}{2} L_{\mathrm{ref}}$

$L_{\mathrm{big}} = T_{\mathrm{big}} = 10.752\,\mathrm{fm}$

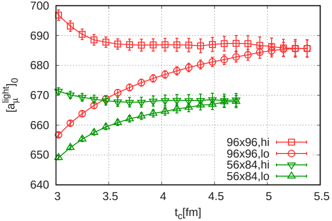


FV correction in two steps

$$a_\mu(\infty,\infty) - a_\mu(L_{\mathrm{ref}}, T_{\mathrm{ref}}) = \\ = [a_\mu(L_{\mathrm{big}}, T_{\mathrm{big}}) - a_\mu(L_{\mathrm{ref}}, T_{\mathrm{ref}})]_{\mathtt{4HEX}} + \\ + [a_\mu(\infty,\infty) - a_\mu(L_{\mathrm{big}}, T_{\mathrm{big}})]_{\mathrm{NNLO}}$$

1. $a_\mu(L_{\mathrm{big}}, T_{\mathrm{big}}) - a_\mu(L_{\mathrm{ref}}, T_{\mathrm{ref}})$
   Choose action with small taste splitting
   - 4 steps of HEX smearing
   - DBW2 gauge action
   - $\beta = 0.73$, $a = 0.112\,\mathrm{fm}$
   - $M_\pi = 104\,\mathrm{MeV}$ and $M_\pi = 121\,\mathrm{MeV}$
   - Interpolate to $M_\pi = 110\,\mathrm{MeV}$

   $$\longrightarrow \quad M_{\pi,\mathrm{HMS}}^{-2} \equiv \frac{1}{16} \sum_\alpha M_{\pi,\alpha}^{-2} = M_{\pi^0,\mathrm{phys}}^{-2}.$$



$$a_\mu(L_{\mathrm{big}}, T_{\mathrm{big}}) - a_\mu(L_{\mathrm{ref}}, T_{\mathrm{ref}}) = 18.1(2.0)_{\mathrm{stat}}(1.4)_{\mathrm{cont}}$$

FV: non-lattice

Comparison to non-lattice approaches

- NLO and NNLO Chiral perturbation theory

  [Gasser & Leutwyler 1985] [Bijnens *et.al.* 1999]

- MLLGS-model

  [Gounaris & Sakurai 1968] [Lellouch & Lüscher 2001] [Meyer 2011] [Francis *et.al.* 2013]

- Hansen–Patella approach

  [Hansen & Patella 2019,2020]

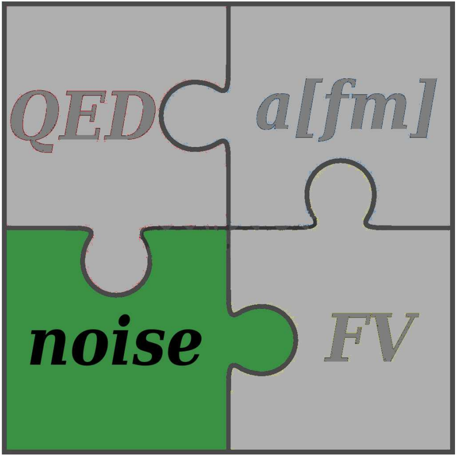
	NLO XPT	NNLO XPT	MLLGS	HP
$a_\mu(L_{\rm big}, T_{\rm big}) - a_\mu(L_{\rm ref}, T_{\rm ref})$	11.6	15.7	17.8	–
$a_\mu(L_{\rm big}, \infty) - a_\mu(L_{\rm ref}, \infty)$	11.2	15.3	17.4	16.3

$$a_\mu(L_{\rm big}, T_{\rm big}) - a_\mu(L_{\rm ref}, T_{\rm ref}) = 18.1(2.0)_{\rm stat}(1.4)_{\rm cont}$$

2. $a_\mu(\infty, \infty) - a_\mu(L_{\rm big}, T_{\rm big})$

	NLO XPT	NNLO XPT	HP
$a_\mu(\infty, \infty) - a_\mu(L_{\rm big}, T_{\rm big})$	0.3	0.6	-
$a_\mu(\infty, \infty) - a_\mu(L_{\rm big}, \infty)$	1.2	1.4	1.4

$$a_\mu(\infty, \infty) - a_\mu(L_{\rm ref}, T_{\rm ref}) = 18.7(2.0)_{\rm stat}(1.4)_{\rm cont}(0.3)_{\rm big}(0.6)_{I=0}(0.1)_{\rm qed}[2.5]$$


