Aspects of Supersymmetry and its Breaking

Thomas T. Dumitrescu[a*] and Zohar Komargodski[b†]

[a]Department of Physics, Princeton University,
Princeton, New Jersey, 08544, USA

[b]School of Natural Sciences, Institute for Advanced Study,
Princeton, New Jersey, 08540, USA

We describe some basic aspects of supersymmetric field theories, emphasizing the structure of various supersymmetry multiplets. In particular, we discuss supercurrents – multiplets which contain the supersymmetry current and the energy-momentum tensor – and explain how they can be used to constrain the dynamics of supersymmetric field theories, supersymmetry breaking, and supergravity. These notes are based on lectures delivered at the Cargése Summer School 2010 on "String Theory: Formal Developments and Applications," and the CERN Winter School 2011 on "Supergravity, Strings, and Gauge Theory."

1. Supersymmetric Theories

1.1. Supermultiplets and Superfields

A four-dimensional theory possesses $\mathcal{N} = 1$ supersymmetry (SUSY) if it contains a conserved spin-$\frac{1}{2}$ charge Q_α which satisfies the anti-commutation relation

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \ . \tag{1}$$

Here $\bar{Q}_{\dot{\alpha}}$ is the Hermitian conjugate of Q_α. (We will use bars throughout to denote Hermitian conjugation.) Unless otherwise stated, we follow the conventions of [1]. In local quantum field theories, the basic objects of interest are well-defined local operators. In SUSY field theories all such operators must be embedded in multiplets of the supersymmetry algebra, or *supermultiplets*. Conserved currents furnish an important class of local operators. Of particular interest is the supersymmetry current $S_{\alpha\mu}$, which satisfies

$$\partial^\mu S_{\alpha\mu} = 0 \ , \qquad Q_\alpha = \int d^3x \, S_\alpha{}^0 \ . \tag{2}$$

In this review we will describe some basic aspects of SUSY field theories, emphasizing the structure of various supermultiplets – especially those containing the supersymmetry current. In particular, we will show how these supercurrents can be used to study the dynamics of supersymmetric field theories, SUSY-breaking, and supergravity.

We begin by recalling basic facts about supermultiplets and superfields. A set of bosonic and fermionic operators $\{\mathcal{O}_i^B(x)\}$ and $\{\mathcal{O}_i^F(x)\}$ furnishes a supermultiplet if these operators satisfy commutation relations of the schematic form

$$\begin{aligned} &\left[Q_\alpha, \mathcal{O}_i^B(x)\right] \sim \mathcal{O}_j^F(x) + \partial\mathcal{O}_k^F(x) + \cdots \ , \\ &\left\{Q_\alpha, \mathcal{O}_i^F(x)\right\} \sim \mathcal{O}_j^B(x) + \partial\mathcal{O}_k^B(x) + \cdots \ , \end{aligned} \tag{3}$$

and likewise for the $\bar{Q}_{\dot{\alpha}}$ commutators, such that the SUSY algebra (1) is satisfied. It is straightforward to show that a supermultiplet must contain equally many independent bosonic and fermionic operators (see, for instance, [2]).

It is always possible (and very convenient) to embed the component fields $\mathcal{O}_i^{B,F}(x)$ of a supermultiplet in a superfield $S(x,\theta,\bar{\theta})$. Here θ_α is an anti-commuting superspace coordinate. (For now we suppress any Lorentz indices carried by S.) The component fields are identified with the x-dependent coefficients when $S(x,\theta,\bar{\theta})$ is expanded as a power series in $\theta, \bar{\theta}$. The commutation rela-

*T.D. is supported by a DOE Fellowship in High Energy Theory, NSF grant PHY-0756966, and a Centennial Fellowship from Princeton University.
†Z.K. is supported by DOE grant DE-FG02-90ER40542. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

tions (3) are succinctly encoded in the formula

$$\left[\xi Q + \bar{\xi}\bar{Q}, S\right] = i\left(\xi \mathcal{Q} + \bar{\xi}\bar{\mathcal{Q}}\right) S \ , \tag{4}$$

which is the defining property of a superfield.[3] Here ξ_α is an arbitrary Grassmann parameter and $\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot\alpha}$ are the superspace differential operators

$$\begin{aligned} \mathcal{Q}_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \ , \\ \bar{\mathcal{Q}}_{\dot\alpha} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \ . \end{aligned} \tag{5}$$

Conversely, the defining property (4) implies that the components of any superfield furnish a supermultiplet. Thus, supermultiplets and superfields are in one-to-one correspondence, and we will treat them synonymously.

To see this in a little more detail, consider the component expansion of a general scalar superfield:

$$S(x,\theta,\bar\theta) = C(x) + i\theta^\alpha\psi_\alpha(x) + \cdots + \theta^2\bar\theta^2 D(x) \ . \tag{6}$$

As explained above, the defining property (4) determines the commutation relations of the supercharges $Q_\alpha, \bar{Q}_{\dot\alpha}$ with the component fields. These commutators show that the supercharges act as raising operators for the component fields.[4] This has two important consequences:

- The superfield $S(x,\theta,\bar\theta)$ can be constructed from its bottom component $C(x)$ by applying the supercharges. Thus, any local operator can be embedded in the bottom component of a superfield. However, it is *not* always possible to embed an operator in a higher component. This will play a crucial role in our analysis of various supercurrents.

- The SUSY variation of the top component $D(x)$ of any superfield is always a total derivative. This fact will enable us to write supersymmetric Lagrangians.

[3] The factor of i in (4) is necessary for Hermiticity in Minkowski space.
[4] For instance, $[Q_\alpha, C] = -\psi_\alpha$.

A general superfield does not furnish an irreducible representation of supersymmetry. To reduce a supermultiplet, we impose supersymmetric constraints. This is most conveniently done in terms of the superspace differential operators

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \ , \\ \bar{D}_{\dot\alpha} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \ . \end{aligned} \tag{7}$$

These anti-commute with the supersymmetry generators $\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot\alpha}$, and thus map superfields to superfields. Hence, any constraint written in terms of $D_\alpha, \bar{D}_{\dot\alpha}$ is automatically supersymmetric. We are now ready to begin exploring various important supermultiplets.

1.2. Chiral Multiplets and Lagrangians

The most familiar representation of supersymmetry is the chiral multiplet. It is the basic building block which enables us to write SUSY Lagrangians describing only scalars and fermions. The bottom component of a chiral multiplet is annihilated by $\bar{Q}_{\dot\alpha}$. For example, the bottom component $\phi(x)$ of a scalar chiral multiplet satisfies

$$\left[\bar{Q}_{\dot\alpha}, \phi(x)\right] = 0 \ . \tag{8}$$

The multiplet obtained from $\phi(x)$ by acting with the supercharges is organized in a superfield which satisfies the constraint $\bar{D}_{\dot\alpha}\Phi = 0$. This constraint can be solved in components:

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) \ , \tag{9}$$

where $y^\mu = x^\mu + \theta\sigma^\mu\bar\theta$. We immediately note two key properties of chiral superfields:

- Any function which depends on the chiral superfields Φ^i, but not their complex conjugates, is again a chiral superfield. Such a function is said to be holomorphic in the Φ^i.

- The SUSY variation of $F(x)$ is a total derivative.

From (9) we see that Φ contains a complex scalar ϕ, a Weyl fermion ψ_α, and another complex scalar F which will turn out to be a non-propagating auxilliary field. Among other things,

F ensures that the chiral multiplet has four (real) bosonic degrees of freedom to match the four fermionic degrees of freedom coming from ψ_α. We will abbreviate this by saying that Φ is a $4+4$ multiplet. As advertised, Φ has exactly the right field content to describe a theory of scalars and fermions, and we would like to write a Lagrangian for such a theory.

Up to total derivatives, a SUSY Lagrangian $\mathscr{L}$ must be a real scalar whose variation under supersymmetry is a total derivative. We can thus take

$$\mathscr{L} = D + F + \bar{F} \ , \tag{10}$$

where D is the top component of a real scalar superfield $K = \bar{K}$ and F is the θ^2-component of a chiral superfield W. This ensures that $\mathscr{L}$ is real and supersymmetric. For reasons that will be explained below, K is known as the Kähler potential, and W is called the superpotential. A particularly simple choice is to take $K = \bar{\Phi}\Phi$ and $W = 0$. It is standard to pick out different components of a superfield using Grassmann integration. For example, we can use $\int d^4\theta$ to pick out the top component of a superfield, or $\int d^2\theta$ to pick out the θ^2 component. We thus write

$$\mathscr{L} = \int d^4\theta \, \bar{\Phi}\Phi \ , \tag{11}$$

up to total derivatives. In components:

$$\mathscr{L} = -\partial^\mu \bar{\phi} \partial_\mu \phi + i\partial_\mu \psi \sigma^\mu \bar{\psi} + \bar{F}F \ . \tag{12}$$

This describes a free complex scalar ϕ, a free Weyl fermion ψ_α, and a non-propagating auxiliary field F. In this example F vanishes by its equations of motion. Because the choice $K = \bar{\Phi}\Phi$ gives rise to the usual kinetic terms for ϕ and ψ_α, it is called a canonical Kähler potential.

This trivial free theory can be readily generalized: consider N chiral superfields Φ^i and consider the Lagrangian

$$\mathscr{L} = \int d^4\theta \, K(\Phi, \bar{\Phi}) \ , \tag{13}$$

where the Kähler potential K may now be a complicated real function of the Φ^i and their complex conjugates. (The superpotential W is still taken to vanish.) After solving for the auxiliary fields F^i, which are now non-zero, the component Lagrangian takes the form

$$\mathscr{L} = -g_{i\bar{j}}\left(\phi, \bar{\phi}\right) \partial_\mu \bar{\phi}^{\bar{j}} \partial^\mu \phi^i + i g_{i\bar{j}} \, \partial_\mu \psi^i \sigma^\mu \bar{\psi}^{\bar{j}} + i g_{i\bar{j}} \Gamma^i_{kl} \left(\partial_\mu \phi^k\right) \psi^l \sigma^\mu \bar{\psi}^{\bar{j}} + \frac{1}{4} R_{i\bar{j}k\bar{l}} \, \psi^i \psi^k \, \bar{\psi}^{\bar{j}} \bar{\psi}^{\bar{l}} \tag{14}$$

This theory is known as the supersymmetric non-linear σ-model. The bosonic part is an ordinary σ-model, whose target space is an N complex dimensional manifold with metric $g_{i\bar{j}}$. Since the Lagrangian (14) does not have a potential term, any point ϕ^i on the target manifold is a supersymmetric vacuum. (Recall that it follows from the SUSY algebra (1) that supersymmetric vacua have zero energy; vacua with positive energy spontaneously break SUSY.) In supersymmetric theories, it is customary to refer to such manifolds of vacua as moduli spaces. As usual, the scalars ϕ^i should be thought of as coordinates on the target space. This is consistent with the fact that the theory is invariant under holomorphic field redefinitions of the form

$$\Phi'^i = f^i(\Phi^j) \ . \tag{15}$$

Such field redefinitions can be thought of as coordinate changes on the target manifold under which the metric transforms in the usual tensorial way.

In the supersymmetric σ-model, the metric is determined by the Kähler potential:

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K \ . \tag{16}$$

Thus, the target space is a Kähler manifold. Note that $g_{i\bar{j}}$ must be positive definite so that the kinetic terms in (14) have the correct sign required by unitarity. The fermions couple to the geometry of this Kähler manifold through the connection Γ^i_{jk} and the curvature tensor $R_{i\bar{j}k\bar{l}}$, which depend on the metric and its derivatives. Under a coordinate chage (15) the fermions transform as $\psi'^i = \frac{\partial \phi'^i}{\partial \phi_j} \psi^j$. They can thus be identified with vectors in the tangent space of the target manifold.

The dynamics of the non-linear σ-model is unchanged under redefinitions of the Kähler potential which take the form

$$K'(\Phi,\bar\Phi) = K(\Phi,\bar\Phi) + F(\Phi) + \bar F(\bar\Phi) \ , \qquad (17)$$

for some holomorphic function F of the Φ^i. This is known as a Kähler transformation. Such a transformation only changes the Lagrangian by a total derivative. This is because $F(\Phi^i)$ is chiral so that its top component is a total derivative. Alternatively, we see from (16) that the metric $g_{i\bar\jmath}$ is unaffected by Kähler transformations, so that the component Lagrangian (14) is also unchanged.

It is important to note that while we are always free to perform Kähler transformations, there are situations in which we are forced to do so. This is the case whenever the target manifold has non-trivial topology and must be covered with several patches which make it impossible to consistently define a single-valued Kähler potential. In this case we are forced to use different Kähler potentials in different patches. These Kähler potentials must be related by a Kähler transformation whenever two patches overlap.

As an example of a supersymmetric σ-model which illustrates this point, we consider a single chiral superfield Φ with Kähler potential

$$K = f_\pi^2 \log\left(1+|\Phi|^2\right) \ . \qquad (18)$$

In this normalization Φ is dimensionless, while the constant f_π has dimensions of mass.[5] The metric is $g_{\phi\bar\phi} = \frac{f_\pi^2}{(1+|\phi|^2)^2}$, which we recognize as the familiar round metric on the two-sphere with radius $\sim f_\pi$. The coordinate values $\phi = 0$ and $\phi = \infty$ correspond to antipodal points. As we explained above, the moduli space of SUSY vacua is just the two-sphere itself. This theory is usually referred to as the $\mathbb{CP}^1$-model. The Kähler potential (18) gives rise to the Fubini-Study metric on $\mathbb{CP}^1$, which is nothing but the round metric on the two-sphere.

