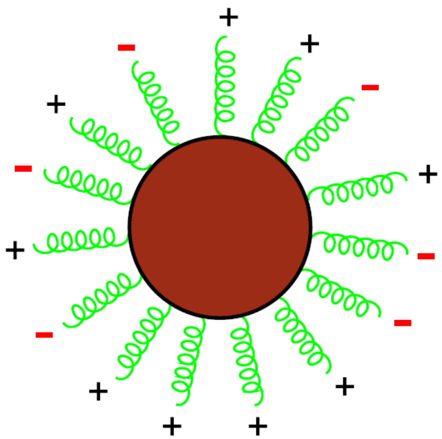
Scattering Amplitudes in Gauge Theory and Gravity

Lecture 2 – Introduction to “On-shell” methods



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CERN Winter School
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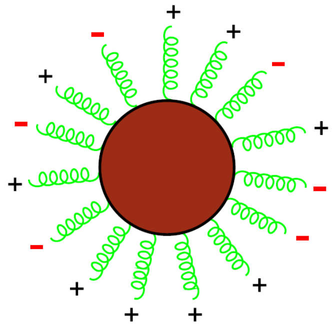
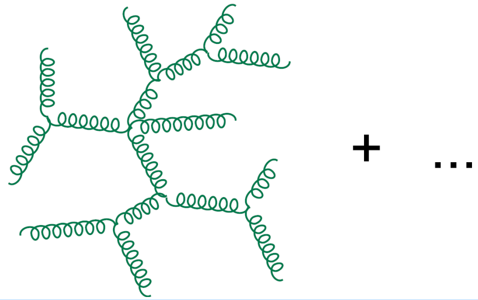
On-Shell Methods

- LHC QCD backgrounds, as well as state-of-art computations in N=4 SYM or N=8 SUGRA, require detailed understanding of perturbative scattering amplitudes for many ultra-relativistic ("massless") particles.



- Long ago, **Feynman** told us how to do this – in principle



- However, **Feynman diagrams**, while very general and **powerful**, are not optimized for these processes
- There are more efficient methods for multi-parton and multi-loop amplitudes, which take full advantage of the **analyticity** of the *S*-matrix, and **recycle** lower loop and lower-point on-shell information:

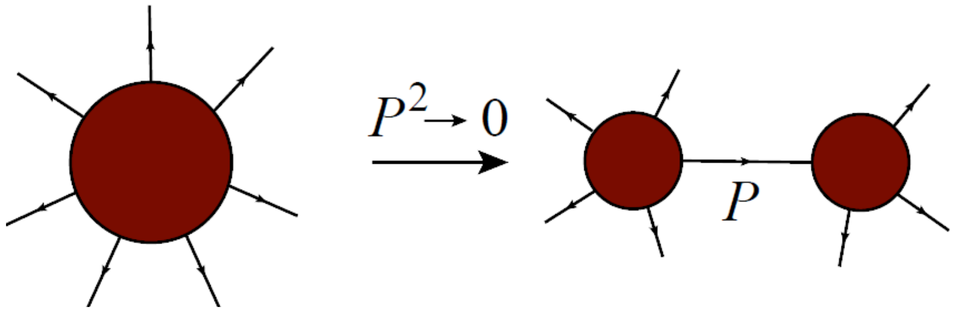
Remembering a Simpler Time...

• In the 1960s there was no QCD, no Lagrangian or Feynman rules for the strong interactions

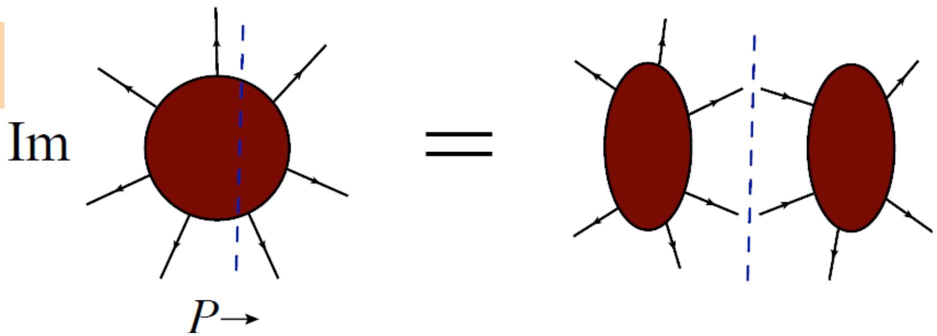
The Analytic *S*-Matrix

Bootstrap program for strong interactions: Reconstruct scattering amplitudes **directly** from **analytic properties: "on-shell" information**

- Poles



- Branch cuts



Landau; Cutkosky; Chew, Mandelstam; Eden, Landshoff, Olive, Polkinghorne; Veneziano; Virasoro, Shapiro; … (1960s)

Analyticity fell out of favor in 1970s with the rise of QCD & Feynman rules

Now **resurrected** for computing amplitudes in **perturbative** QCD
– as **alternative to Feynman diagrams!**
Perturbative information now assists analyticity.