QED
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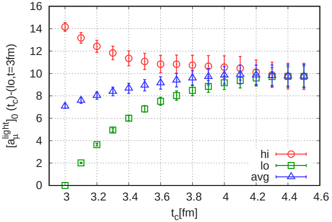
Upper and lower bounds on $\langle JJ \rangle$

- Connected light propagator [Lehner 2016] [Borsanyi *et.al.* 2017]

$$0 \leq G^{\text{light}}(t) \leq G^{\text{light}}(t_c)\frac{\varphi(t)}{\varphi(t_c)} \qquad t_c = 4.0\,\text{fm}$$

for finite-T: $$\varphi(t) = \frac{\cosh[E_{2\pi}(t - T/2)] + 1}{\cosh(E_{2\pi}T/2) - 1}$$



- Disconnected propagator

$$0 \leq -G^{\text{disc}}(t) \leq \frac{1}{10}G^{\text{light}}(t_c)\frac{\varphi(t)}{\varphi(t_c)} + G^{\text{strange}}(t) + G^{\text{charm}}(t)$$

$t_c = 2.5\,\text{fm}$

Low Mode Averaging

Connected light propagator [Neff *et.al.* 2001] [Giusti et.al. 2004] [Li *et.al.* 2010] ...

$$C(t,\bar{t}) = -\frac{1}{12L^3}\sum_{\mu=1}^{3}\mathrm{Re}\,\mathrm{Tr}\left[S_{\mu,t}M^{-1}S_{\mu,\bar{t}}M^{-1}\right]$$

$S_{\mu,t}$: covariant shift in timeslice t

Using lowest eigenvectors v_i and eigenvalues λ_i, split M^{-1}

$$M^{-1} = M_\mathrm{e}^{-1} + M_\mathrm{r}^{-1},$$

$$M_\mathrm{e}^{-1} = \sum_i \frac{1}{\lambda_i} v_i v_i^\dagger \qquad \text{and} \qquad M_\mathrm{r}^{-1} = M^{-1}\left(1 - \sum_i v_i v_i^\dagger\right),$$

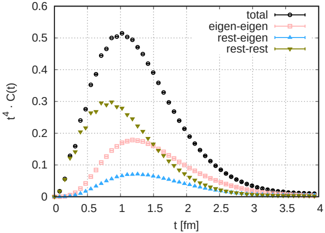
$$\begin{aligned} C_\mathrm{ee} &\sim \mathrm{Tr}\, SM_\mathrm{e}^{-1}SM_\mathrm{e}^{-1} &\longrightarrow&\quad \text{computed exactly} \\ C_\mathrm{re} &\sim \mathrm{Tr}\, SM_\mathrm{r}^{-1}SM_\mathrm{e}^{-1} &\longrightarrow&\quad v_i^\dagger SM_\mathrm{r}^{-1}Sv_i \\ C_\mathrm{rr} &\sim \mathrm{Tr}\, SM_\mathrm{r}^{-1}SM_\mathrm{r}^{-1} &\longrightarrow&\quad \xi^\dagger SM_\mathrm{r}^{-1}S\xi \quad \text{with random sources } \xi \end{aligned}$$

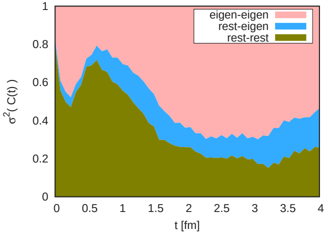
C_re and C_rr: AMA/Truncated Solver method [Bali *et.al.* 2010] [Blum *et.al.* 2013]

- Stop after 400 iterations
- Compute correction to high precision on every 32$^\mathrm{nd}$ vector