To describe the point at infinity, we must perform a change of variables $\Phi \to 1/\Phi$. This induces a Kähler transformation (17) with $F(\Phi) = f_\pi^2\log(\Phi)$. It is, in fact, impossible to cover the two-sphere (or, as we will later explain, any other compact manifold) with patches in such a way that K is a globally well-defined scalar.

So far we have only discussed theories without a superpotential W, such as the SUSY σ-model (13). These theories did not have any scalar potential. The simplest example with a superpotential is a single chiral superfield Φ with canonical Kähler potential and with superpotential $W = \frac{1}{2}m\Phi^2$. After integrating out the auxiliary field F, this leads to the component Lagrangian

$$\begin{aligned}\mathscr{L} =& -\partial^\mu\bar\phi\partial_\mu\phi - |m|^2|\phi|^2 + i\partial_\mu\psi\sigma^\mu\bar\psi \\ & - \frac{1}{2}m\psi^2 - \frac{1}{2}m^*\bar\psi^2 \ . \end{aligned} \qquad (19)$$

Thus, ϕ and ψ both acquire a mass $|m|$.

More generally, we can take the SUSY σ-model (13) and add a superpotential $W(\Phi^i)$. Since W must be chiral, it must be holomorphic in the Φ^i. The resulting theory is called a Wess-Zumino model:

$$\mathscr{L} = \int d^4\theta\, K(\Phi,\bar\Phi) + \int d^2\theta\, W(\Phi) + \text{c.c.} \ . \qquad (20)$$

After integrating out the auxiliary fields, this adds to the component Lagrangian (14) of the σ-model a scalar potential and Yukawa-type interactions:

$$\begin{aligned}\mathscr{L} \supset & - g^{i\bar\jmath}\partial_i W \partial_{\bar\jmath}\bar W \\ & - \frac{1}{2}\left(\left(\partial_i\partial_j W - \Gamma^k_{ij}\partial_k W\right)\psi^i\psi^j + \text{c.c.}\right) \ . \end{aligned} \qquad (21)$$

The vacua of this theory are given by the minima of the scalar potential which appears in the first line of (21). Since supersymmetric vacua have zero energy and the metric $g_{i\bar\jmath}$ is positive definite, the space of SUSY vacua is specified by the N complex equations $\partial_i W = 0$ in the N complex unknowns ϕ^i. This implies that without any further structure Wess-Zumino models generally have SUSY vacua. In section 3, we will explore some simple theories whose vacua spontaneously break supersymmetry.

1.3. Vector Multiplets and Gauge Theories

In this section we will describe the vector multiplet, which will enable us to write Lagrangians

[5] This notation is due to the fact that f_π is the analogue of the pion decay constant in the chiral Lagrangian for QCD. Our example (18) describes the coset manifold $SU(2)/U(1) = \mathbb{CP}^1$.

for supersymmetric gauge theories. For simplicity we will limit ourselves to the Abelian case. A vector multiplet is a real superfield $V = \bar{V}$, subject to the gauge transformations

$$V \to V + i\left(\Lambda - \bar{\Lambda}\right) \ , \tag{22}$$

where Λ is a chiral superfield. These gauge transformations allow us to fix Wess-Zumino gauge in which the low-lying components of V vanish:

$$V = -\left(\theta\sigma^\mu\bar{\theta}\right) A_\mu + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D \ . \tag{23}$$

Here A_μ is a real gauge field, the Weyl fermion λ_α is its gaugino superpartner, and D is a real scalar auxiliary field. The residual gauge freedom which remains in Wess-Zumino gauge just consists of ordinary gauge-transformations for A_μ.

To capture the gauge-invariant information in V, we can define the superfield

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V \ , \tag{24}$$

which is invariant under (22). From this definition, we immediately see that W_α satisfies the constraints

$$\bar{D}_{\dot{\alpha}} W_\alpha = 0 \ , \qquad D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \ . \tag{25}$$

In components:

$$\begin{aligned} W_\alpha = & - i\lambda_\alpha(y) + \theta_\beta\bigg(\delta_\alpha{}^\beta D(y) \\ & - i(\sigma^{\mu\nu})_\alpha{}^\beta F_{\mu\nu}(y)\bigg) + \theta^2\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}}(y) \ , \end{aligned} \tag{26}$$

where the closed, real two-form $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of A_μ. We therefor refer to W_α as a field strength superfield. We see that W_α is a $4+4$ multiplet, corresponding to the $4+4$ gauge-invariant degrees of freedom in V.

For future reference, we note that any spinor superfield W_α which satisfies the constraints (25) has the component expansion (26) for some closed, real two-form $F_{\mu\nu}$. Locally, we can always express such a two-form as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for some vector A_μ, and we can consequently write W_α as in (24) for some vector superfield V. However, in general A_μ and V are not well-defined: they may undergo gauge-transformations.

The superfield W_α has exactly the right field content to write supersymmetric kinetic terms for the gauge field and the gaugino. Since W_α is chiral, we can take

$$\begin{aligned} \mathscr{L} &= \frac{1}{4e^2}\int d^2\theta\, W^\alpha W_\alpha + \text{c.c.} \\ &= -\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{e^2}\partial_\mu\lambda\sigma^\mu\bar{\lambda} + \frac{1}{2e^2}D^2 \ . \end{aligned} \tag{27}$$

Here e is the gauge coupling. In this simple free theory the auxiliary field D vanishes by its equations of motion.

In Abelian gauge theories, it is possible to add a so-called Fayet-Iliopoulos (FI) term to the Lagrangian:

$$\mathscr{L}_{\text{FI}} = \xi\int d^4\theta V \ . \tag{28}$$

Here the FI-parameter ξ is real. The FI-term is gauge invariant because the top-component of a chiral superfield is a total derivative. Note that the gauge-invariance of the FI-term resembles the invariance of the Wess-Zumino Lagrangian (20) under Kähler transformations. This connection will resurface in later sections.

2. Global Symmetries

2.1. Global Current Supermultiplets

In local quantum field theories, the Noether theorem guarantees that each continuous global symmetry gives rise to a conserved current j^μ satisfying

$$\partial^\mu j_\mu = 0 \ , \qquad Q = \int d^3x\, j^0 \ . \tag{29}$$

Here Q is the conserved charge corresponding to j^μ. In this review, we will not discuss R-symmetries, so that Q is just an ordinary global symmetry.[6] In SUSY theories, we must embed j^μ in some supermultiplet. In order to decide what constraints this superfield will satisfy, we are guided by the following two observations:

[6] An R-symmetry is a global symmetry which does not commute with SUSY: $[R, Q_\alpha] = -Q_\alpha$.

- Because Q is an ordinary global symmetry charge, it commutes with SUSY, $[Q_\alpha, Q] = 0$. This implies the current algebra

  $$[Q_\alpha, j_\mu] = (\text{S.T.}) \ . \tag{30}$$

  The right-hand-side of this equation is conserved when acting with ∂^μ on both sides, and it is a total space-derivative when $\mu = 0$ so that it integrates to zero; it is known as a Schwinger Term (S.T.).

- We expect j^μ to be the highest spin component in its supermultiplet. If this is not the case, we could not gauge j^μ without introducing higher-spin gauge fields. This is not expected to be consistent in rigid field theories without gravity.

We thus posit that j^μ is embedded in a real, scalar superfield $\mathcal{J}$ which schematically takes the form

$$\mathcal{J} = J + i\theta j - i\bar\theta\bar j + \left(\theta\sigma^\mu\bar\theta\right)(j_\mu + \cdots) + \cdots \ . \tag{31}$$

Here J is a real scalar, and j_α is a Weyl fermion. The dots are terms which are not uniquely fixed by the requirements outlined above. Different choices of Schwinger terms in (30) terms lead to different ways of completing the multiplet. Although there is a constructive way of writing the most general solution, we will not follow this approach here. Instead, we will simply write down a set of constraints for $\mathcal{J}$ and explore the resulting component structure.

Conventionally, the global current supermultiplet $\mathcal{J}$ is taken to satisfy

$$D^2\mathcal{J} = \bar D^2\mathcal{J} = 0 \ . \tag{32}$$

A real superfield obeying these constraints is called a linear multiplet. In components, such a multiplet takes the form

$$\mathcal{J} = J + i\theta j - i\bar\theta\bar j + \left(\theta\sigma^\mu\bar\theta\right) j_\mu + \frac{1}{2}\theta^2\bar\theta\bar\sigma^\mu\partial_\mu j - \frac{1}{2}\bar\theta^2\theta\sigma^\mu\partial_\mu\bar j - \frac{1}{4}\theta^2\bar\theta^2\partial^2 J \ , \tag{33}$$

where j_μ is the conserved vector current. We immediately see that $\mathcal{J}$ contains no higher-spin components. Moreover, we can use (4) on (33) to extract the commutation relation

$$[Q_\alpha, j_\mu] = -2i(\sigma_{\mu\nu})_\alpha{}^\beta\partial^\nu j_\beta \ , \tag{34}$$

whose right-hand-side is indeed a pure Schwinger term. Having familiarized ourself with the structure of the global current multiplet (33), we can now explore how it arises in theories with global symmetries.

We begin by revisiting the free theory of a chiral superfield (11), which is invariant under $U(1)$ phase rotations of Φ. This corresponds to the invariance of the component theory (12) under $U(1)$ phase rotations of ϕ, ψ_α and F. The equation of motion $D^2\Phi = 0$ implies that $\mathcal{J} = \bar\Phi\Phi$ is a linear multiplet, which in the normalization of (33) contains the conserved vector current

$$j_\mu = i\left(\bar\phi\partial_\mu\phi - \partial_\mu\bar\phi\phi\right) + \bar\psi\bar\sigma_\mu\psi \ . \tag{35}$$

This is the usual $U(1)$ phase current which gives charge $+1$ to ϕ and ψ_α.[7]

More generally, we can consider continuous global symmetries of the SUSY σ-model (13). Assume that this theory is invariant under the infinitesimal global symmetry transformation

$$\delta\Phi^i = \varepsilon^{(a)} X^{(a)i}(\Phi) \ . \tag{36}$$

Here the $X^{(a)i}$ are holomorphic in the Φ^i and the $\varepsilon^{(a)}$ are infinitesimal real parameters; we will use indices $a, b, \ldots$ to label different global symmetries. The transformation (36) must leave the metric $g_{i\bar j}$ invariant.[8] However, the Lagrangian (13) may pick up a Kähler transformation:

$$\delta\mathscr{L} = \int d^4\theta\, \varepsilon^{(a)}\left(F^{(a)}(\Phi) + \bar F^{(a)}(\bar\Phi)\right) \ , \tag{37}$$

where the functions $F^{(a)}$ are holomorphic in the Φ^i. This only changes the Lagrangian by a total derivative.

[7] In our convention, an operator $\mathcal{O}$ has charge q under the symmetry generated by Q if it satisfies $[Q, \mathcal{O}] = q\mathcal{O}$.

[8] Geometrically, this means that the $X^{(a)i}$ are the components of holomorphic Killing vector fields $X^{(a)} = X^{(a)i}\partial_i$.

To find the conserved currents corresponding to the symmetries (36), we perform the superspace analog of the usual Noether procedure: replace each infinitesimal transformation parameter $\varepsilon^{(a)}$ with an arbitrary infinitesimal chiral superfield $\Lambda^{(a)}$. (The fact that the $\Lambda^{(a)}$ are chiral ensures that the variations of the chiral superfields Φ^i are still chiral.) To linear order in the $\Lambda^{(a)}$, the change in the Lagrangian must now take the form

$$\delta\mathscr{L} = \int d^4\theta\, \Lambda^{(a)} \left(F^{(a)} - i\mathcal{J}^{(a)}\right) + \text{c.c.}\ , \tag{38}$$

for some real superfields $\mathcal{J}^{(a)}$. Note that (38) reduces to (37) if we set $\Lambda^{(a)} = \varepsilon^{(a)}$. However, using the explicit form (36) of the infinitesimal transformation with $\varepsilon^{(a)}$ replaced by $\Lambda^{(a)}$, we can also write the change in the Lagrangian as

$$\delta\mathscr{L} = \int d^4\theta\, \Lambda^{(a)} X^{(a)i} \partial_i K + \text{c.c.}\ . \tag{39}$$

Since (38) and (39) must agree for all chiral superfields $\Lambda^{(a)}$, we can identify the Noether currents

$$\mathcal{J}^{(a)} = iX^{(a)i}\partial_i K - iF^{(a)}\ . \tag{40}$$

These are guaranteed to be real and conserved, as can be checked using the sigma model equations of motion $D^2\partial_{\bar{i}} K = 0$. Note that the Noether currents (40) depend on the functions $F^{(a)}$ which appear in the Kähler transformation (37). This is not surprising: when the Lagrangian changes by a total derivative under a symmetry transformation, then the Noether current acquires an additional piece.

Such a situation can arise even in the trivial theory (11). This theory has a shift symmetry $\delta\Phi = \varepsilon$ under which the Lagrangian undergoes a Kähler transformation with $F = \Phi$. By (40), the Noether current for this symmetry is given by

$$\mathcal{J} = i(\bar{\Phi} - \Phi)\ . \tag{41}$$

An interesting complication occurs if we compactify the target space of this model to a cylinder by identifying $\Phi \sim \Phi + i$. As we go around the cylinder once, the bottom component of $\mathcal{J}$ shifts by a constant. It is thus not a good operator in the theory. In this case, we cannot embed the vector current corresponding to the shift symmetry in the linear multiplet $\mathcal{J}$. This also means that we cannot gauge the shift symmetry in the usual way (see subsection 2.2). This difficulty can be overcome by embedding j_μ in a different multiplet which is well-defined. (The details of this procedure are beyond the scope of this review.) Conceptually, this situation has an exact analogue for multiplets containing the supersymmetry current $S_{\alpha\mu}$ and the energy-momentum tensor $T_{\mu\nu}$, to be discussed in detail below.