For Efficient Computation

Reduce
the number of “diagrams”

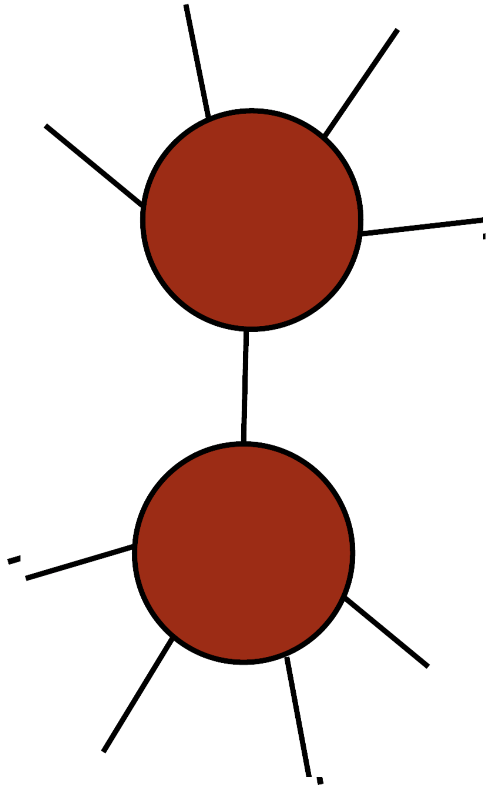
Reuse
building blocks over & over

Recycle
lower-point (1-loop) & lower-loop (tree)
on-shell amplitudes

Recurse

Recycling “Plastic” Amplitudes

Amplitudes fall apart into simpler ones in special limits – pole information



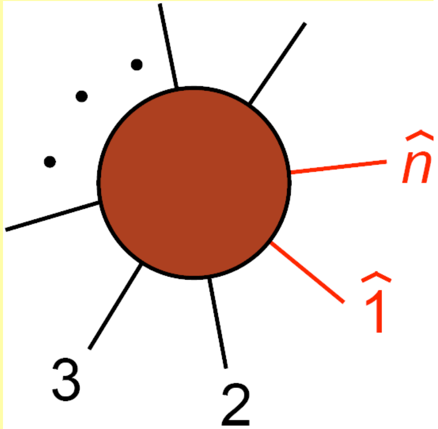
On-shell recursion at tree level

Britto, Cachazo, Feng, hep-th/0412308; Britto, Cachazo, Feng, Witten, hep-th/0501052

- BCFW consider a family of on-shell amplitudes $A_n(z)$ depending on a complex parameter z which smoothly deforms the momenta.
- Best described using spinor variables.
- For example, the $[n,1\rangle$ shift:

$$\lambda_n \to \lambda_n \qquad \tilde{\lambda}_n \to \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n - z\tilde{\lambda}_1$$

$$\lambda_1 \to \hat{\lambda}_1 = \lambda_1 + z\lambda_n \qquad \tilde{\lambda}_1 \to \tilde{\lambda}_1$$



- On-shell condition:

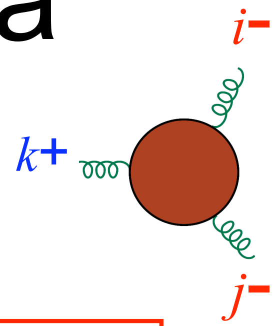
$$(\hat{k}_1)^\mu(\hat{k}_1)_\mu = (\hat{k}_1)^{\alpha\dot{\alpha}}(\hat{k}_1)_{\dot{\alpha}\alpha} = \langle(\lambda_1 + z\lambda_n)(\lambda_1 + z\lambda_n)\rangle[1\,1] = 0$$

similarly, $\hat{k}_n^2 = 0$

- Momentum conservation:

$$\hat{k}_1 + \hat{k}_n = (\lambda_1 + z\lambda_n)\tilde{\lambda}_1 + \lambda_n(\tilde{\lambda}_n - z\tilde{\lambda}_1) = k_1 + k_n$$

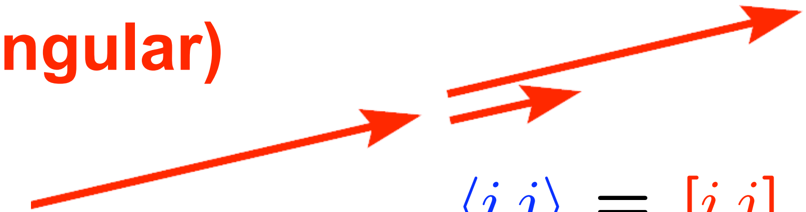
Special Complex Momenta



- Make sense of most basic process: all 3 particles massless

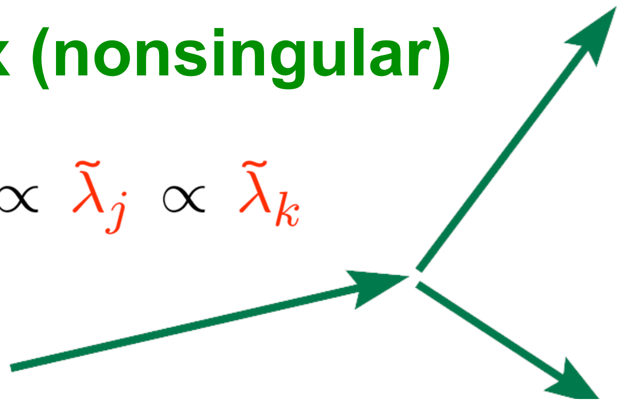
$$s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2 = 0 \quad \forall\ i, j \qquad \langle i\, j\rangle [j\, i] = s_{ij}$$

real (singular)



$$\langle i\, j\rangle = [i\, j] = s_{ij} = 0 \qquad \forall\ i, j$$

complex (nonsingular)



$$\tilde\lambda_i \propto \tilde\lambda_j \propto \tilde\lambda_k$$

$[i\, j] = 0$ but $\langle i\, j\rangle \neq 0$

$$\frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle \langle 2\, 3\rangle \langle 3\, 1\rangle}$$
makes sense

use conjugate kinematics for (++-): $\lambda_i \propto \lambda_j \propto \lambda_k \quad \langle i\, j\rangle = 0\,,\ [i\, j] \neq 0$

MHV example

- Apply the $[n,1\rangle$ shift $\qquad \lambda_1 \to \lambda_1 + z\lambda_n \qquad \tilde{\lambda}_n \to \tilde{\lambda}_n - z\tilde{\lambda}_1$

to the Parke-Taylor (MHV) amplitudes:

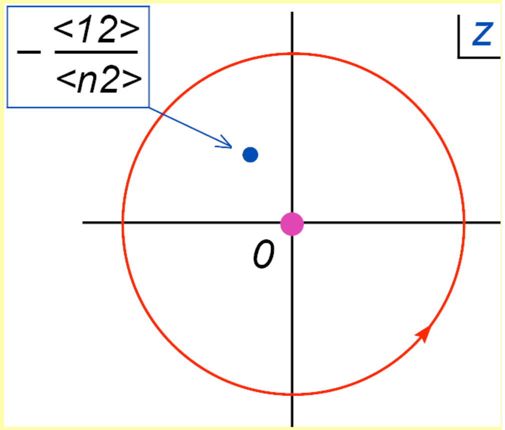
$$A_n(z=0) = A_n^{jn,\,\mathrm{MHV}} = \frac{\langle j\, n\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle}$$

$$\begin{aligned}\langle n\, 1\rangle &= \lambda_n\lambda_1 \to \lambda_n(\lambda_1 + z\lambda_n) = \langle n\, 1\rangle + z\langle n\, n\rangle = \langle n\, 1\rangle \\ \langle 1\, 2\rangle &= \lambda_1\lambda_2 \to (\lambda_1 + z\lambda_n)\lambda_2 = \langle 1\, 2\rangle + z\langle n\, 2\rangle\end{aligned}$$

- So $\quad A_n(z) = \dfrac{\langle j\, n\rangle^4}{(\langle 1\, 2\rangle + z\langle n\, 2\rangle)\langle 2\, 3\rangle \cdots \langle n\, 1\rangle}$

- Consider: $\quad \dfrac{1}{2\pi i}\displaystyle\oint_C dz \frac{A_n(z)}{z}$

- 2 poles, opposite residues



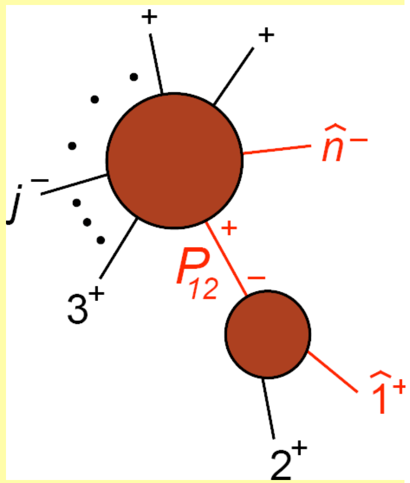
MHV example (cont.)

- MHV amplitude obeys: $$A_n(0) = - \underset{z=-\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}}{\text{Res}} \frac{A_n(z)}{z}$$

- Compute residue using factorization
- At $$z = -\frac{\langle 1\,2\rangle}{\langle n\,2\rangle} = -\frac{\langle 1\,2\rangle[2\,1]}{\langle n\,2\rangle[2\,1]} = -\frac{s_{12}}{\langle n^-|(1+2)|1^-\rangle}$$

  kinematics are complex collinear

$$\langle \hat{1}\,2\rangle = \langle 1\,2\rangle + z\langle n\,2\rangle = 0 \qquad [\hat{1}\,2] = [1\,2] \neq 0$$
$$s_{\hat{1}2} = \langle \hat{1}\,2\rangle[2\,\hat{1}] = 0$$



- so $$- \underset{z=-\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}}{\text{Res}} \frac{A_n(z)}{z} = A_{n-1}(\hat{P}_{12}^+, 3^+, \ldots, j^-, \ldots, n^-)$$
$$\times \left[- \underset{z=-\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}}{\text{Res}} \frac{1}{z} \frac{1}{\hat{P}_{12}^2(z)} \right] A_3(\hat{1}^+, 2^+, -\hat{P}_{12}^-)$$

note
$A_3(+,+,+) = 0$

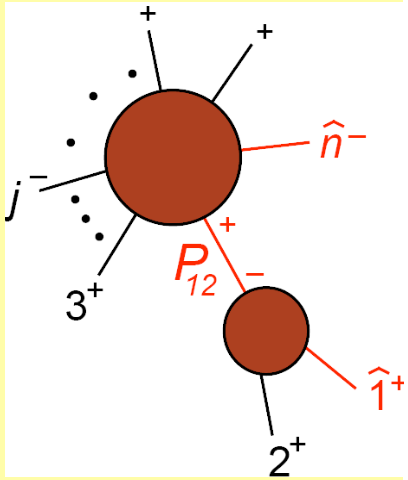
Evaluate the ingredients

- Since $\hat{P}_{12}^2(z) \;=\; (k_1 + k_2 + z\lambda_n\tilde{\lambda}_1)^2 = s_{12} + z\langle n^-|(1+2)|1^-\rangle$

$$-\underset{z=-\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}}{\text{Res}} \frac{1}{z}\frac{1}{\hat{P}_{12}^2(z)} \;=\; -\frac{\langle n^-|(1+2)|1^-\rangle}{s_{12}}\frac{1}{\langle n^-|(1+2)|1^-\rangle} = \frac{1}{s_{12}}$$

- So

$$A_n(0) \;=\; A_{n-1}(\hat{P}_{12}^+, 3^+, \ldots, j^-, \ldots, n^-)\frac{1}{s_{12}}A_3(\hat{1}^+, 2^+, -\hat{P}_{12}^-)$$



- Check this explicitly:

$$A_n(0) \;=\; \frac{\langle j\,\hat{n}\rangle^4}{\langle \hat{P}\,3\rangle\langle 3\,4\rangle\cdots\langle n-1,\hat{n}\rangle\langle \hat{n}\,\hat{P}\rangle}\frac{1}{s_{12}}\frac{[\hat{1}\,2]^3}{[2\,\hat{P}][\hat{P}\,\hat{1}]}$$

$$= \frac{\langle j\,n\rangle^4}{\langle \hat{P}\,3\rangle\langle 3\,4\rangle\cdots\langle n-1,n\rangle\langle n\,\hat{P}\rangle}\frac{1}{s_{12}}\frac{[1\,2]^3}{[2\,\hat{P}][\hat{P}\,1]}$$

MHV check (cont.)

