1. Finite-size effect on TMR correlator: non-interacting pions

NLO ChPT prediction. Here the expression from [Mainz/CLS 1705.01775]:

$$
\begin{aligned}
G(t,L) - G(t,\infty) &= \frac{1}{3}\left[\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right] \frac{\vec{k}^2}{\vec{k}^2 + m_\pi^2}\; e^{-2t\sqrt{\vec{k}^2+m_\pi^2}} \\
&= \frac{m_\pi^4 t}{3\pi^2} \sum_{\vec{n}\neq 0} \left\{ \frac{K_2(m_\pi\sqrt{L^2\vec{n}^2 + 4t^2})}{m_\pi^2(L^2\vec{n}^2 + 4t^2)} \right. \\
&\quad \left. - \frac{1}{m_\pi L|\vec{n}|} \int_1^\infty dy\; K_0(m_\pi y\sqrt{L^2\vec{n}^2 + 4t^2})\; \sinh(m_\pi L|\vec{n}|(y-1)) \right\}.
\end{aligned}
$$

- for given L, first expression (in terms of energy eigenstates) is adequate at large t, the second one (winding-number expansion) at small t.
- fixed t, $L \to \infty$: second expression $\Rightarrow$ finite-size effect is O($e^{-m_\pi L}$). NNLO ChPT, or Hansen-Patella treatment, provides the prefactor more accurately, as well as subleading exponentials.
- at large t however, the finite volume effect goes to zero more slowly than $e^{-m_\pi L}$ for, say, $3 < m_\pi L < 5$.
- 't is large' means that only a handful of states in the box are contributing significantly to the correlator.
- for $m_\pi L = 4$, it turns out that a sizeable part of $a_\mu^{\rm hvp}$ (and most of FSE) comes from 'large t' $\rightsquigarrow$ can we improve the first expression in that regime?

2. Interacting pions

- finite-volume correlator:

$$G(t,L) = \sum_n |A_n|^2 \, e^{-\omega_n t}$$

- infinite-volume correlator:

$$\begin{aligned} G(t,\infty) &= \int_0^\infty d\omega \, \omega^2 \, \rho(\omega^2) \, e^{-\omega t}, \\ \rho(\omega^2) &= \frac{1}{48\pi^2}\left(1 - \frac{4m_\pi^2}{\omega^2}\right)^{\frac{3}{2}} |F_\pi(\omega)|^2 + \text{other channels.} \end{aligned}$$

$\rightsquigarrow$ need the connection between $(|A_n|^2, \omega_n) \longleftrightarrow F_\pi(\omega)$.

3. The connection

Lüscher relation between the discrete energy levels in the box and the infinite-volume phase shift [NPB364 (1991) 237]:

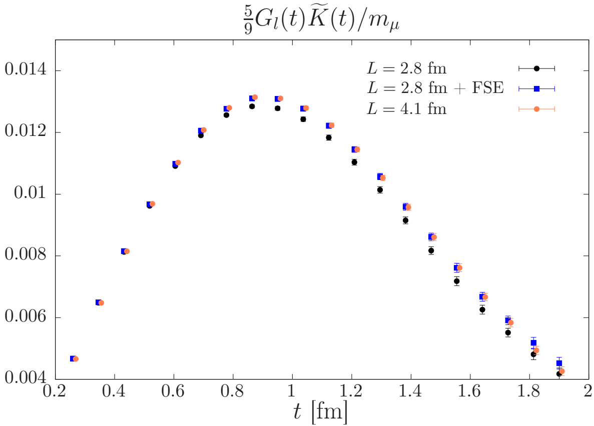
$$\delta_{11}(k) + \phi\left(\tfrac{kL}{2\pi}\right) = n\pi, \qquad n = 1, 2, \ldots; \qquad \omega \equiv 2\sqrt{m_\pi^2 + k^2}.$$

The corresponding finite-volume matrix elements are given by [HM, 1105.1892]

$$\begin{aligned} |F_\pi(\omega)|^2 &= \mathbb{L}(k)\frac{3\pi\omega^2}{2k^5}|A|^2, \\ \mathbb{L}(k) &\equiv \left[z\phi'(z)\right]_{z=\frac{kL}{2\pi}} + k\frac{\partial\delta_{11}(k)}{\partial k}. \qquad \text{(a Lellouch-Lüscher factor).} \end{aligned}$$

The function $\phi(z)$ is defined by $\tan\phi(z) = -\frac{\pi^{3/2}z}{\mathcal{Z}_{00}(1;z^2)}$, where $\mathcal{Z}_{00}(1;z^2)$ is the analytic continuation in s of $\mathcal{Z}_{00}(s;z^2) = \frac{1}{\sqrt{4\pi}}\sum_{\boldsymbol{n}\in\mathbb{Z}^3}\frac{1}{(\boldsymbol{n}^2 - z^2)^s}$.

4. Practical tests of the finite-size correction



[Mainz/CLS 1904.03120]

Other successful checks of FSE by RBC/UKQCD, and BMW-20 (at 10% level).