2.2. Gauging Global Symmetries

In order to couple matter to a gauge field A_μ, we add to the Lagrangian a term

$$\delta\mathscr{L} \sim j_\mu A^\mu + \cdots\ . \tag{42}$$

Here j_μ is a conserved matter current, so that this interaction is invariant under gauge transformations of A_μ. If the current j_μ itself transforms under gauge transformations, then (42) contains additional terms (represented by the dots) with higher powers of A_μ to ensure gauge invariance. They will not be important for us, and we will not discuss them.

In SUSY theories, the analogue of (42) is given by the coupling

$$\delta\mathscr{L} = \int d^4\theta \mathcal{J} V\ , \tag{43}$$

where $\mathcal{J}$ is a linear multiplet and V is a vector multiplet. Since $D^2\mathcal{J} = 0$, this interaction term is invariant under gauge transformations (22) of V. In addition to the term in (42), the interaction (43) contains Yukawa-type interactions for the gaugino λ_α, as well as a coupling of the bottom component J of $\mathcal{J}$ to the auxiliary field D of the vector multiplet V. When D is integrated out, it gives rise to a new contribution to the scalar potential:

$$V_D = \frac{e^2}{8}\left(J + \xi\right)^2\ , \tag{44}$$

where e is the gauge coupling and ξ is an FI-term, which may be present. We refer to (44) as a D-term potential, since it arises from integrating out the D-component of a vector multiplet. This is to be contrasted with the scalar potentials discussed at the end of subsection 1.2, which were the result of integrating out the F-components of chiral superfields. These are known as F-term potentials.

As a simple example, which will reappear in later sections, consider a theory of two chiral superfields $\Phi_{1,2}$ with canonical Kähler potential and charge +1 under $U(1)$ phase rotations.[9] The corresponding conserved current is given by

$$\mathcal{J} = \bar{\Phi}_1\Phi_1 + \bar{\Phi}_2\Phi_2 \ . \qquad (45)$$

This gives rise to the D-term potential

$$V_D = \frac{e^2}{8}\left(|\phi_1|^2 + |\phi_2|^2 + \xi\right)^2 \ . \qquad (46)$$

If $\xi > 0$, then the vacuum energy is positive: there are no SUSY vacua. If $\xi < 0$, then the moduli space of vacua is given by the three-sphere $|\phi_1|^2 + |\phi_2|^2 = -\xi$ modulo $U(1)$ gauge transformations acting on $\phi_{1,2}$, which is nothing but $\mathbb{CP}^1$. The corresponding low-energy theory describing the massless moduli is the $\mathbb{CP}^1$-model, which we introduced at the end of subsection 1.2.

For future use, we now briefly describe how to gauge a general global symmetry of the non-linear σ-model. As in the examples above, we simply add to the Lagrangian (13) a term

$$\mathscr{L}' = \int d^4\theta\, \mathcal{J}^{(a)} V^{(a)} \ . \qquad (47)$$

Here the $\mathcal{J}^{(a)}$ are the Noether currents (40). Under a gauge-transformation with chiral gauge parameter $\Lambda^{(a)}$, the vector fields transform in the usual way, while the change in the σ-model Lagrangian $\mathscr{L}$ is given in (38). The complete Lagrangian is now gauge-invariant up to total derivatives:

$$\delta\left(\mathscr{L} + \mathscr{L}'\right) = \int d^4\theta\, \Lambda^{(a)} F^{(a)} + \text{c.c.} + \cdots \ , \qquad (48)$$

where the dots represent unimportant higher-order terms and the $F^{(a)}$ are the holomorphic functions which appear in the Kähler transformation (37). The D-term scalar potential in this theory is given by

$$V_D = \frac{1}{8}\sum_a g_a^2 \left(J^{(a)}\right)^2 \ . \qquad (49)$$

Here the g_a are the gauge coupling constants which arise from the kinetic terms of the different $V^{(a)}$, and the $J^{(a)}$ are the bottom components of the Noether currents $\mathcal{J}^{(a)}$.

3. SUSY-Breaking

3.1. Simple Examples

If supersymmetry is to play any role in describing nature, then it must be spontaneously broken. As we already mentioned, SUSY is spontaneously broken if the vacuum energy is positive. Moreover, broken supersymmetry always leads to a massless fermion, the Goldstino, which is the exact analogue of the Goldstone bosons which appear when ordinary global symmetries are spontaneously broken. We will now explore a few simple examples which break SUSY at tree-level.

The simplest SUSY-breaking theory consists of one chiral superfield Φ with a linear term in the superpotential

$$\mathscr{L} = \int d^4\theta\, \bar{\Phi}\Phi + \int d^2\theta\, f\Phi + \text{c.c.} \ . \qquad (50)$$

The scalar potential in this theory is simply a constant $V = |f|^2$. Thus, the spectrum consists of a massless fermion ψ_α – the Goldstino – and a massless scalar ϕ. Since there is no potential for ϕ, the theory has infinitely many SUSY-breaking vacua with nonzero energy $|f|^2$, and since the theory is free these vacua are not lifted.

We can make this model more interesting by adding an additional term to the Kähler potential:

$$\mathscr{L} = \int d^4\theta\, \left(\bar{\Phi}\Phi - \frac{1}{4\Lambda^2}\bar{\Phi}^2\Phi^2\right) + \int d^2\theta\, f\Phi + \text{c.c.} \ . \qquad (51)$$

[9] The fact that this theory is quantum mechanically anomalous will not be important for us.

This is an effective theory below the high cutoff scale Λ. The scalar potential is now given by

$$V = \frac{|f|^2}{1-\frac{1}{\Lambda^2}|\phi|^2} = |f|^2\left(1+\frac{|\phi|^2}{\Lambda^2}+\cdots\right) \ . \quad (52)$$

Now there is a single SUSY-breaking vacuum at $\phi = 0$. The boson ϕ has acquired a mass $|f|^2/\Lambda^2$, while the massless fermion ψ_α is again identified with the Goldstino.

The model (51) is not renormalizable. The simplest non-trivial, renormalizable model of spontaneous SUSY-breaking is the O'Raifeartaigh model [3]. The model has three chiral superfields $X, \Phi, \tilde{\Phi}$ with canonical Kähler potential and superpotential

$$W = X\Phi^2 + m\Phi\tilde{\Phi} + fX \ . \quad (53)$$

The conditions for the existence of a SUSY vacuum are $\phi = 0$, $\phi^2 = f$, and $m\tilde{\phi} = 2x\phi$, where we denote by $x, \phi, \tilde{\phi}$ the bottom components of $X, \Phi, \tilde{\Phi}$. Clearly these conditions are inconsistent and there is no supersymmetric vacuum. By studying minima of the full scalar potential V, we see that the theory has a SUSY-breaking vacuum with energy $V = |f|^2$ at $\phi = \tilde{\phi} = 0$. (This vacuum is only stable if $2|f| < |m|^2$; if this is not the case, the structure of the vacuum is somewhat different, but SUSY is still broken.) For $\phi = 0$, the scalar potential is independent of x. The field x is thus massless and can take on any vacuum expectation value. Just like the trivial theory (50), the O'Raifeartaigh model thus has infinitely many SUSY-breaking vacua at tree level. (This is a very general property of such theories [4,5].) However, unlike the previous theory which was free, it is now possible for radiative corrections to lift this vacuum degeneracy: x becomes massive and is stabilized at the origin by the one-loop correction to the scalar potential:

$$V^{(1)} \sim \frac{1}{16\pi^2}\frac{|f|^2}{m^2}|x|^2 + \cdots \ . \quad (54)$$

Note that after integrating out the heavy fields $\Phi, \tilde{\Phi}$ of mass $\sim m$, the low-energy dynamics of X is governed by an effective theory of the form (51) with the cutoff given by $\Lambda \sim m$.

In the three examples above SUSY was broken because the vacuum energy was positive due to non-vanishing F-terms coming from chiral superfields. It is also possible to break supersymmetry through non-vanishing D-terms coming from vector superfields. The prototypical such example is the FI-model:

$$\mathscr{L} = \frac{1}{4e^2}\int d^2\theta\, W^\alpha W_\alpha + \text{c.c.} + \int \xi d^4\theta\, V \ . \quad (55)$$

This model is free, but has a vacuum energy $V = e^2\xi^2/8$. The free gauge field is necessarily massles, while the massless gaugino is the Goldstino.

3.2. Comments on Dynamical SUSY-Breaking

Unlike other symmetries, the spontaneous breaking of supersymmetry is highly constrained. It is a consequence of powerful non-renormalization theorems [6,7] that *supersymmetric vacua cannot be lifted by radiative corrections*: if there is a SUSY vacuum in the classical theory, then it cannot disappear in perturbation theory. Thus SUSY can only be broken in two ways:

- The classical theory has no SUSY vacua. In this case we say that it breaks SUSY at tree-level.
- The classical theory has SUSY vacua, but they are lifted by non-perturbative effects. This is known as dynamical SUSY-breaking.

All theories discussed in the previous subsection break SUSY at tree-level. As was emphasized by Witten [8], such models are unappealing because they force us to introduce dimensionful parameters by hand so that we still have to explain the large hierarchy between the electroweak scale and the Planck or GUT scale.

On the other hand, the option of dynamical SUSY-breaking is very appealing. The scale $\sqrt{f}$ of supersymmetry breaking can now arise through dimensional transmutation

$$\sqrt{f} \sim \Lambda_{UV}\ e^{-\frac{8\pi^2}{g^2}} \ , \quad (56)$$

with $g \lesssim 1$ an asymtotically free gauge coupling, and thus $\sqrt{f}$ can naturally be exponentially smaller than the cutoff Λ_{UV}. This might explain why the weak scale is so much smaller than the Planck or GUT scale. We should thus be studying theories with strong dynamics in the IR which can trigger SUSY-breaking. This is a vast subject (see [9] and references therein), and we will restrict ourselves to a few general comments.

Roughly speaking, dynamical theories fall into three classes:

- Theories in which supersymmetry breaking is triggered by non-perturbative effects, but the vacuum is in the semiclassical regime.
- Theories in which the vacuum is in the strongly-coupled regime of the original degrees of freedom, but can still be analyzed via Seiberg duality [10].
- Strongly coupled theories in which little can be said about the vacuum, but for which there are convincing (usually indirect) arguments that SUSY is broken.

The first two types of models are referred to as calculable. At low energies, they are described by Wess-Zumino models, possibly with IR-free gauge fields and effective FI-terms. By studying such models we may hope to shed light on the dynamics of interesting dynamical models.

In the next section we will continue our study of interesting supermultiplets. We will then explain how these multiplets can be used to constrain the dynamics and low-energy behavior of various theories – both with and without SUSY breaking.

4. Supercurrents

4.1. The Ferrara-Zumino Multiplet

Four-dimensional theories with $\mathcal{N} = 1$ supersymmetry possess a conserved supersymmetry current $S_{\alpha\mu}$ satisfying (2). Just like any other well-defined local operator, we have to embed $S_{\alpha\mu}$ in some multiplet. In order to write down sensible constraints for such a supercurrent multiplet, we are guided by the following two observations:

- The SUSY-algebra (1) gives rise to the current algebra

$$\{\bar{Q}_{\dot{\alpha}}, S_{\alpha\nu}\} = 2\sigma^{\mu}_{\alpha\dot{\alpha}} T_{\mu\nu} + (\mathrm{S.T.}) \ , \qquad (57)$$

where (S.T.) are Schwinger Terms, and $T_{\mu\nu}$ is a conserved, symmetric energy momentum tensor:

$$\partial^{\nu} T_{\mu\nu} = 0 \ , \qquad P_{\mu} = \int d^3x \, T_{\mu}{}^{0} \ . \qquad (58)$$

The supersymmetry currrent and the energy-momentum tensor must thus belong to the same supermultiplet – a supercurrent.

- We expect $T_{\mu\nu}$ to be the highest spin component in the supercurrent multiplet. If this is not the case, then it might be problematic to couple the theory to supergravity.

We conclude that the supersymmetry current and the energy-momentum tensor are embedded in a real vector superfield $\mathcal{J}_\mu$ which schematically takes the form

$$\begin{aligned}\mathcal{J}_\mu = & \, j_\mu - i\theta\left(S_\mu + \cdots\right) + i\bar{\theta}\left(\bar{S}_\mu + \cdots\right) \\ & + \left(\theta\sigma^\nu\bar{\theta}\right)\left(2T_{\nu\mu} + \cdots\right) + \cdots \ . \end{aligned} \qquad (59)$$

The real vector j_μ is generally not conserved. As in the case of the global current multiplet, the dots are terms which are not uniquely fixed by the requirements outlined above, and different choices of Schwinger terms in in (57) give rise to different ways of completing the multiplet. Again we will not discuss the constructive approach to writing the most general solution [11]. Instead, we will begin by exploring the most conventional set of constraints for the supercurrent multiplet, only amending them when we are forced to do so.

The simplest and most widely known supercurrent is called the Ferrara-Zumino (FZ) multiplet [12]. It is defined by the equations[10]

$$\bar{D}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}} = D_\alpha X \ , \qquad \bar{D}_{\dot{\alpha}} X = 0 \ . \qquad (60)$$

[10] Our convention for switching between vectors and bispinors is $\ell_{\alpha\dot{\alpha}} = -2\sigma^{\mu}_{\alpha\dot{\alpha}}\ell_\mu$, $\ell_\mu = \frac{1}{4}\bar{\sigma}^{\dot{\alpha}\alpha}_{\mu}\ell_{\alpha\dot{\alpha}}$.