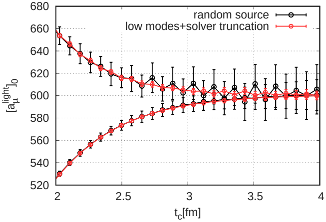
Low Mode Averaging

- $L = 6\,\mathrm{fm}$ ≈ 1000 eigenvectors up to $\approx m_s/2$

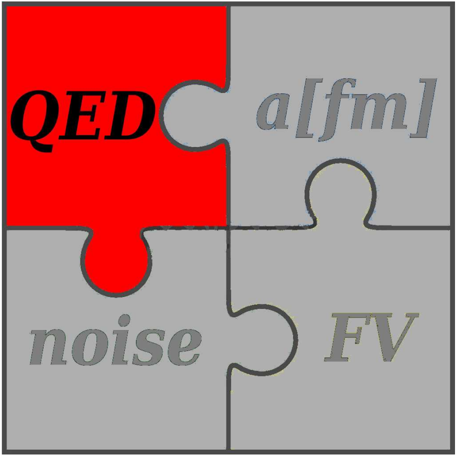




- factor 5 gain in precision
- bounding t_c: $3\,\mathrm{fm} \to 4\,\mathrm{fm}$



- $L = 11\,\mathrm{fm}$ ≈ 6000 eigenvectors


QED
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FV

QCD+QED

- Rewrite dynamical QED as quenched QED expectation values

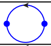
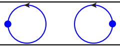
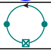
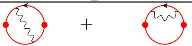
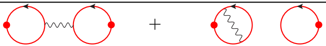
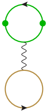
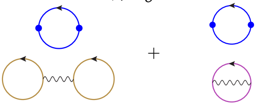
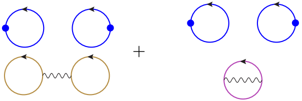
$$\langle \mathcal{O} \rangle_{\text{QCD+QED}} = \frac{\left\langle \left\langle \mathcal{O}(U,A) \frac{\det M(U,A)}{\det M(U,0)} \right\rangle_{A,\text{quenched}} \right\rangle_U}{\left\langle \left\langle \frac{\det M(U,A)}{\det M(U,0)} \right\rangle_{A,\text{quenched}} \right\rangle_U}$$

- Expansion up to $\mathcal{O}(e^4, e^2\delta m, \delta m^2)$ e_v: valence charge, e_s: sea charge
  - $\mathcal{O}(U,A) \approx \mathcal{O}_0(U) + \frac{\delta m}{m_l} \cdot \mathcal{O}'_m(U) + e_v \cdot \mathcal{O}'_1(U,A) + e_v^2 \cdot \mathcal{O}''_2(U,A)$
  - $\left(\prod_{f=u,d,s,c} \frac{\det M^{(f)}[U,A]}{\det M^{(f)}[U,0]} \right)^{1/4} \approx 1 + e_s \cdot \frac{d_1(U,A)}{d_0(U)} + e_s^2 \cdot \frac{d_2(U,A)}{d_0(U)}$
  - $m_u = m_l - \frac{\delta m}{2}, \quad m_d = m_l + \frac{\delta m}{2} \quad \longrightarrow \quad \mathcal{O}(\delta m)$ sea effect vanishes
- Strategy:
  - Take isospin symmetric gluon configurations: U
  - Measure $\mathcal{O}_0(U)$ and $\mathcal{O}'_m(U)$
  - For each gluon field, generate *quenched* photon fields: A
  - Measure $\mathcal{O}'_1(U,A)$, $\mathcal{O}''_2(U,A)$, $\frac{d_1(U,A)}{d_0(U)}$ and $\frac{d_2(U,A)}{d_0(U)}$

[De Divitiis *et.al.* 2013] [Eichten *et.al.* 1997]

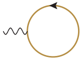
Contributions

$$\langle\mathcal{O}\rangle_{\mathrm{QCD+QED}} \approx \langle\mathcal{O}\rangle\big|_0 + \frac{\delta m}{m_l}\cdot\langle\mathcal{O}\rangle'_m + e_v^2\cdot\langle\mathcal{O}\rangle''_{20} + e_v e_s\cdot\langle\mathcal{O}\rangle''_{11} + e_s^2\cdot\langle\mathcal{O}\rangle''_{02}$$

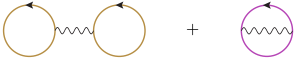
	quark-connected HVP	quark-disconnected HVP
$\langle\mathcal{O}\rangle\big\|_0 = \langle\mathcal{O}_0(U)\rangle_U$		
$\langle\mathcal{O}\rangle'_m = \left\langle\mathcal{O}'_m(U)\right\rangle_U$		
$\langle\mathcal{O}\rangle''_{20} = \left\langle\left\langle\mathcal{O}''_2(U,A)\right\rangle_{A,\mathrm{q.}}\right\rangle_U$		
$\langle\mathcal{O}\rangle''_{11} = \left\langle\left\langle\mathcal{O}'_1(U,A)\cdot\frac{d_1(U,A)}{d_0(U)}\right\rangle_{A,\mathrm{q.}}\right\rangle_U$		
$\langle\mathcal{O}\rangle''_{02} = \left\langle\left(\mathcal{O}_0(U) - \langle\mathcal{O}_0(U)\rangle_U\right)\cdot\left\langle\frac{d_2(U,A)}{d_0(U)}\right\rangle_{A,\mathrm{q.}}\right\rangle_U$		