- Using
$$\langle n\,\hat{P}\rangle[\hat{P}\,2] = \langle n^-|(1+2)|2^-\rangle + z\langle n\,n\rangle[1\,2] = \langle n\,1\rangle[1\,2]$$
$$\langle 3\,\hat{P}\rangle[\hat{P}\,1] = \langle 3^-|(1+2)|1^-\rangle + z\langle 3\,n\rangle[1\,1] = \langle 3\,2\rangle[2\,1]$$

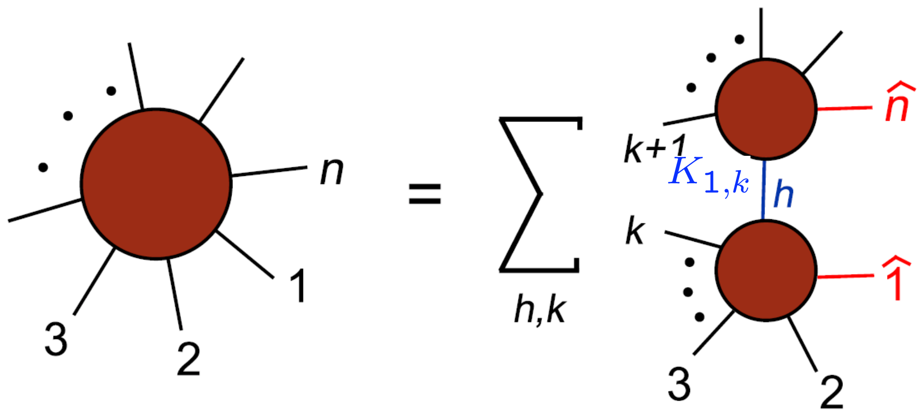
one confirms

$$\begin{aligned} A_n(0) &= \frac{\langle j\,n\rangle^4}{\langle \hat{P}\,3\rangle\langle 3\,4\rangle\cdots\langle n-1,n\rangle\langle n\,\hat{P}\rangle}\frac{1}{s_{12}}\frac{[1\,2]^3}{[2\,\hat{P}][\hat{P}\,1]} \\ &= \frac{\langle j\,n\rangle^4[1\,2]^3}{(\langle 1\,2\rangle[2\,1])([1\,2]\langle 2\,3\rangle)(\langle n\,1\rangle[1\,2])\langle 3\,4\rangle\cdots\langle n-1,n\rangle} \\ &= \frac{\langle j\,n\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n-1,n\rangle\langle n\,1\rangle} \\ &= A_n^{jn,\,\mathrm{MHV}} \end{aligned}$$

The general case

Britto, Cachazo, Feng, hep-th/0412308;
Britto, Cachazo, Feng, Witten, hep-th/0501052

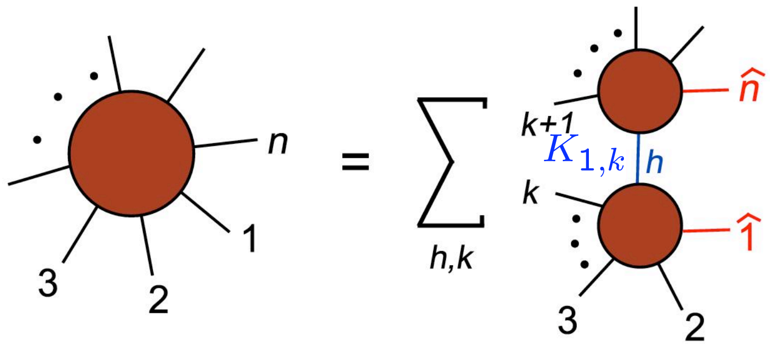
$$A_n(1,2,\ldots,n) = \sum_{h=\pm}\sum_{k=2}^{n-2} A_{k+1}(\hat{1},2,\ldots,k,-\hat{K}_{1,k}^{-h}) \times \frac{i}{K_{1,k}^2} A_{n-k+1}(\hat{K}_{1,k}^{h},k+1,\ldots,n-1,\hat{n})$$



A_{k+1} and A_{n-k+1} are on-shell tree amplitudes with fewer legs, evaluated with 2 momenta shifted by a **complex** amount

Momentum shift

Shift for k^{th} term comes from setting $z = z_k$, where



$$z_k = -\frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^- \rangle}$$

is the solution to

$$\hat{K}_{1,k}^2(z) = 0 = (K_{1,k} + z\lambda_n\tilde{\lambda}_1)^2 = K_{1,k}^2 + z\lambda_n^a (K_{1,k})_{a\dot{a}} \tilde{\lambda}_1^{\dot{a}}$$

plugging in, shift is:

$$\hat{\lambda}_1 = \lambda_1 - \frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^- \rangle} \lambda_n \qquad \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1$$

$$\hat{\lambda}_n = \lambda_n \qquad \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n + \frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^- \rangle} \tilde{\lambda}_1$$

Proof of on-shell recursion relations

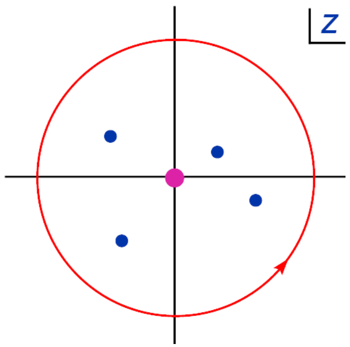
Britto, Cachazo, Feng, Witten, hep-th/0501052