In components, the FZ-multiplet takes the form

$$\begin{aligned}\mathcal{J}_\mu &= j_\mu - i\theta\left(S_\mu + \frac{1}{3}\sigma_\mu\bar{\sigma}^\nu S_\nu\right) + \frac{i}{2}\theta^2\partial_\mu\bar{x}\\&+ i\bar{\theta}\left(\bar{S}_\mu + \frac{1}{3}\bar{\sigma}_\mu\sigma^\nu\bar{S}_\nu\right) - \frac{i}{2}\bar{\theta}^2\partial_\mu x\\&+ \left(\theta\sigma^\nu\bar{\theta}\right)\left(2T_{\mu\nu} - \frac{2}{3}\eta_{\mu\nu}T - \frac{1}{2}\varepsilon_{\nu\mu\rho\sigma}\partial^\rho j^\sigma\right)\\&+ \cdots \ . \end{aligned} \tag{61}$$

Here $T = T^\lambda{}_\lambda$ is the trace of the energy-momentum tensor. The higher components of $\mathcal{J}_{\alpha\dot{\alpha}}$ only contain derivatives of components which we have already displayed. The chiral superfield X takes the form

$$X = x(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) \ , \tag{62}$$

with

$$\begin{aligned}\psi &= \frac{i\sqrt{2}}{3}\sigma^\mu_{\alpha\dot{\alpha}}\bar{S}^{\dot{\alpha}}_\mu \ ,\\ F &= \frac{2}{3}T + i\partial^\mu j_\mu \ . \end{aligned} \tag{63}$$

We see that the FZ-multiplet contains $12 = 2\ [x] + 4\ [j_\mu] + 6\ [T_{\mu\nu}]$ independent bosonic components, which match the 12 fermionic components of $S_{\alpha\mu}$. Thus, the FZ-multiplet is a $12+12$ multiplet. It is straightforward to check that the explicit component expressions (61) and (62) lead to a SUSY current algebra of the form (57):

$$\begin{aligned}\{\bar{Q}_{\dot{\alpha}}, S_{\alpha\mu}\} = \sigma^\nu_{\alpha\dot{\alpha}}\Big(&2T_{\nu\mu} + i\partial_\nu j_\mu\\ &- i\eta_{\mu\nu}\partial^\lambda j_\lambda - \frac{1}{2}\varepsilon_{\nu\mu\rho\lambda}\partial^\rho j^\lambda\Big) \ . \end{aligned} \tag{64}$$

When X vanishes, then we see from (63) that the energy-momentum tensor is traceless, and that j_μ is now a conserved current. In this case, the theory is superconformal and the FZ-multiplet only has $8+8$ components.

As the simplest example, we again consider the free theory (11) of a single chiral superfield Φ. Using the superspace equation of motion $D^2\Phi = 0$ it is straightforward to check that the multiplet

$$\mathcal{J}_{\alpha\dot{\alpha}} = 2D_\alpha\Phi\bar{D}_{\dot{\alpha}}\bar{\Phi} - \frac{2}{3}[D_\alpha, \bar{D}_{\dot{\alpha}}]\bar{\Phi}\Phi \tag{65}$$

satisfies

$$\bar{D}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = 0 \ . \tag{66}$$

As expected, the FZ-multiplet reduces to the superconformal multiplet for the free scalar. By expanding Φ in components and comparing with (61), we see that the energy-momentum tensor embedded in $\mathcal{J}_{\alpha\dot{\alpha}}$ reads

$$\begin{aligned}T_{\mu\nu} &= \partial_\mu\bar{\phi}\partial_\nu\phi + (\mu \leftrightarrow \nu) - \eta_{\mu\nu}\partial^\lambda\bar{\phi}\partial_\lambda\phi\\ &- \frac{1}{3}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2\right)|\phi|^2 + (\text{fermions}) \ . \end{aligned} \tag{67}$$

The first line is the familiar energy-momentum tensor for a free, massless scalar. The second line is an improvement term: it is automatically conserved (without using the equations of motion), and it does not affect the conserved charge P_μ because setting $\nu = 0$ turns it into a total space derivative. We are free to add these kind of improvement terms to any conserved current. In this case the constraint $\bar{D}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = 0$ fixes the improvement term in a particular way: it guarantees that $T_{\mu\nu}$ is traceless. The relation between supersymmetric constraints and improvement terms will play a pivotal role in what follows.

More generally, we can consider the Wess-Zumino model (20). Using the equations of motion $\bar{D}^2\partial_i K = 4\partial_i W$, we can check that

$$\begin{aligned}\mathcal{J}_{\alpha\dot{\alpha}} &= 2g_{i\bar{j}}D_\alpha\Phi^i\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{j}} - \frac{2}{3}[D_\alpha, \bar{D}_{\dot{\alpha}}]K \ ,\\ X &= 4W - \frac{1}{3}\bar{D}^2 K \end{aligned} \tag{68}$$

satisfy the defining equation (60) of the FZ-multiplet. We will make use of this formula on several occasions.

Finally, the FZ-multiplet for the FI-model (55) readily follows from the equation of motion $D^\alpha W_\alpha = e^2\xi$. It is given by:

$$\begin{aligned}\mathcal{J}_{\alpha\dot{\alpha}} &= -\frac{4}{e^2}W_\alpha\bar{W}_{\dot{\alpha}} - \frac{2}{3}\xi[D_\alpha, \bar{D}_{\dot{\alpha}}]V \ ,\\ X &= -\frac{\xi}{3}\bar{D}^2 V \ . \end{aligned} \tag{69}$$

It is similarly straightforward to obtain the FZ-multiplet for gauge theories coupled to matter (we will not write down the most general expression here).

Looking at (55) we make an unsettling observation: in the presence of an FI-term ξ, neither $\mathcal{J}_{\alpha\dot{\alpha}}$ nor X are invariant under the usual gauge-transformation $\delta V = i(\Lambda - \bar{\Lambda})$, with Λ chiral [13]. (This conclusion is not changed by the inclusion of matter.) Thus, the FZ-multiplet is not gauge-invariant, and consequently its components are not well-defined operators. From this, we might mistakenly conclude that models with FI-terms are inherently ill-defined, because they do not have a well-defined supersymmetry current or energy-momentum tensor.

To see why this is not so, we examine the gauge non-invariance of the FZ-multiplet (69) in more detail. Under a gauge transformation, this multiplet transforms as follows:

$$\delta\mathcal{J}_{\alpha\dot{\alpha}} = -\frac{2}{3}\xi\partial_{\alpha\dot{\alpha}}\left(\Lambda + \bar{\Lambda}\right) \ ,$$
$$\delta X = \frac{i}{3}\xi\bar{D}^2\bar{\Lambda} \ . \tag{70}$$

By expanding these expressions in components and comparing with (61), we see that the bottom component $j_\mu \sim \xi A_\mu +$ (fermions) explicitly depends on the gauge field A_μ and is simply not gauge-invariant. However, the gauge non-invariance of the energy-momentum tensor takes the special form

$$\delta T_{\mu\nu} = \frac{2}{3}\xi\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2\right)\mathrm{Im}\Lambda| \ , \tag{71}$$

where $\Lambda|$ denotes the bottom component of the superfield Λ. We see that the transformation of $T_{\mu\nu}$ in (70) is a pure improvement term. As we explained above, this means that it is automatically conserved and does not affect the conserved charged P_μ. Completely analogously, it can be checked that the supersymmetry current $S_{\alpha\mu}$ also only shifts by a pure improvement term. The FI-model thus has well-defined, gauge-invariant operators P_μ, Q_α, which make it a well-defined supersymmetric field theory.

How, then, do we interpret the fact that the supersymmetry current and the energy-momentum tensor embedded in the FZ-multiplet (61) are not gauge-invariant and do not exist as well-defined operators? The answer is that $S_{\alpha\mu}$ and $T_{\mu\nu}$ are not unique; they are only defined up to improvement terms. Different choices of improvement terms lead to different local currents, but do not affect the corresponding charges. It is usually possible to chose the improvement terms in such a way that the currents are well-defined operators. However, a problem could arise when attempting to embed these operators into supermultiplets. In this case there may be a clash between the existence of certain supermultiplets and the requirement that these multiplets be gauge invariant. This is precisely what happens in the FI-model: forcing $S_{\alpha\mu}$ and $T_{\mu\nu}$ into an FZ-multiplet requires us to pick gauge non-invariant improvement terms. Conversely, it is possible to pick gauge-invariant improvement terms for $S_{\alpha\mu}$ and $T_{\mu\nu}$, but this makes it impossible to embed these operators into an FZ-multiplet. In this case it is necessary (and possible) to find different supermultiplets into which we can embed gauge-invariant choices for $S_{\alpha\mu}$ and $T_{\mu\nu}$. Such multiplets will be discussed in section 6.

In light of the preceding discussion, we are led to carefully reconsider the existence of the FZ-multiplet (68) for the Wess-Zumino model. We see that the Kähler potential K formally appears in the same way as the vector field V in the FZ-multiplet for the FI-model. Morover, Kähler transformations of K are take the same form as gauge-transformations of V. We conclude that the FZ-multiplet of the Wess-Zumino model is not invariant under Kähler transformations. Again, these Kähler transformations only change the supersymmetry current and the energy-momentum tensor by improvement terms.

However, unlike gauge transformations, Kähler transformations are not an absolute necessity. It often does not matter if the multiplet transforms under Kähler transformations. The only become essential when the target space $\mathcal{M}$ of the sigma model needs to be covered with several patches and Kähler transformations are needed to switch between the patches. This happens, for instance, in the $\mathbb{CP}^1$ model described in subsection 1.2. In such theories, the FZ-multiplet is not a well-defined operator. Equivalent ways of describing the target space of the theory result in energy-momentum tensors which differ by improvement terms, even though they should be identical, and likewise for the supersymmetry currents. In other words, there is no unambiguous way of fixing these operators.

We would like to understand the precise mathematical conditions under which the FZ-multiplet for Wess-Zumino models is well-defined. From the metric $g_{i\bar{j}}$ we can construct the Kähler form

$$\omega = i g_{i\bar{j}} d\phi^i \wedge d\bar{\phi}^{\bar{j}} \ . \tag{72}$$

Since $g_{i\bar{j}}$ is locally derived from a Kähler potential, ω is closed: $d\omega = 0$. In every patch we can thus find a one-form $\mathcal{A}$ such that $\omega = d\mathcal{A}$. Locally, this Kähler connection is given by $\mathcal{A} \sim i\partial_i K d\phi^i + \text{c.c.}$. In general, ω is not exact and the Kähler connection $\mathcal{A}$ is not globally well-defined. The obstruction to the global existence of $\mathcal{A}$ is measured by the cohomology class $[\omega] \in H^2(\mathcal{M})$. If ω vanishes in $H^2(\mathcal{M})$, then $\mathcal{A}$ is globally well-defined and thus a good operator the theory.

The bottom component of (68) is given by

$$j_\mu = \frac{2i}{3}\partial_i K \partial_\mu \phi^i + \text{c.c.} - \frac{1}{3} g_{i\bar{j}} \bar{\psi}^{\bar{j}} \bar{\sigma}_\mu \psi^i \ . \tag{73}$$

While the fermionic piece only depends on the metric and is thus invariant under Kähler transformations, we recognize the bosonic piece as the pull-back to space-time of the Kähler connection $\mathcal{A}$. This is is well-defined only if ω is exact and vanishes in $H^2(\mathcal{M})$. If this condition is not satisfied, then j_μ is not a good local operator in the theory, since it depends on the choice of patch in target space. In this case, the entire FZ-multiplet (68) is not well-defined [14].

As a special case, note that the Kähler form of a compact manifold can never be exact. If this were the case, then the volume form $\omega^{\dim(\mathcal{M})/2}$ would also be exact and thus integrate to zero on the compact manifold; this is a contradiction. Thus, a Wess-Zumino model with compact target manifold, such as the $\mathbb{CP}^1$-model (18), cannot have a well-defined FZ-multiplet.

In this section we have explored the FZ-multiplet, in which the supersymmetry current and the energy-momentum tensor are conventionally embedded. This multiplet exists in the vast majority of SUSY theories, including asymptotically free gauge theories, such as supersymmetric QCD (SQCD). As far as we know, the FZ-multiplet only fails to exist when one of the following conditions holds:

- Some $U(1)$ gauge group has an FI-term.
- The target-space has non-trivial topology, so that the Kähler form is not exact.

In the first case, the FZ-multiplet is not gauge-invariant, while in the second case it is not globally well-defined. These insights will enable us to derive exact results about the dynamics of SUSY field theories, SUSY-breaking, and supergravity.

4.2. Consequences for SUSY Theories

It is a general fact about quantum field theory that operator equations are invariant under renormalization group (RG) flow as long as the operators involved are well-defined and gauge-invariant. This means that if some operator relation holds in the UV (where field theories are usually defined), then it continues to hold along the entire RG-flow. Applying this reasoning to the FZ-multiplet immediately leads to constraints on the RG-flow of supersymmetric field theories.