Measurement

- $\mathcal{O}_1'(U,A)$ and $\mathcal{O}_2''(U,A)$: compute as finite differences
  - Measure $\mathcal{O}(0)$, $\mathcal{O}(\frac{1}{3}e_*)$, $\mathcal{O}(-\frac{1}{3}e_*)$
- $\mathcal{O}_m'(U)$
  - light connected: as derivative
  - disconnected: as finite difference: $\mathcal{O}(m_l,0)$, $\mathcal{O}(0.9\,m_l,0)$
- $\frac{d_1(U,A)}{d_0(U)}$ =



  - One photon field on each gluon configuration: same as in $\mathcal{O}_1'(U,A)$
  - Exact trace on low-lying eigenspace
  - 12000 random sources
  - Reduce UV noise by exact rewriting using HPE
- $\frac{d_2(U,A)}{d_0(U)}$ =



  - 2000 photon fields on each gluon configuration
  - 12 random sources on each photon field
  - Contact term is treated as d_1/d_0

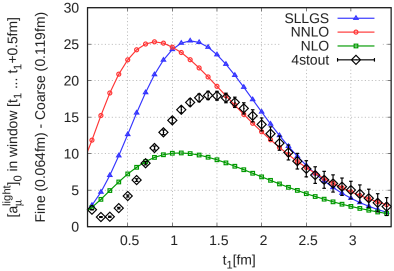
Continuum limit

Taste improvement: how to describe cutoff effects

- Leading cutoff effect in staggered: pion tower shrinking as $a \to 0$
  $a = 0.134\,\text{fm}$: 135 – 240 – 310 – 370 – 420 MeV
  $a = 0.064\,\text{fm}$: 135 – 145 – 155 – 165 – 175 MeV
- For rho meson this effect is far smaller (relatively)

Discretization effects are mostly taste effects, which are mass effects.

- Let us look at the infrared region to describe the cutoff (=mass) effects, two independent methods
  1. NNLO Staggered XPT [Lee, Sharpe, Van de Water, Bailey *et.al.*]
  $\longrightarrow$ depends only on 1 LEC (l_6) and masses of pion tastes
  2. Staggered MLLGS
  $\longrightarrow$ apply taste averaging on MLLGS



Taste improvement: long distance region

- We are not dealing with the total value of a_μ but just with the a_μ in the long distance region (window 3).
- We are not even dealing with the total value of a_μ at long distances (window 3) but merely with its lattice spacing dependence = pion mass dependence.
- One expects that this procedure describes most of the physics behind taste violation. This expectation is confirmed by our data.
- The remaining small effects are taken care of by a linear plus quadratic extrapolation in a^2.

Global fit procedure

- For full result: physical point is set via
  $M_\Omega, \quad M^2_{K_\chi} = \frac{1}{2}\left(M^2_{K_0} + M^2_{K_+} - M^2_{\pi_+}\right), \quad \Delta M^2_K, \quad M^2_{\pi_0}$ $\longleftarrow$ Type-I
- For IB-decomposition: physical point is set via
  $w_0, \quad M^2_{ss}, \quad \Delta M^2 = M^2_{dd} - M^2_{uu}, \quad M^2_{\pi_\chi} = \frac{1}{2}\left(M^2_{uu} + M^2_{dd}\right)$ $\longleftarrow$ Type-II
- Expand observable around physical point

$$Y = A + BX_l + CX_s + DX_{\delta m} + Ee_v^2 + Fe_v e_s + Ge_s^2$$

- One χ^2 fit for all components

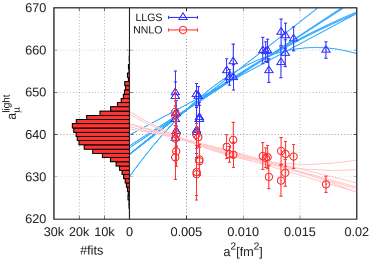
$$\begin{aligned}
[Y]_0 &= [A + BX_l + CX_s]_0 \\
[Y]'_m &= [DX_{\delta m}]'_m \\
[Y]''_{20} &= [A + BX_l + CX_s + DX_{\delta m}]''_{20} + [E]_0 \\
[Y]''_{11} &= [A + BX_l + CX_s + DX_{\delta m}]''_{11} + [F]_0 \\
[Y]''_{02} &= [A + BX_l + CX_s + DX_{\delta m}]''_{02} + [G]_0
\end{aligned}$$

$$\begin{aligned}
A &= A_0 + A_2 a^2 + A_4 a^4 \\
B &= B_0 + B_2 a^2 \\
C &= C_0 + C_2 a^2 \\
D &= D_0 + D_2 a^2 + D_l X_l + D_s X_s \\
E &= E_0 + E_2 a^2 + E_l X_l + E_s X_s \\
F &= F_0 + F_2 a^2 \\
G &= G_0 + G_2 a^2
\end{aligned}$$