Same analysis as above – Cauchy's theorem + **amplitude factorization**

Let complex momentum shift depend on *z*. Use analyticity in *z*.

$$\hat{\lambda}_1 = \lambda_1 + z\lambda_n \qquad \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 \qquad \Rightarrow \quad A(0) \;\to\; A(z)$$

$$\hat{\lambda}_n = \lambda_n \qquad \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n - z\tilde{\lambda}_1$$

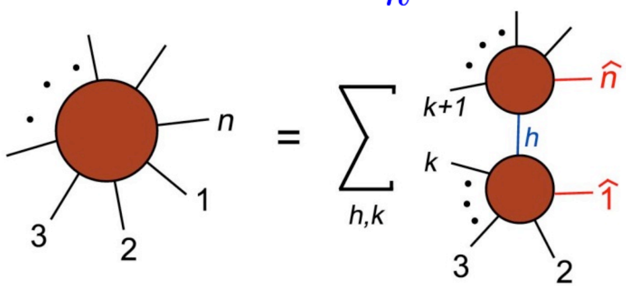


Cauchy: If $A(\infty) = 0$ then

$$0 = \frac{1}{2\pi i}\oint dz\,\frac{A(z)}{z} = A(0) + \sum_k \text{Res}\left[\frac{A(z)}{z}\right]\Big|_{z=z_k}$$

poles in *z*: physical factorizations $\hat{K}^2_{1,k} = 0$

residue at $z_k = -\frac{K^2_{1,k}}{\langle n^-|\not{K}_{1,k}|1^-\rangle}$ = [k^{th} term]



To show: $A(\infty) = 0$

Britto, Cachazo, Feng, Witten, hep-th/0501052

Propagators:

$$\frac{1}{\hat{K}^2_{1,k}(z)} = \frac{1}{K^2_{1,k} + z\lambda^a_n (K_{1,k})_{a\dot{a}} \tilde{\lambda}^{\dot{a}}_1} \sim \frac{1}{z}$$

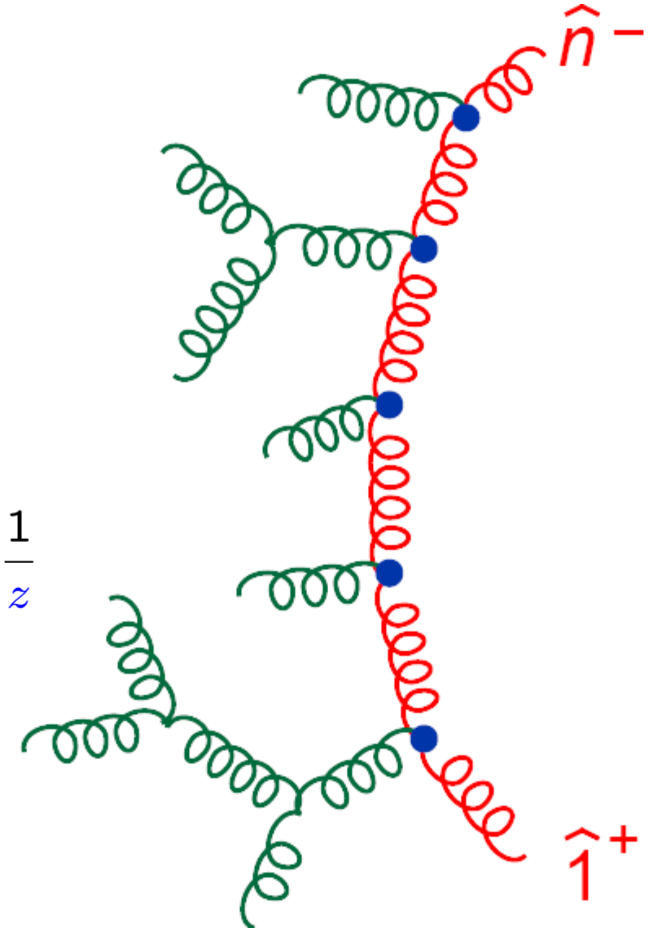
3-point vertices: $\propto \ \hat{k}^\mu(z) \propto z$

Polarization vectors:

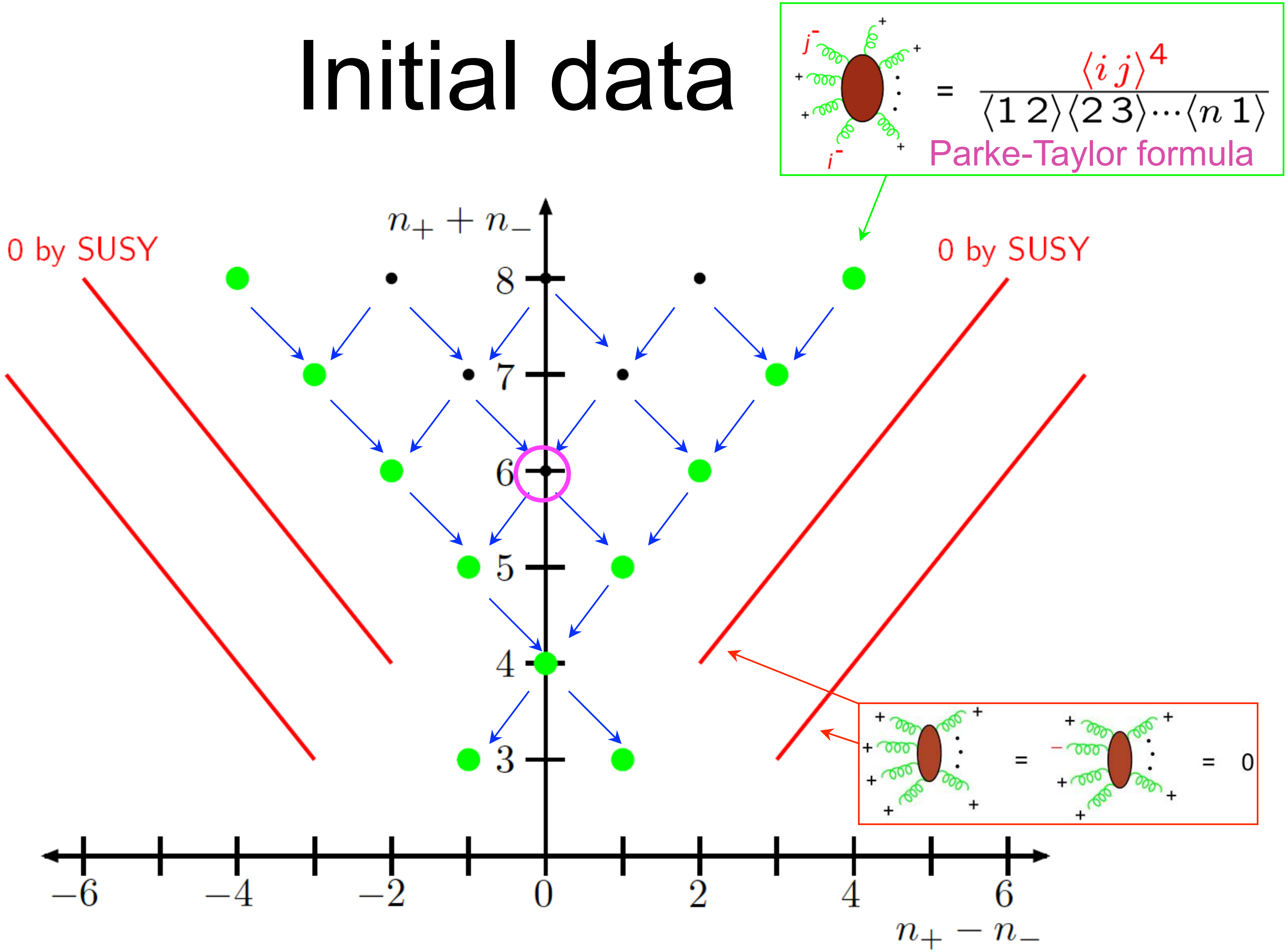
$$\not{\epsilon}^+_1 \ \propto \ \frac{\tilde{\lambda}_1 \lambda_q}{\langle \lambda_1 \, \lambda_q \rangle} \ \propto \ \frac{1}{z} \qquad \not{\epsilon}^-_n \ \propto \ \frac{\lambda_n \tilde{\lambda}_q}{\langle \tilde{\lambda}_n \, \tilde{\lambda}_q \rangle} \ \propto \ \frac{1}{z}$$

Total:

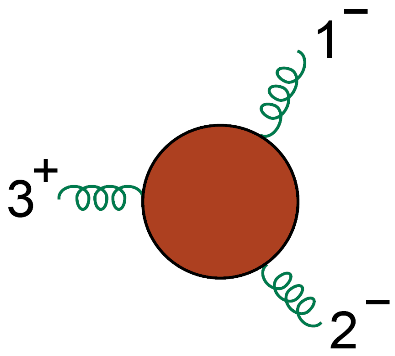
$$\frac{1}{z} \times \left(z \frac{1}{z} \right)^r z \times \frac{1}{z} = \frac{1}{z}$$



Initial data

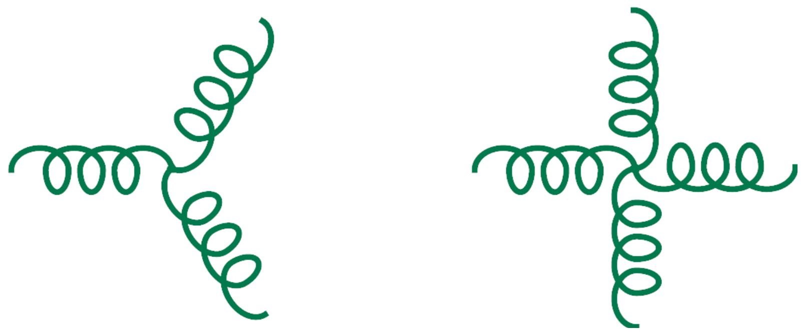


All gluon tree amplitudes built from:



$$= \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}$$

(In contrast to Feynman vertices, it is on-shell, gauge invariant.)