Consider a supersymmetric field theory which has no FI-terms in the UV. This means that this theory has a well-defined, gauge invariant FZ-multiplet. This multiplet must therefore persist along the entire RG-flow. We conclude that FI-terms cannot be generated along the flow; this is true at any order in perturbation theory and even non-perturbatively. For instance, it might happen that the theory flows through a regime with strong dynamics, and that a $U(1)$ gauge symmetry emerges in some weakly-coupled dual description. In this case the gauge-invariance of the FZ-multiplet prevents this emergent $U(1)$ gauge theory from having an FI-term. It is even possible to argue that if an FI-term is present in the UV, then its value is not renormalized. Other derivations of these results can be found in [15–18].

Completely analogously, let us consider a theory which has a well-defined, gauge-invariant FZ-multiplet in the UV. For example, we could consider an ordinary SUSY gauge theory, with canonical Kähler potential for the matter fields and no FI-terms. At low energies, such field theories are often described by a weakly-coupled σ-model (perhaps with IR-free gauge fields). It is expected that this will happen whenever the field theory has a strong coupling scale Λ, below which it is described by massless moduli. Since this theory has a well-defined FZ-multiplet in the UV, and this multiplet persists along the entire RG-flow, we conclude that the low-energy σ-model must have have an exact Kähler form ω. This severely constrains the quantum moduli space of the low-energy theory: the integral $\int \omega \wedge \omega \wedge \cdots$ over any compact, even-dimensional cycle must vanish. As we already explained in the previous subsection, the means in particular that the moduli space cannot be compact (of course, it can be a set of points). It is particularly interesting to test this theorem in SQCD with the same number of colors and flavors. In this case the topology of the modul-space changes under RG-flow and becomes non-trivial in the IR [19]. However, in accordance with the general result above, the Kähler form of this deformed quantum moduli space is exact.

The formal similarity between the two theorems described above is not coincidental. On the one hand, they are both consequences of the RG-invariance of the FZ-multiplet. On the other hand, it may be that if a theory does not have an FZ-multiplet in the UV due to the presence of an FI-term, it may flow to a σ-model which fails to have an FZ-multiplet because its target space has a non-exact Kähler form.

The simplest example of this phenomenon is the theory discussed at the end of subsection 2.2. In the UV, this theory starts out as an abelian gauge theory with an FI-term ξ and two chiral matter fields of charge $+1$. When $\xi < 0$, it follows from the scalar potential (46) that the low-energy theory which describes the moduli space of SUSY-vacua is given by the $\mathbb{CP}^1$-model, whose target space is compact.

5. Applications to SUSY-Breaking

In this section we will consider two applications of the FZ-multiplet to SUSY-breaking. The first application rests on the relation of the FZ-multiplet to the Goldstino. This will enable us to derive exact results about such SUSY-breaking theories. It will also lead to a useful formalism for writing Lagrangians with non-linearly realized supersymmetry. The second application concerns the interplay of F-terms and D-terms in SUSY-breaking theories. We show under very broad assumptions that there can be no SUSY-breaking vacua (even meta-stable ones) in which the D-terms are parametrically larger than the F-terms.

5.1. The FZ-Multiplet and the Goldstino

A universal prediction of spontaneous SUSY-breaking is the existence of a massless Weyl fermion, the Goldstino G_α. We will not do justice to the extensive literature on the Goldstino. Most of the references can be found in [20], which is also the main reference for this subsection.

When a global symmetry is spontaneously broken, the corresponding conserved charge is not a well-defined operator because its correlation functions are IR divergent. As a result, the physical states of the theory are not in linear representations of the symmetry. However, the conserved current and even commutators of local operators with the conserved charge do exist. In contrast to the states, the local operators do in fact furnish a linear representation of the symmetry. These statements carry over without modification when supersymmetry is spontaneously broken in infinite volume: the supercharge Q_α does not exist, but the supersymmtry current and (anti-) commutators with Q_α do exist. (When SUSY is spontaneously broken in finite volume, even Q_α exists.) This means that operators still reside in supermultiplets, and that we can use the formalism of superspace and superfields without modification.

Recall the defining equation (60) of the FZ-multiplet:

$$\bar{D}^{\dot\alpha} \mathcal{J}_{\alpha\dot\alpha} = D_\alpha X \ , \tag{74}$$

where X is chiral. In this subsection we will only discuss theories which admit a well-defined, gauge-invariant FZ-multiplet. This multiplet requires the existence of a complex scalar x, the bottom component of X, which is a well-defined operator in the theory. If we are given a microscopic description of the theory in the UV, we can express x in terms of elementary fields.

The chiral superfield X can be consistently followed along the RG-flow down to the IR, even when SUSY is spontaneously broken. We can see this in the simplest SUSY-breaking model (50), which only has a single chiral superfield Φ. In this theory, the F-term is non-zero, but ϕ and ψ_α are free, massless fields. As was discussed in subsection 3.1, we identify ψ_α with the Goldstino G_α. Since this theory is a Wess-Zumino model with $K = \bar{\Phi}\Phi$ and $W = f\Phi$, we find from (20) that $X = \frac{8}{3}f\Phi$. In this simple example, the operator X is thus proportional to Φ, so that the θ-component of X is proportional to the Goldstino.

More generally, standard arguments about symmetry breaking show that when supersymmetry is spontaneously broken, the low-energy supersymmetry current is expressed in terms of the Goldstino G_α as

$$S_{\alpha\mu} = \sqrt{2}f\sigma_{\mu\alpha\dot{\alpha}}\bar{G}^{\dot{\alpha}} + f'(\sigma_{\mu\nu})_\alpha{}^\beta\partial^\nu G_\beta + \cdots \ , \quad (75)$$

for some constants f and f'. Note that the term with f' is an improvement term and can be ignored. We see that at low energies, the spin-$\frac{3}{2}$ component of the supersymmetry current essentially decouples, while the spin-$\frac{1}{2}$ component is proportional to the Goldstino. Since we know from (63) that the θ-component of X is given by the spin-$\frac{1}{2}$ projection of $S_{\alpha\mu}$, we again conclude that at long distance the θ-component of X is proportional to the Goldstino.

In the trivial example of a free chiral superfield Φ, the lowest component x of the superfield X is proportional to the free scalar ϕ. In general, the low-energy Goldstino is not accompanied by a massless scalar. Then, the bosonic operator x cannot create a-one particle state. Instead, the simplest bosonic state it can create contains two Goldstinos. When supersymmetry is broken in finite volume, the states in the Hilbert space are in linear supersymmetry multiplets. The supersymmetric partner of a one Goldstino state $\bar{Q}_{\dot{\alpha}}|0\rangle$ is a two Goldstino state $\bar{Q}_{\dot{1}}\bar{Q}_{\dot{2}}|0\rangle$. At infinite volume the supercharge does not exist and the zero momentum Goldstino state is not normalizable. However, the finite volume intuition is still valid. The operator ψ_ϕ creates a one Goldstino state and its superpartner $\phi \sim x$ creates a two-Goldstino state.

We denote this non-linear superfield which contains the Goldstino bilinear in the bottom component X_{NL}, and x_{NL} denotes the bottom component itself. Consistency of this superfield under supersymmetry transformations (which, again, are legal even if SUSY is spontaneously broken) fixes it to take the form

$$X_{NL} = \frac{G^2}{2F} + \sqrt{2}\theta G + \theta^2 F \ , \quad (76)$$

where all the fields are functions of y^μ. Clearly, (76) satisfies the interesting operator identity

$$X_{NL}^2 = 0 \ . \quad (77)$$

(Other realizations of nonlinear supersymmetry can be found, for example, in [21,22])

We conclude that the Goldstino resides in a chiral superfield X_{NL} which satisfies $X_{NL}^2 = 0$. Furthermore, this chiral superfield is the IR limit of the microscopic superfield X. Surprisingly, this result is true both for F-term and for D-term breaking because it relies only on the existence of the chiral operator X. (The only exception is the situation with pure D-term breaking which occurs when a tree-level FI-term is present for an unbroken $U(1)$ gauge theory. In this case the operator X is not gauge invariant and we do not discuss it here.[11] Thus, our discussion is applicable in all the interesting models of dynamical SUSY breaking.

We will momentarily see that this identification of the low-energy limit of the operator X, which exists in all field theories we are interested in, allows us to derive some non-perturbative results.

It is instructive to compare the situation here with the theory of ordinary Goldstone bosons like pions. Clearly, the decay constant f here is analogous to the decay constant f_π of pion physics. Both of them are well defined. However, in pion

[11] However, our discussion does apply in the Higgs phase of the FI-model [23], where the low-energy limit of X can be rendered gauge invariant by dressing it with fields that obtain expectation values.

physics the order parameter $\langle\psi\psi\rangle$ for chiral symmetry breaking does not have a universal definition with a precise normalization (it suffers from wavefunction renormalization). In our case the order parameter for supersymmetry breaking is the energy-momentum tensor which resides in the same multiplet as the supersymmetry current. Hence, it has a well-defined normalization. Therefore, our X is completely well-defined. Correspondingly, the operator $\psi\psi$ of pion physics acts as an interpolating field for pions, but its normalization is not meaningful. In our case X is used both as an order parameter for supersymmetry breaking and as an interpolating field for Goldstinos with well-defined normalization.

The analogy with pion physics also clarifies the meaning of our constraint (77). It arises from removing the massless scalar in X. This is analogous to describing pion physics by starting with a linear sigma model with a σ-field. Removing the σ-field is implemented by imposing a constraint $UU^\dagger = 1$, which is analogous to our $X_{NL}^2 = 0$.

Since in every microscopic theory we can identify X in the ultraviolet, and if SUSY is spontaneously broken we also know its universal low-energy limit, we can calculate various correlation functions of operators at large separations even in strongly coupled models. Hence even "incalculable" models of SUSY breaking (like the $SU(5)$ theory of [24] or the $SO(10)$ theory of [25]) have a solvable sector at long distances. The operator x interpolates between the vacuum and a state with two Goldstinos, with a universal normalization. This is because at long distances the leading behavior of x is proportional to that of G^2. Therefore,

$$\lim_{|\mathbf{r}|\to\infty} \langle x(\mathbf{r})\bar{x}(0)\rangle = \left(\frac{4}{3\pi^2}\right)^2 \frac{1}{|\mathbf{r}|^6} \ . \tag{78}$$

This is independent of the details of the microscopic theory and its coupling constants. In a similar fashion we can calculate the long distance limit of any correlation function of x and $\bar{x}$.

This idea can be pushed further and many more non-perturbative results of this form can be obtained. In fact, a pretty rich set of exact results on SUSY-breaking theories arises with interesting underlying mathematical structure, but we will not elaborate on this any further here.

We will now explain how to use X_{NL} to write supersymmetric effective Lagrangians. We start without including derivatives. At that level, the most general supersymmetric Lagrangian subject to the constraint $X_{NL}^2 = 0$ is

$$\int d^4\theta\, \bar{X}_{NL} X_{NL} + \int d^2\theta\, f X_{NL} + \text{c.c.}\ , \tag{79}$$

where without loss of generality we take f to be real. This looks like the free chiral multiplet except that the superfield is constrained. This constraint removes the massless scalar field and introduces nonlinearities.

More explicitly, the constraint can be solved as in (76). Substituting this in (79), we derive the component Lagrangian

$$i\partial_\mu \bar{G}\bar{\sigma}^\mu G + \bar{F}F + \frac{\bar{G}^2}{2\bar{F}}\partial^2\left(\frac{G^2}{2F}\right) + (fF + \text{c.c.})\ . \tag{80}$$

The equations of motion of the auxiliary fields $F, \bar{F}$ can be solved and upon substituting this back in the component Lagrangian we find

$$\begin{aligned}\mathcal{L} = &-f^2 + i\partial_\mu \bar{G}\bar{\sigma}^\mu G + \tfrac{1}{4f^2}\bar{G}^2\partial^2 G^2 \\ &- \tfrac{1}{16f^6} G^2\bar{G}^2\partial^2 G^2 \partial^2 \bar{G}^2\ .\end{aligned} \tag{81}$$

This is equivalent to the Akulov-Volkov Lagrangian [26]. (See also [27–29].) Here we have given a fully off-shell supersymmetric description of this theory.

The real advantage of this approach is the simplifications in writing higher derivative corrections to (79) and, more interestingly, couplings to possibly light matter fields (such as MSSM fields, Goldstone bosons, 't Hooft fermions etc.). Indeed such problems are easily solved in this formalism since we have an off-shell description and so we can use all the familiar superspace techniques for writing Lagrangians (the number of allowed operators is smaller because of the various constraints).

This toolbox can be put to use in many possible contexts and some recent examples include [30–37]

The last thing we would like to demonstrate before closing this subsection is the way the nilpotent equation (77) arises from the dynamics in a simple example. In order to remove the massless scalar we include a non-canonical Kähler potential

$$K = \bar{\Phi}\Phi - \frac{c}{M^2}\bar{\Phi}^2\Phi^2 \ , \tag{82}$$

and

$$W = f\Phi \ . \tag{83}$$

Here c is a dimensionless positive number of order one. Such a theory can arise as the low-energy Lagrangian below some scale M after neglecting higher order terms in $\frac{1}{M}$. (For example, the low-energy limit of the standard O'Raifeartaigh model looks like (82).) It is valid for

$$\sqrt{f} \ll E \ll M \ . \tag{84}$$

Let us integrate out the massive bosons. Remembering that the Lagrangian contains interaction terms of the form $-\frac{c}{M^2}\left|2\phi F_\phi - \psi^2\right|^2$, the zero-momentum equation of motion of ϕ sets it to

$$\phi = \frac{\psi^2}{2F_\phi} \ . \tag{85}$$

Note that it is independent of c, or in other words, it is independent of the details of the high energy physics (this is an example of our universality).