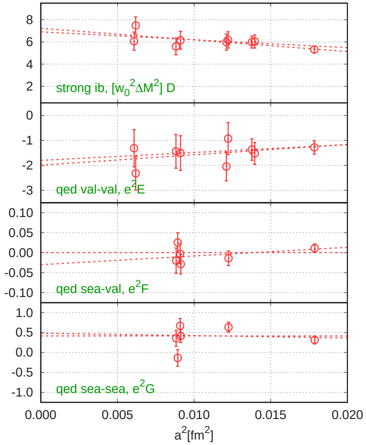
- Several thousand analyses, combined using histogram method
  - functional form
  - cuts in lattice spacing
  - hadron mass fit ranges
  - taste improvement type
  - window borders
  - ...

Result: a_μ^{light}

phys. point: $M_\Omega, M^2_{K_\chi}, \Delta M^2_K$



phys. point: $w_0, M^2_{ss}, \Delta M^2$

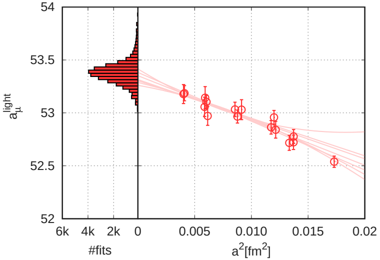


258048 fits

$$a_\mu^{\text{light}}(L_{\text{ref}}, T_{\text{ref}}) = 640.3(2.6)(3.6)[4.4]$$

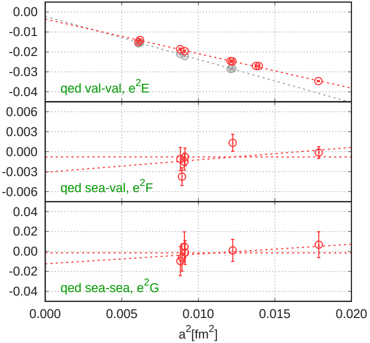
Result: a_μ^{strange}

phys. point: $M_\Omega, M_{K_\chi}^2, \Delta M_K^2$



32256 fits

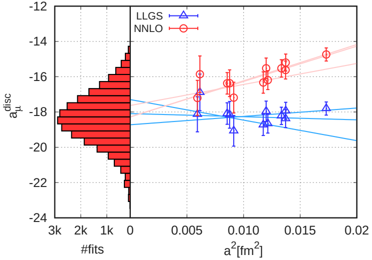
phys. point: $w_0, M_{ss}^2, \Delta M^2$



$$a_\mu^{\text{strange}}(L_{\text{ref}}, T_{\text{ref}}) = 53.379(89)(67)[111]$$

Result: $a_\mu^{\rm disc}$

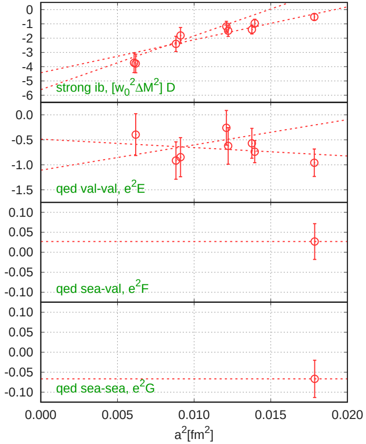
phys. point: $M_\Omega, M^2_{K_\chi}, \Delta M^2_K$