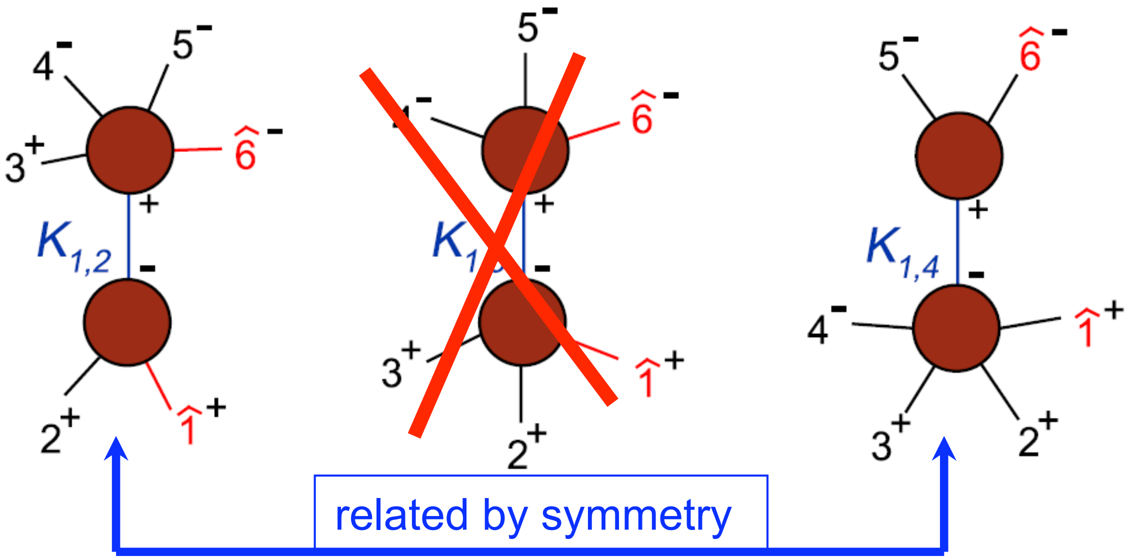
A 6-gluon example

220 Feynman diagrams for *gggggg*

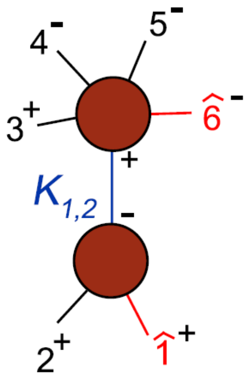
Helicity + color + MHV results + symmetries

$\Rightarrow$ only $A_6(1^+,2^+,3^+,4^-,5^-,6^-)$, $A_6(1^+,2^+,3^-,4^+,5^-,6^-)$

3 BCF diagrams $\rightarrow$ 2 $\rightarrow$ 1



The one $A_6(1^+,2^+,3^+,4^-,5^-,6^-)$ diagram



$$= -\frac{i}{s_{12}} \frac{[\hat{1}\, 2]^3}{[2\, \hat{K}][\hat{K}\, \hat{1}]} \frac{[\hat{K}\, 3]^3}{[3\,4][4\,5][5\,\hat{6}][\hat{6}\, \hat{K}]}$$

$$= -\frac{i}{s_{12}} \frac{[1\, 2]^3}{([2\, \hat{K}]\langle \hat{K}\, 6\rangle)(\langle 6\, \hat{K}\rangle[\hat{K}\, 1])} \frac{(\langle 6\, \hat{K}\rangle[\hat{K}\, 3])^3}{[3\,4][4\,5][5\,\hat{6}]([\hat{6}\, \hat{K}]\langle \hat{K}\, 6\rangle)}$$

$$= i\, \frac{\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\langle 1\,2\rangle[3\,4][4\,5]s_{612}\langle 2^-|(6+1)|5^-\rangle}$$

$$\langle 6\, \hat{K}\rangle[\hat{K}\, a] = \langle 6\, 1\rangle\, [1\, a] + \langle 6\, 2\rangle\, [2\, a]$$
$$= \langle 6^-|(1+2)|a^-\rangle$$

$$[5\, \hat{6}] = [5\, 6] + \frac{s_{12}[5\, 1]}{\langle 6\, 2\rangle[2\, 1]} = \frac{\langle 5^+|(6+1)|2^+\rangle}{\langle 6\, 2\rangle}$$

$$[\hat{6}\, \hat{K}]\langle \hat{K}\, 6\rangle = \langle 6^+|(1+2)|6^+\rangle + s_{12} = s_{612}$$

Simple final form

$$-iA_6(1^+,2^+,3^+,4^-,5^-,6^-) = \frac{\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\langle 1\,2\rangle[3\,4][4\,5]\,s_{612}\langle 2^-|(6+1)|5^-\rangle} + \frac{\langle 4^-|(5+6)|1^-\rangle^3}{\langle 2\,3\rangle\langle 3\,4\rangle[5\,6][6\,1]\,s_{561}\langle 2^-|(6+1)|5^-\rangle}$$

Simpler than form found in 1980s despite (because of?) spurious singularities $\langle 2^-|(6+1)|5^-\rangle$

Mangano, Parke, Xu (1988)

$$-iA_6(1^+,2^+,3^+,4^-,5^-,6^-) = \frac{([1\,2]\langle 4\,5\rangle\langle 6^-|(1+2)|3^-\rangle)^2}{s_{61}s_{12}s_{34}s_{45}s_{612}} + \frac{([2\,3]\langle 5\,6\rangle\langle 4^-|(2+3)|1^-\rangle)^2}{s_{23}s_{34}s_{56}s_{61}s_{561}} + \frac{s_{123}[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle\langle 6^-|(1+2)|3^-\rangle\langle 4^-|(2+3)|1^-\rangle}{s_{12}s_{23}s_{34}s_{45}s_{56}s_{61}}$$

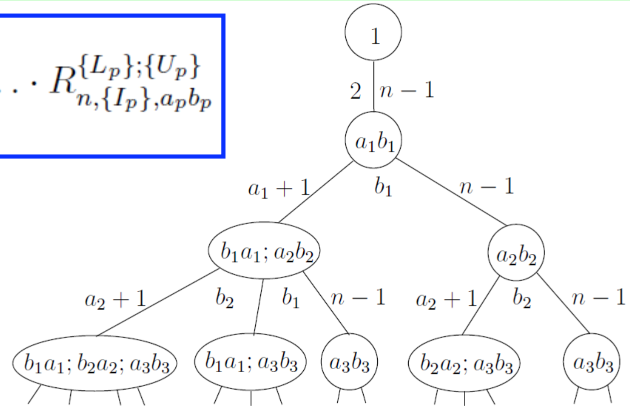
Relative simplicity even more striking for *n>6*

Bern, Del Duca, LD, Kosower (2004)