Upon substituting (85) back into Φ, we discover that in the $\Phi^2 = 0$. Hence, it satisfies the same constraint as X_{NL}. To make the relation more precise, we can go through a careful calculation of the supercurrent. The main point, however, is that the nilpotentcy equation arises from integrating out heavy degrees of freedom.

5.2. Restrictions on Large D-Terms

In subsection 3.1 we encountered several simple models of F-term and D-term SUSY-breaking. While models with FI-terms lead to many tree-level examples of D-term SUSY-breaking, most known calculable models of dynamical supersymmetry breaking are predominately F-term driven. We would like to understand whether this has to hold in general, or whether there are in fact dynamical models with large D-terms. In this subsection, we will show that such models do not exist: under broad assumptions, the D-terms are necessarily parametrically smaller than the F-terms [36].

To get an intuitive picture, we can loosely identify D-term breaking with the presence of FI-terms. In dynamical models, one usually starts with a well-behaved, asymptotically-free gauge theory without FI-terms in the UV. Since such theories has a well-defined FZ-multiplet, the discussion of subsection 4.2 show that they cannot acquire FI-terms at low energies. Roughly speaking, the absence of low-energy FI-terms implies the absence of large D-terms.

We can make this intuitive picture precise by using the tools we have developed so far. Since we are interested in calculable models with parametrically small F-terms, we consider theories in which the low-energy dynamics responsible for SUSY-breaking is described by a σ-model (13) and the F-terms are set to zero in first approximation. In addition, we will include IR-free gauge fields by gauging some global symmetries of the σ-model. If the D-term potential which results from this gauging leads to SUSY-breaking vacua (even meta-stable ones), then these putative vacua will have parametrically large D-terms which are larger than the F-terms by inverse powers of the small, IR-free gauge couplings. We will now show that such vacua do not exist in theories which have a well-defined, gauge-invariant FZ-multiplet. (Therefore they cannot exist in interesting dynamical models.)

As was discussed at the end of subsection 2.2, the Lagrangian for the gauged σ-model is given by making the substitution

$$K \to K + \mathcal{J}^{(a)}V^{(a)} \ , \tag{86}$$

and including conventional kinetic terms for the gauge fields. By making an identical substitution in expression (68) for the FZ-multiplet of the σ-model, we see that a gauge-transformation (48) changes this new FZ-multiplet $\mathcal{J}_{\alpha\dot{\alpha}}$ by an amount

$$\delta\mathcal{J}_{\alpha\dot{\alpha}} \sim i\partial_{\alpha\dot{\alpha}}\Lambda^{(a)}F^{(a)} + \text{c.c.} \ . \tag{87}$$

Since we assumed that the UV theory had a well-defined FZ-multiplet, the same must be true for the low-energy gauged σ-model. Thus, all functions $F^{(a)}$ must vanish. The Noether currents $\mathcal{J}^{(a)}$ now take the simple form

$$\mathcal{J}^{(a)} = iX^{(a)i}\partial_i K \ . \tag{88}$$

Using this expression, it is straightforward to check that the D-term potential (49) of the gauged σ-model satisfies the identity

$$V_D = \frac{1}{2}g^{i\bar{j}}\partial_i K \partial_{\bar{j}} V_D \ . \tag{89}$$

This identity immediately implies that a critical point of the potential can only occur when $V_D = 0$. In other words, every critical point must be a supersymmetric minimum and this D-term potential does not admit SUSY-breaking vacua – not even metastable ones.

The argument we have given above explains the absence of dynamical models with parametrically large D-terms. It is, however, possible to build models of dynamical SUSY-breaking with comparable D-terms and F-terms (see [36] and references therein).

6. More Supercurrent Multiplets

We have seen that there are well-defined supersymmetric field theories which do not admit an FZ-multiplet. In these theories, we must search for other, well-defined supercurrent multiplets into which the energy-momentum tensor and the supersymmetry current can be embedded. Note that the existence of such alternative multiplets has no effect on the results we extracted in previous sections, which were based on the existence of a well-defined FZ-multiplet.

There is indeed a supercurrent multiplet $\mathcal{S}_{\alpha\dot{\alpha}}$ which exists in theories with FI-terms or non-exact Kähler form, which do not admit an FZ-multiplet. This supercurrent multiplet satisfies the defining equations

$$\begin{aligned} &\bar{D}^{\dot{\alpha}}\mathcal{S}_{\alpha\dot{\alpha}} = D_\alpha X + \chi_\alpha \ , \qquad \bar{D}_{\dot{\alpha}} X = 0 \ , \\ &\bar{D}_{\dot{\alpha}}\chi_\alpha = 0 \ , \qquad D^\alpha \chi_\alpha = \bar{D}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \ . \end{aligned} \tag{90}$$

Note that X appears exactly as in the FZ-multiplet, while χ_α satisfies the same constraints as the field-strength superfield W_α discussed in subsection 1.3. In components, the $\mathcal{S}$-multiplet takes the form

$$\begin{aligned} \mathcal{S}_\mu &= j_\mu - i\theta\left(S_\mu - \frac{i}{\sqrt{2}}\sigma_\mu\bar{\psi}\right) + \frac{i}{2}\theta^2\partial_\mu\bar{x} \\ &+ i\bar{\theta}\left(\bar{S}_\mu - \frac{i}{\sqrt{2}}\bar{\sigma}_\mu\psi\right) - \frac{i}{2}\bar{\theta}^2\partial_\mu x + \left(\theta\sigma^\nu\bar{\theta}\right)\left(2T_{\mu\nu}\right. \\ &\left. - \eta_{\mu\nu}Z + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\rho j^\sigma + \frac{1}{8}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\right) + \dots \ . \end{aligned} \tag{91}$$

The chiral superfields X and χ_α are given by

$$\begin{aligned} X &= x(y) + \sqrt{2}\theta\psi(y) + \theta^2\left(Z(y) + i\partial^\nu j_\nu(y)\right) \ , \\ \chi_\alpha &= -i\lambda_\alpha(y) + \theta_\beta\left(\delta_\alpha{}^\beta D(y) - i(\sigma^{\mu\nu})_\alpha{}^\beta F_{\mu\nu}(y)\right) \\ &+ \theta^2\sigma^\nu_{\alpha\dot{\alpha}}\partial_\nu\bar{\lambda}^{\dot{\alpha}}(y) \ , \end{aligned} \tag{92}$$

with

$$\begin{aligned} D &= -4T + 6Z \ , \\ \lambda &= 2\sigma^\mu\bar{S}_\mu + 3i\sqrt{2}\psi \ . \end{aligned} \tag{93}$$

As before, the operator j_μ is not in general conserved, while the supersymmetry current $S_{\alpha\mu}$ and the symmetric energy-momentum tensor $T_{\mu\nu}$ are conserved. The $\mathcal{S}$-multiplet has 4+4 more degrees of freedom than the FZ-multiplet. They consists of the real scalar Z, the closed, real two-form $F_{\mu\nu}$ and the Weyl fermion ψ_α. The $\mathcal{S}$-multiplet is thus a $16 + 16$ multiplet.[12]

As in the case of the FZ-multiplet, the component expressions (91)and (92) lead to a SUSY current algebra of the form (57):

$$\begin{aligned} \{\bar{Q}_{\dot{\alpha}}, S_{\alpha\mu}\} = \sigma^\nu_{\alpha\dot{\alpha}}\Big(2T_{\mu\nu} - \frac{1}{8}\epsilon_{\nu\mu\rho\sigma}F^{\rho\sigma} + i\partial_\nu j_\mu \\ - i\eta_{\nu\mu}\partial^\rho j_\rho - \frac{1}{2}\epsilon_{\nu\mu\rho\sigma}\partial^\rho j^\sigma\Big) \ . \end{aligned} \tag{94}$$

However, the Schwinger Terms are now mor complicated. The supersymmetry algebra (1) follows

[12] Setting $X = 0$ in the $\mathcal{S}$-multiplet results in the so-called $\mathcal{R}$-multiplet. It is a $12 + 12$ multiplet whose bottom component j_μ is a conserved R-current. We will not discuss the $\mathcal{R}$-multiplet in this review, although we will occasionally refer to it in passing.

provided that $\int d^3x\, F_{ij}$ vanishes for all spatial indices i, j.

All known supersymmetric field theories admit a well-defined $\mathcal{S}$-multiplet.[13] Those theories which do not have FI-terms or non-exact Kähler form also admit a well-defined FZ-multiplet.[14] For instance, consider a Wess-Zumino model with arbitrary K and W. If the Kähler form is not exact, then the FZ-multiplet does not exist. However, the $\mathcal{S}$-multiplet exists, and takes the form

$$\mathcal{S}_{\alpha\dot\alpha} = 2g_{i\bar j} D_\alpha \Phi^i \bar D_{\dot\alpha} \bar\Phi^{\bar j} \ , \tag{95}$$

with $X = 4W$ and $\chi_\alpha = \bar D^2 D_\alpha K$.

The existence of the $\mathcal{S}$-multiplet in theories without an FZ-multiplet has important consequences for supergravity theories. In the next section we will explain how to construct supergravity starting from the supercurrent, and this will lead to additional applications.

7. Elements of Linearized Supergravity and Constraints on Moduli

In this section we study the couplings of supercurrent multiplets to supergravity theories. We are only interested in linearized supergravity, namely the leading order in $\frac{1}{M_p}$. This approximate analysis is sufficient to derive several general results.

This approach to supergravity is taken, for example, in [2]. We begin with a review of the coupling of the FZ-multiplet to supergravity. We then explain the coupling of the $\mathcal{S}$-multiplet to supergravity.[15]

7.1. Gauging the FZ-Multiplet

We start by reviewing the coupling of the FZ-multiplet to linearized gravity. The FZ-multiplet contains a conserved energy-momentum tensor and supercurrent and can therefore be coupled to supergravity. The supergravity multiplet is embedded in a real vector superfield $H_{\alpha\dot\alpha}$. The $\theta\bar\theta$ component of $H_{\alpha\dot\alpha}$ contains the metric field, $h_{\mu\nu}$, a two form field $B_{\mu\nu}$, and a real scalar. The coupling of gravity to matter is dictated at leading order by

$$\int d^4\theta\, \mathcal{J}_{\alpha\dot\alpha} H^{\alpha\dot\alpha} \ . \tag{96}$$

We should impose gauge invariance, namely, the invariance under coordinate transformations and local supersymmetry transformations. The gauge parameters are embedded in a complex superfield L_α, which so far obey no constraints. We assign a transformation law to the supergravity fields of the form

$$H'_{\alpha\dot\alpha} = H_{\alpha\dot\alpha} + D_\alpha \bar L_{\dot\alpha} - \bar D_{\dot\alpha} L_\alpha \ . \tag{97}$$

where $\bar L_{\dot\alpha}$ is the complex conjugate of L_α, and thus this maintains the reality condition.

Requiring that (96) be invariant under these coordinate transformations, we get a constraint on the superfield L_α. Indeed, invariance requires that

$$0 = \int d^4\theta\, \bar D^{\dot\alpha} \mathcal{J}_{\alpha\dot\alpha} L^\alpha = \int d^4\theta\, X D^\alpha L_\alpha \ . \tag{98}$$

Since X is an unconstrained chiral superfield we get the complex equation

$$\bar D^2 D^\alpha L_\alpha = 0 \ . \tag{99}$$

The analog of the Wess-Zumino gauge is that the lowest components of H_μ vanish, i.e.

$$H_\mu\big| = H_\mu\big|_\theta = H_\mu\big|_{\bar\theta} = 0 \ , \tag{100}$$

as well as the fact that $H_\mu\big|_{\theta\sigma^\nu\bar\theta}$ is symmetric in μ and ν.

There is also some residual gauge freedom:

- $H_\mu\big|_{\theta^2}$ can be shifted by any complex divergenceless vector. This leaves only one complex degree of freedom, $\partial^\mu H_\mu\big|_{\theta^2}$.

- The metric field $h_{\mu\nu}$ transforms as

$$\delta h_{\mu\nu} = \partial_\nu \xi_\mu + \partial_\mu \xi_\nu \ , \tag{101}$$

  where ξ_μ is a real vector.

[13] There are, however, theories in which $D_\alpha X$ is well-defined, but X itself is not.
[14] R-symmetric theories always admit a well-defined $\mathcal{R}$-multiplet.
[15] The case of the $\mathcal{R}$-multiplet is very similar but will not be discussed in detail because gravitational theories with global symmetries do not seem to be relevant according to our current understanding of quantum gravity.

- The gravitino transforms as

$$\delta\Psi_{\mu\alpha} = \partial_\mu \omega_\alpha \ . \tag{102}$$

In this Wess-Zumino gauge the components containing the gravitino and metric take the form

$$H_\mu\big|_{\theta\sigma^\nu\bar\theta} = h_{\mu\nu} - \eta_{\mu\nu} h \ , \tag{103}$$

and

$$H_\mu\big|_{\bar\theta^2\theta} = \Psi_{\mu\alpha} + \sigma_\mu \bar\sigma^\rho \Psi_\rho \ . \tag{104}$$

The top component of H_μ is a vector field which survives in the Wess-Zumino gauge. The bosonic off-shell degrees of freedom in H_μ consist of the complex scalar $\partial^\mu H_\mu\big|_{\theta^2}$, six real degrees of freedom in the graviton and the four real degrees of freedom in the top component of H_μ, for a total of 12 off-shell bosons. For the fermions, we have only the gravitino. It has $16 - 4 = 12$ off-shell degrees of freedom. This is the old minimal multiplet of supergravity [38–40]. This is in accordance with the 12 degrees of freedom in the FZ-multiplet.