27648 fits

$$a_\mu^{\rm disc}(L_{\rm ref}, T_{\rm ref}) = -18.37(1.15)(1.09)[1.58]$$

phys. point: $w_0, M^2_{ss}, \Delta M^2$



Contributions

- IB contributions

	$a_\mu^{\rm strange}(L_{\rm ref}, T_{\rm ref})$	$a_\mu^{\rm light}(L_{\rm ref}, T_{\rm ref})$	$a_\mu^{\rm disc}(L_{\rm ref}, T_{\rm ref})$
total	53.379(89)(67)	640.3(2.6)(3.6)	-18.37(1.15)(1.09)
iso	53.393(89)(68)	634.6(2.7)(3.7)	-13.15(1.28)(1.29)
qed	-0.0136(86)(76)	-0.92(34)(43)	-0.58(14)(10)
qed-vv	-0.0086(42)(41)	-1.28(40)(33)	-0.55(15)(11)
qed-sv	-0.0014(11)(14)	-0.0080(85)(98)	0.011(24)(14)
qed-ss	-0.0031(76)(69)	0.42(20)(19)	-0.047(33)(23)
sib	–	6.59(63)(53)	-4.63(54)(69)

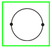
- Other contributions

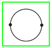
strange, $a_\mu^{\rm strange}(L_{\rm ref}, T_{\rm ref})$	53.379(89)(67)	
light, $a_\mu^{\rm light}(L_{\rm ref}, T_{\rm ref})$	640.3(2.6)(3.6)	
disconnected, $a_\mu^{\rm disc}(L_{\rm ref}, T_{\rm ref})$	-18.37(1.15)(1.09)	
finite-size, $a_\mu(\infty, \infty) - a_\mu(L_{\rm ref}, T_{\rm ref})$	18.7(2.5)	
charm iso, $[a_\mu^{\rm charm}]_{\rm iso}$	14.6(0.0)(0.1)	[Borsanyi *et.al.* 2017]
charm qed, $[a_\mu^{\rm charm}]_{\rm qed}$	0.0182(36)	[Giusti *et.al.* 2019]
charm effect on $a_\mu^{\rm disc}$	<0.1	[Borsanyi *et.al.* 2017]
bottom, $a_\mu^{\rm bottom}$	0.271(37)	[Colquhoun *et.al.* 2015]
perturbative, $a_\mu^{\rm pert}$	0.16	[Borsanyi *et.al.* 2017]
one-photon-reducible subtraction, $-a_\mu^{1\gamma\rm R}$	-0.321(11)	[Chakraborty *et.al.* 2018]

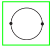
- $$a_\mu = 708.7(2.8)(4.5)[5.3]$$

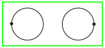
Contributions

Isospin symmetric

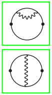

connected light
634.6(2.7)(3.7)

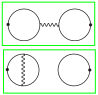

connected strange
53.393(89)(68)


connected charm
14.6(0)(1)

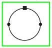

disconnected
-13.15(1.28)(1.29)

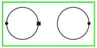
QED isospin-breaking: valence


connected -1.27(40)(33)

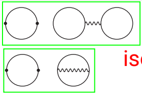

disconnected -0.55(15)(11)

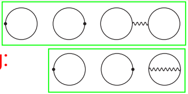
Strong isospin-breaking


connected
6.59(63)(53)


disconnected
-4.63(54)(69)

QED isospin-breaking: sea


connected 0.42(20)(19)

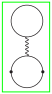

disconnected -0.047(33)(23)

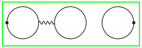
Etc.

bottom; higher order; perturbative

0.11(4)

QED isospin-breaking: mixed


connected -0.0095(86)(99)


disconnected 0.011(24)(14)

Finite-size effects

isospin-symmetric
18.7(2.5)

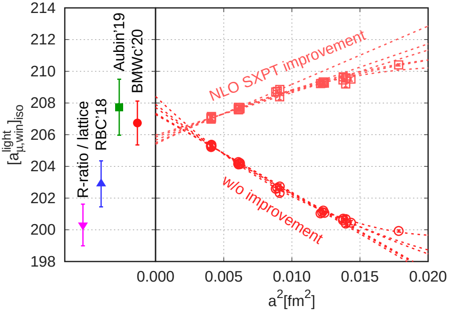
isospin-breaking
0.0(0.1)

$$10^{10} \times a_\mu^{\text{LO-HVP}} = 708.7(2.8)_{\text{stat}}(4.5)_{\text{sys}}[5.3]_{\text{tot}}$$

Window

Window observable

- Restrict correlator to window between $t_1 = 0.4\,\text{fm}$ and $t_2 = 1.0\,\text{fm}$
- Less challenging than full a_μ



$$[a^{\text{light}}_{\mu,\text{win}}]_{\text{iso}} = 206.7(0.2)(1.4)[1.4]$$

Overlap crosscheck

- $L = 3\,\text{fm}$
- Valence: overlap
- Sea: `4stout` staggered