A simple consistency check is to verify the leading couplings of the graviton and gravitino to matter. Recalling the formula for $\mathcal{J}_{\alpha\dot\alpha}$ (61) we find

$$\mathscr{L} \sim h_{\mu\nu} T^{\mu\nu} \ , \tag{105}$$

as expected. Similarly, for the coupling of the gravitino to matter we get

$$\begin{aligned}\mathscr{L} &\sim \epsilon^{\alpha\beta} \left(\Psi_\mu + \sigma_\mu \bar\sigma^\rho \Psi_\rho\right)_\alpha \left(S^\mu + \tfrac{1}{3}\sigma^\mu\bar\sigma^\rho S_\rho\right)_\beta \\ &= \Psi_{\mu\alpha} S^{\alpha\mu} \ . \end{aligned} \tag{106}$$

The last ingredient is the kinetic term for the graviton and gravitino. We begin by constructing a real superfield $E^{FZ}_{\alpha\dot\alpha}$ by covariantly differentiating $H_{\alpha\dot\alpha}$

$$\begin{aligned} E^{FZ}_{\alpha\dot\beta} &= \bar D_{\dot\tau} D^2 \bar D^{\dot\tau} H_{\alpha\dot\beta} + \bar D_{\dot\tau} D^2 \bar D_{\dot\beta} H^{\dot\tau}_\alpha \\ &+ D^\gamma \bar D^2 D_\alpha H_{\gamma\dot\beta} - 2\partial_{\alpha\dot\beta}\partial^{\gamma\dot\tau} H_{\gamma\dot\tau} \ . \end{aligned} \tag{107}$$

The reader can easily check that this expression is real. It is equivalent to a different-looking expression in [41]. The gauge transformations (97) act as

$$E'^{FZ}_{\alpha\dot\beta} = E^{FZ}_{\alpha\dot\beta} + [D_\alpha, \bar D_{\dot\beta}] \left(D^2 \bar D_{\dot\alpha} \bar L^{\dot\alpha} + \bar D^2 D^\beta L_\beta\right) \ . \tag{108}$$

We see that $E^{FZ}_{\alpha\dot\beta}$ is invariant if (99) is imposed.

The superfield $E^{FZ}_{\alpha\dot\beta}$ satisfies another important algebraic equation

$$\bar D^{\dot\beta} E^{FZ}_{\alpha\dot\beta} = D_\alpha \left(\bar D^2 [D^\gamma, \bar D^{\dot\tau}] H_{\gamma\dot\tau}\right) \ . \tag{109}$$

Here, the expression in parenthesis is chiral. Note the similarity of the equation above to the defining property of the FZ-multiplet itself (60). The fact that E^{FZ} is invariant and satisfies an equation identical to the supercurrent superfield guarantees that the Lagrangian

$$\mathscr{L}_{\rm kin} \sim M_P^2 \int d^4\theta \, H^\mu E^{FZ}_\mu \tag{110}$$

is invariant. This contains in components the linearized Einstein and Rarita-Schwinger terms. The six additional supergauge-invariant bosons, $\partial^\mu H_\mu\big|_{\theta^2}$, $H_\mu\big|_{\theta^4}$ are auxiliary fields which are easily integrated out yielding $\partial^\mu H_\mu\big|_{\theta^2} \sim ix$, $H_\mu\big|_{\theta^4} \sim j_\mu$ where x and j_μ are the matter operators in the supercurrent multiplet.

We conclude that theories which have a well-defined FZ-multiplet can be coupled to supergravity in this fashion. The coupling to supergravity adds to the original theory a propagating graviton and gravitino. No other propagating fields are present in theory other than the original ones and the graviton and gravitino. This is important to remember in order to appreciate the point we will make soon. Note that if the FZ multiplet is not well defined (for example if it is not gauge invariant or if it is not globally well defined) the procedure above of constructing minimal supergravity cannot be carried out.[16]

7.2. Supergravity from the $\mathcal{S}$-Multiplet

We emphasized in the previous sections that various supersymmetric field theories do not have

[16] If there is no FZ-multiplet but there is an R-symmetry, we can construct the $\mathcal{R}$-multiplet and couple it to supergravity. For example, a free supersymmetric $U(1)$ theory with an FI-term can be coupled to supergravity in this fashion. For more comments on this case see also [42,43]. This procedure, however, gives rise to a supergravity theory with a continuous global R-symmetry.

an FZ-multiplet and the energy-momentum tensor and the supersymmetry current must be embedded in a larger multiplet $\mathcal{S}_{\alpha\dot{\alpha}}$. In such a case the construction of the previous subsection cannot be accomplished and the only possible supergravity theory is the one in which the $\mathcal{S}_{\alpha\dot{\alpha}}$ is gauged.

In this section we analyze this theory and as in the previous subsection, we limit ourselves to the analysis of the linearized theory. Since we have already understood how to do such things in the previous sections, we will be somewhat briefer now.

We begin from the coupling to matter

$$\int d^4\theta\, \mathcal{S}_{\alpha\dot{\alpha}} H^{\alpha\dot{\alpha}} \ . \tag{111}$$

For this to be invariant under (97), we need to impose the constraints

$$\bar{D}^2 D^\alpha L_\alpha = 0 \ , \qquad \bar{D}_{\dot{\alpha}} D^2 \bar{L}^{\dot{\alpha}} = D_\alpha \bar{D}^2 L^\alpha \ . \tag{112}$$

The first of them already appeared in the gauging of the FZ-multiplet but the second one is new. Since L_α is more constrained here than in the previous subsection, we will find more gauge invariant degrees of freedom. Some of them will even propagate.

Using an arbitrary L_α subject to these constraints we can choose the Wess-Zumino gauge

$$H_\mu\big| = H_\mu\big|_\theta = H_\mu\big|_{\bar{\theta}} = 0 \ . \tag{113}$$

The residual gauge transformations allow us to transform $H_\mu\big|_{\theta^2}$ by any divergence-less vector so we remain with one complex gauge invariant operator $\partial^\mu H_\mu\big|_{\theta^2}$. Another important residual transformation is

$$\delta H_\mu\big|_{\theta\sigma^\nu\bar{\theta}} + \delta H_\nu\big|_{\theta\sigma^\mu\bar{\theta}} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \ , \tag{114}$$

with $\partial^\nu \xi_\nu = 0$. This means that the trace part of this symmetric tensor is invariant under the residual symmetries and therefore, the $\theta\bar{\theta}$ component contains the usual graviton but also an additional invariant scalar. The antisymmetric piece enjoys the usual gauge transformation of a two-form

$$\delta B_{\mu\nu} = \partial_\nu \omega_\nu - \partial_\nu \omega_\mu \ . \tag{115}$$

We also note that the top component of H_μ is invariant. Thus, we see that we have 16 off-shell bosonic degrees of freedom. The fermion is in the $\theta^2\bar{\theta}$ component (and its complex conjugate). It has residual gauge symmetry

$$\delta\Psi_{\mu\alpha} = i\partial_\mu \omega_\alpha \ , \qquad \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \bar{\omega}^{\dot{\alpha}} = 0 \ . \tag{116}$$

Since ω_α satisfies the Dirac equation it cannot be used to set any further components to zero. Therefore, our theory includes a gravitino as well as an additional Weyl fermion. Thus, we have 16 off-shell fermionic degrees of freedom.

We conclude that the theory has $16+16$ fields. This is in accord with the $16+16$ operators in the multiplet $\mathcal{S}_{\alpha\dot{\alpha}}$. This supergravity multiplet has been recognized in the supergravity literature [44–46].

It is easy to construct a kinetic term; in fact $E^{FZ}_{\alpha\dot{\beta}}$ defined in (107) is still invariant because the set of transformations here is smaller than when the FZ-multiplet is gauged. However, this theory has another invariant. It is easy to see that $[D^\beta, \bar{D}^{\dot{\beta}}] H_{\beta\dot{\beta}}$ is invariant. We can use this observation to write an invariant kinetic term

$$\int d^4\theta \, \left([D,\bar{D}]H\right)^2 \ , \tag{117}$$

in addition to the one we have already included in the discussion of the FZ-multiplet.

To summarize, we find that this theory admits two independent kinetic terms. Thus there is one free real parameter r, and the most general kinetic term is

$$\int d^4\theta \left(H^{\alpha\dot{\alpha}} E^{FZ}_{\alpha\dot{\alpha}} + \frac{1}{2r} H^{\alpha\dot{\alpha}} [D_\alpha, \bar{D}_{\dot{\alpha}}][D^\beta, \bar{D}^{\dot{\beta}}] H_{\beta\dot{\beta}} \right) \ . \tag{118}$$

Our goal now is to identify the on-shell degrees of freedom in this theory and study their couplings to matter fields. There are many ways to do this, here we will choose a somewhat peculiar way that will make some very important facts transparent. We enlarge the gauge symmetry, relaxing either one of the two constraints (112) or both, and add compensator fields.

In order to contrast the situation with that in the previous subsection we choose to keep the first constraint in (112) and relax the second one by adding a chiral compensator field λ_α which transforms as

$$\delta\lambda_\alpha = \frac{3}{2}\bar{D}^2 L_\alpha \ . \tag{119}$$

First, the non-invariance of the coupling to matter $\int d^4\theta\, H^{\alpha\dot\alpha}\mathcal{S}_{\alpha\dot\alpha}$ can be corrected by a adding to the Lagrangian the term $-\frac{1}{6}\int d^2\theta\,\lambda^\alpha\chi_\alpha + \text{c.c.}$. Next, we move to the kinetic terms (118). The first term is invariant, but the second term is not. This is easily fixed by adding more terms to the Lagrangian. We end up with the invariant Lagrangian

$$\begin{aligned}\mathscr{L} = &\int d^4\theta\big(H^{\alpha\dot\alpha}E^{FZ}_{\alpha\dot\alpha} + \frac{1}{2r}H^{\alpha\dot\alpha}[D_\alpha,\bar D_{\dot\alpha}]\big([D,\bar D]H\big)\\ &+ H^{\alpha\dot\alpha}\mathcal{S}_{\alpha\dot\alpha}\big) - \big(\frac{1}{6}\int d^2\theta\lambda^\alpha\chi_\alpha + \text{c.c.}\big)\\ &-\frac{1}{r}\int d^4\theta\Big(\big(D^\gamma\lambda_\gamma + \bar D_{\dot\gamma}\bar\lambda^{\dot\gamma}\big)[D,\bar D]H\\ &-\frac{1}{2}\big(D^\gamma\lambda_\gamma + \bar D_{\dot\gamma}\bar\lambda^{\dot\gamma}\big)^2\Big) \ .\end{aligned} \tag{120}$$

The first term in the second line corrects the non-invariance of the coupling to matter and the other two terms fix the transformations of the kinetic terms.

In order to read out the spectrum we denote $G = D^\gamma\lambda_\gamma + \bar D_{\dot\gamma}\bar\lambda^{\dot\gamma}$ which is a real linear superfield (i.e. it satisfies $D^2G=0$). We also express $\chi_\alpha = -\frac{3}{2}\bar D^2 D_\alpha U$, with a real U. We should remember that this U might not be well-defined; e.g. it might not be globally well-defined or might not be gauge invariant. In fact, the need of gauging the $\mathcal{S}$-multiplet arises precisely when this U is not well-defined. The Lagrangian above becomes

$$\begin{aligned}\mathscr{L} = \int d^4\theta\big(H^{\alpha\dot\alpha}E^{FZ}_{\alpha\dot\alpha} + \tfrac{1}{2r}H^{\alpha\dot\alpha}[D_\alpha,\bar D_{\dot\alpha}][D,\bar D]H\\ +H^{\alpha\dot\alpha}\mathcal{S}_{\alpha\dot\alpha}\big)\\ -\tfrac{1}{r}\int d^4\theta\Big(G\big([D,\bar D]H - rU\big) - \tfrac{1}{2}G^2\Big) \ .\end{aligned} \tag{121}$$

Now we can dualize G. This is done by viewing it as an arbitrary real superfield and imposing the constraint $D^2G=0$ by a Lagrange multiplier term $\int d^4\theta\,\big(\Phi+\bar\Phi\big)\,G$ where Φ is a chiral superfield. This makes it easy to integrate out G using its equation of motion $G = r^2\,\big(\Phi+\bar\Phi\big) + r^2U + [D,\bar D]H$ to find the Lagrangian

$$\begin{aligned}\mathscr{L} = \int d^4\theta\Big(H^{\alpha\dot\alpha}E^{FZ}_{\alpha\dot\alpha} + \big(\Phi+\bar\Phi+U\big)[D,\bar D]H\\ -\tfrac{r}{2}\big(\Phi+\bar\Phi+U\big)^2 + H^{\alpha\dot\alpha}\mathcal{S}_{\alpha\dot\alpha}\Big) \ .\end{aligned} \tag{122}$$

In this presentation the theory looks like a standard supergravity theory based on the FZ-multiplet which is coupled to a matter system which includes the original matter as well as the chiral superfield Φ. This is consistent with the counting of degrees of freedom ($4+4$ degrees of freedom in addition to ordinary supergravity) and with the identification of the 16/16 supergravity as an ordinary supergravity coupled to a chiral superfield. Note that even though the new superfield Φ originated from the gravity multiplet, its couplings are not completely determined. At the linear order we have freedom in the dimensionless parameter r and we expect additional freedom at higher orders.

The linear multiplet G, or equivalently the chiral superfield Φ, are easily recognized as the dilaton multiplet in string theory. There the graviton and the gravitino are accompanied by a dilaton, a two-form field and a fermion (dilatino). These are the degrees of freedom in G. After a duality transformation this multiplet turns into a chiral superfield Φ. Furthermore, as in string models, the second term in (122) mixes the dilaton and the trace of the linearized graviton h^μ_μ. Both this term and the term quadratic in Φ lead to the dilaton kinetic term.

As we mentioned above, the need for the multiplet $\mathcal{S}_{\alpha\dot\alpha}$ arises when the operator U is not a good operator in the theory. In this case the current $\mathcal{J}_{\alpha\dot\alpha}$ does not exist. The couplings in (122) explain how the chiral field Φ fixes this problem. Even though U is not a good operator, $\hat U = \Phi + \bar\Phi + U$ is a good operator. If U is not gauge invariant, Φ transforms under gauge transformations such that $\hat U$ is gauge invariant. And if U is not globally well-defined because it undergoes Kähler transformations, Φ has similar Kähler transformations such that $\hat{U}$ is well-defined.

The result of this discussion can be presented in two different ways. First, as we did here, we started with a rigid theory without an FZ-multiplet and we had to gauge the $\mathcal{S}$-multiplet. This has led us to the Lagrangian (122). Alternatively, we could add the chiral superfield Φ to the original rigid theory such that the combined theory does have an FZ-multiplet. Then, this new rigid theory can be coupled to standard supergravity by gauging the FZ-multiplet.

Our discussion makes it clear that if we want to couple the theory to supergravity, the additional chiral superfield Φ is not an option – it must be added, and it is propagating.

7.3. Summary and Constraints on Moduli

Of particular interest to us in this section was the coupling of theories without an FZ-multiplet to supergravity. Here we have limited ourselves to supersymmetric field theories in which all dimensionful parameters are fixed and we have studied the limit $M_p \to \infty$. We have not studied theories in which the matter couplings depend on M_p. These have been recently discussed in [47–51] but we will not elaborate on such "intrinsically gravitational" theories here.

In case where the FZ-multiplet does not exist, we have to gauge the $\mathcal{S}$-multiplet. The upshot of the analysis of this gauging is the following. We add to the rigid theory a chiral superfield Φ whose couplings are such that the combined system including Φ has an FZ-multiplet. This determines some but not all of the couplings of Φ to the matter fields. In the case of the FI-term Φ Higgses the symmetry and in the case of nontrivial target space geometry of the rigid theory it creates a larger total space in which the topology is simpler. Now that we have an FZ-multiplet we can simply gauge it using standard supergravity techniques. In particular, at the linearized level the couplings of Φ depend on only one free parameter: the normalization of its kinetic term.

Our results fit nicely with the many known examples of string vacua. We see that the ubiquity of moduli in string theory is a result of low energy consistency conditions in supergravity. As we emphasized above, the chiral superfield Φ is similar to the dilaton superfield in four dimensional supersymmetric string vacua. We often have field theory limits without an FZ-multiplet. For example, we can have a theory on a brane with an FI-term. The field theory limit does not have an FZ-multiplet and correspondingly, $U \sim \xi V$ is not gauge invariant. This problem is fixed, as in (122), by coupling the matter theory to Φ which is not gauge invariant (note the similarity to the way this is realized in string theory [52]). Similarly, we often consider field theory limits with a target space whose Kähler form is not exact. This happens, for instance, on D3-branes at a point in a Calabi-Yau manifold. If the latter is non-compact we find a supersymmetric field theory on the brane which typically does not have an FZ-multiplet because U is not globally well-defined. Coupling this system to supergravity corresponds to making M_p finite. In this case this is achieved by making the Calabi-Yau compact. Then in addition to the graviton, various moduli of the Calabi-Yau space become dynamical. They include fields like our Φ which couple as in (122), thus avoiding the problems with the FZ-multiplet and making the supergravity theory consistent.

This discussion has direct implications for moduli stabilization. It is often desirable to stabilize some moduli at energies above the supersymmetry breaking scale. In this case we have to make sure that the resulting supergravity theory is still consistent. In particular, it is impossible to stabilize Φ in a supersymmetric way and be left with a low energy theory without an FZ-multiplet.

For example, if the low energy theory includes a $U(1)$ gauge field with an FI-term, this term must be Φ dependent. Furthermore, if the mass of Φ is above the scale of supersymmetry breaking, it must be the same as the mass of the gauge field it Higgses. Consequently, there is no regime in which it is meaningful to say that there is an FI-term. Similar comments hold for theories with a compact target space. It is impossible to stabilize the Kähler moduli while allowing moduli for the positions of branes to remain massless without supersymmetry breaking.

The comments above have applications to

many popular string constructions including D-inflation, flux compactifications, and sequestering.

REFERENCES

1. J. Wess and J. Bagger, "Supersymmetry and supergravity," *Princeton, USA: Univ. Pr. (1992) 259 p*
2. S. Weinberg, "The quantum theory of fields. Vol. 3: Supersymmetry," Cambridge, UK: Univ. Pr. (2000) 419 p.
3. L. O'Raifeartaigh, "Spontaneous Symmetry Breaking For Chiral Scalar Superfields," Nucl. Phys. B **96**, 331 (1975).
4. S. Ray, "Some properties of meta-stable supersymmetry-breaking vacua in Wess-Zumino models," Phys. Lett. B **642**, 137 (2006) [arXiv:hep-th/0607172].
5. Z. Komargodski, D. Shih, "Notes on SUSY and R-Symmetry Breaking in Wess-Zumino Models," JHEP **0904**, 093 (2009). [arXiv:0902.0030 [hep-th]].
6. M. T. Grisaru, W. Siegel, M. Rocek, "Improved Methods for Supergraphs," Nucl. Phys. **B159**, 429 (1979).
7. N. Seiberg, "Naturalness versus supersymmetric nonrenormalization theorems," Phys. Lett. **B318**, 469-475 (1993). [hep-ph/9309335].
8. E. Witten, "Dynamical Breaking of Supersymmetry," Nucl. Phys. **B188**, 513 (1981).
9. K. A. Intriligator, N. Seiberg, "Lectures on Supersymmetry Breaking," Class. Quant. Grav. **24**, S741-S772 (2007). [hep-ph/0702069].
10. N. Seiberg, "Electric - magnetic duality in supersymmetric nonAbelian gauge theories," Nucl. Phys. **B435**, 129-146 (1995). [hep-th/9411149].
11. T. T. Dumitrescu, to appear.
12. S. Ferrara, B. Zumino, "Transformation Properties of the Supercurrent," Nucl. Phys. **B87**, 207 (1975).
13. Z. Komargodski, N. Seiberg, "Comments on the Fayet-Iliopoulos Term in Field Theory and Supergravity," JHEP **0906**, 007 (2009). [arXiv:0904.1159 [hep-th]].
14. Z. Komargodski, N. Seiberg, "Comments on Supercurrent Multiplets, Supersymmetric Field Theories and Supergravity," JHEP **1007**, 017 (2010). [arXiv:1002.2228 [hep-th]].
15. W. Fischler, H. P. Nilles, J. Polchinski *et al.*, "Vanishing Renormalization of the D Term in Supersymmetric U(1) Theories," Phys. Rev. Lett. **47**, 757 (1981).
16. M. A. Shifman, A. I. Vainshtein, "Solution of the Anomaly Puzzle in SUSY Gauge Theories and the Wilson Operator Expansion," Nucl. Phys. **B277**, 456 (1986).
17. M. Dine, "Fields, Strings and Duality: TASI 96," eds. C. Efthimiou and B. Greene (World Scientific, Sigapore, 1997).
18. S. Weinberg, "Nonrenormalization theorems in nonrenormalizable theories," Phys. Rev. Lett. **80**, 3702-3705 (1998). [hep-th/9803099].
19. N. Seiberg, "Exact results on the space of vacua of four-dimensional SUSY gauge theories," Phys. Rev. **D49**, 6857-6863 (1994). [hep-th/9402044].
20. Z. Komargodski, N. Seiberg, "From Linear SUSY to Constrained Superfields," JHEP **0909**, 066 (2009). [arXiv:0907.2441 [hep-th]].
21. M. Rocek, "Linearizing the Volkov-Akulov Model," Phys. Rev. Lett. **41**, 451-453 (1978).
22. S. Samuel, J. Wess, "A Superfield Formulation Of The Nonlinear Realization Of Supersymmetry And Its Coupling To Supergravity," Nucl. Phys. **B221**, 153 (1983).
23. P. Fayet, J. Iliopoulos, "Spontaneously Broken Supergauge Symmetries and Goldstone Spinors," Phys. Lett. **B51**, 461-464 (1974).
24. I. Affleck, M. Dine, N. Seiberg, "Dynamical Supersymmetry Breaking In Chiral Theories," Phys. Lett. **B137**, 187 (1984).
25. I. Affleck, M. Dine, N. Seiberg, "Exponential Hierarchy From Dynamical Supersymmetry Breaking," Phys. Lett. **B140**, 59 (1984).
26. D. V. Volkov, V. P. Akulov, "Is the Neutrino a Goldstone Particle?," Phys. Lett. **B46**, 109-110 (1973).
27. H. Luo, M. Luo, L. Wang, "The Goldstino Field in Linear and Nonlinear Realizations of Supersymmetry," Phys. Lett. **B685**, 338-340 (2010). [arXiv:0911.2836 [hep-th]].
28. A. A. Zheltukhin, "On equivalence of the

Komargodski-Seiberg action to the Volkov-Akulov action," [arXiv:1009.2166 [hep-th]].
29. S. M. Kuzenko, S. J. Tyler, "Relating the Komargodski-Seiberg and Akulov-Volkov actions: Exact nonlinear field redefinition," [arXiv:1009.3298 [hep-th]].
30. M. Dine, G. Festuccia, Z. Komargodski, "A Bound on the Superpotential," JHEP **1003**, 011 (2010). [arXiv:0910.2527 [hep-th]].
31. L. Alvarez-Gaume, C. Gomez, R. Jimenez, "Minimal Inflation," Phys. Lett. **B690**, 68-72 (2010). [arXiv:1001.0010 [hep-th]].
32. C. Cheung, Y. Nomura, J. Thaler, "Goldstini," JHEP **1003**, 073 (2010). [arXiv:1002.1967 [hep-ph]].
33. I. Antoniadis, M. Buican, "Goldstinos, Supercurrents and Metastable SUSY Breaking in N=2 Supersymmetric Gauge Theories," [arXiv:1005.3012 [hep-th]].
34. I. Antoniadis, E. Dudas, D. M. Ghilencea *et al.*, "Non-linear MSSM," Nucl. Phys. **B841**, 157-177 (2010). [arXiv:1006.1662 [hep-ph]].
35. N. Craig, J. March-Russell, M. McCullough, "The Goldstini Variations," JHEP **1010**, 095 (2010). [arXiv:1007.1239 [hep-ph]].
36. T. T. Dumitrescu, Z. Komargodski and M. Sudano, "Global Symmetries and D-Terms in Supersymmetric Field Theories," arXiv:1007.5352 [hep-th].
37. H. -C. Cheng, W. -C. Huang, I. Low *et al.*, "Goldstini as the decaying dark matter," [arXiv:1012.5300 [hep-ph]].
38. K. S. Stelle, P. C. West, "Minimal Auxiliary Fields for Supergravity," Phys. Lett. **B74**, 330 (1978).
39. S. Ferrara, P. van Nieuwenhuizen, "The Auxiliary Fields of Supergravity," Phys. Lett. **B74**, 333 (1978).
40. E. S. Fradkin, M. A. Vasiliev, "S Matrix For Theories That Admit Closure Of The Algebra With The Aid Of Auxiliary Fields: The Auxiliary Fields In Supergravity," Lett. Nuovo Cim. **22**, 651 (1978).
41. S. Ferrara, B. Zumino, "Structure of Conformal Supergravity," Nucl. Phys. **B134**, 301 (1978).
42. K. R. Dienes, B. Thomas, "On the Inconsistency of Fayet-Iliopoulos Terms in Supergravity Theories," Phys. Rev. **D81**, 065023 (2010). [arXiv:0911.0677 [hep-th]].
43. S. M. Kuzenko, "The Fayet-Iliopoulos term and nonlinear self-duality," Phys. Rev. **D81**, 085036 (2010). [arXiv:0911.5190 [hep-th]].
44. G. Girardi, R. Grimm, M. Muller *et al.*, "Antisymmetric Tensor Gauge Potential In Curved Superspace And A (16+16) Supergravity Multiplet," Phys. Lett. **B147**, 81 (1984).
45. W. Lang, J. Louis, B. A. Ovrut, "(16+16) Supergravity Coupled To Matter: The Low-energy Limit Of The Superstring," Phys. Lett. **B158**, 40 (1985).
46. W. Siegel, "16/16 Supergravity," Class. Quant. Grav. **3**, L47-L48 (1986).
47. E. Witten, J. Bagger, "Quantization of Newton's Constant in Certain Supergravity Theories," Phys. Lett. **B115**, 202 (1982).
48. N. Seiberg, "Modifying the Sum Over Topological Sectors and Constraints on Supergravity," JHEP **1007**, 070 (2010). [arXiv:1005.0002 [hep-th]].
49. J. Distler, E. Sharpe, "Quantization of Fayet-Iliopoulos Parameters in Supergravity," [arXiv:1008.0419 [hep-th]].
50. T. Banks, N. Seiberg, "Symmetries and Strings in Field Theory and Gravity," [arXiv:1011.5120 [hep-th]].
51. S. Hellerman, E. Sharpe, "Sums over topological sectors and quantization of Fayet-Iliopoulos parameters," [arXiv:1012.5999 [hep-th]].
52. M. Dine, N. Seiberg, E. Witten, "Fayet-Iliopoulos Terms in String Theory," Nucl. Phys. **B289**, 589 (1987).