All trees

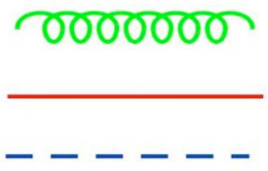
- BCFW recursion relations can easily be implemented numerically
Dinsdale, Ternick, Weinzierl, hep-ph/0602204;…
- Computationally quite fast, although for very large n other approaches can be faster. Berends, Giele, NPB306 (1988) 759; …
- In N=4 SYM, a similar recursion relation can be derived by shifting also Grassmann parameters $\eta^A{}_i$ associated with supersymmetry
Arkani-Hamed, Cachazo, Kaplan, 0808.1446; Bianchi, Elvang, Freedman, 0805.0757; Brandhuber, Heslop, Travaglini, 0807.4097; Elvang, Freedman, Kiermaier, 0808.1720
- And this relation can be solved analytically for all n
in terms of paths through "rooted trees" Drummond, Henn, 0808.2475

$$\mathcal{A}_n^{\mathrm{N}^p\mathrm{MHV}} = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle} \sum_{\text{all paths of length } p} 1 \cdot R_{n,a_1b_1} \cdot R_{n,\{I_2\},a_2b_2}^{\{L_2\};\{U_2\}} \cdot \ldots \cdot R_{n,\{I_p\},a_pb_p}^{\{L_p\};\{U_p\}}$$



N=4 SYM

massless spin 1 gluon
4 massless spin 1/2 gluinos
6 massless spin 0 scalars



all states in adjoint representation, all linked by N=4 supersymmetry

- Interactions uniquely specified by gauge group, say SU(N_c), 1 coupling g

- Exactly scale-invariant (conformal) field theory: $\beta(g) = 0$ for all g

On-shell N=4 superfield $\Phi(\eta)$ ◊ compact superamplitudes Nair (1988),…

$$\Phi(\eta) = g^+ + \eta^A \tilde{g}_A^+ + \frac{1}{2}\eta^A\eta^B\phi_{AB} + \frac{1}{6}\eta^A\eta^B\eta^C\epsilon_{ABCD}\tilde{g}^{D-} + \frac{1}{24}\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}g^-$$

N=4 SYM trees (cont.)

For example, MHV superamplitude is

$$\mathcal{A}_n^{\mathrm{MHV}} = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

where

$$p = \sum_{i=1}^{n} k_i$$

total momentum

$$q^{\alpha,A} = \sum_{i=1}^{n} \lambda_i^{\alpha}\,\eta_i^{A}$$

total fermionic momentum

Extract components using

$$g_i^+ \to \eta_i^A = 0\,, \quad g_i^- \to \int d^4\eta_i = \int d\eta_i^1\, d\eta_i^2\, d\eta_i^3\, d\eta_i^4\,, \quad \tilde{g}_{i,A} \to \int d\eta^A\,, \quad \bar{\tilde{g}}_i^A \to -\int d^4\eta_i\,\eta_i^A$$

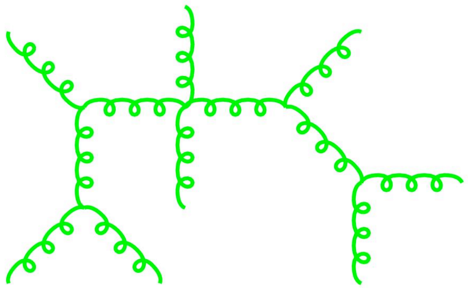
→ MHV n-gluon (i-,j-) numerator factor must contain

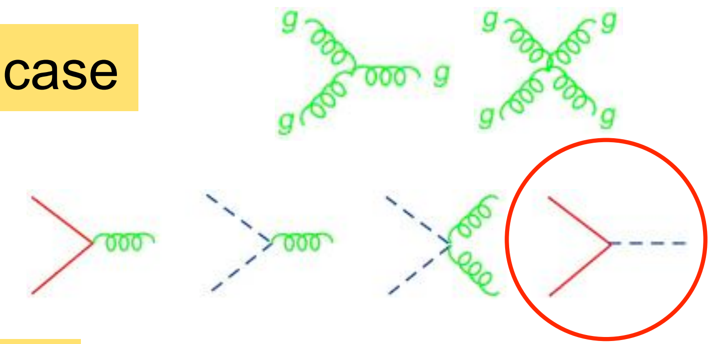
$$(\lambda_i)^4\,(\lambda_j)^4 = \langle i\,j\rangle^4$$

Exercise: work out other MHV components

N=4 → QCD at tree level

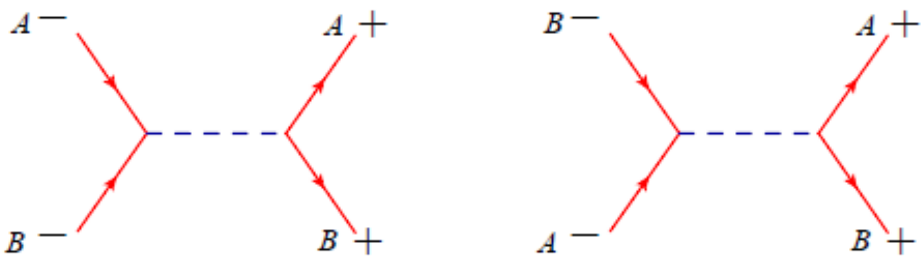
Same amplitudes, clearly, for pure gluon case





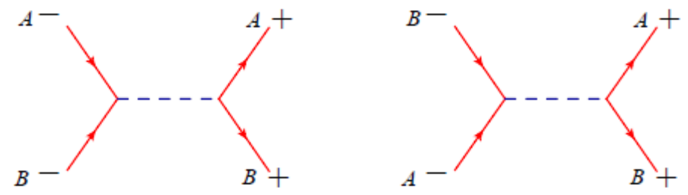
or one fermion line

Potential problems with unwanted scalar exchange begin with 2 fermion lines

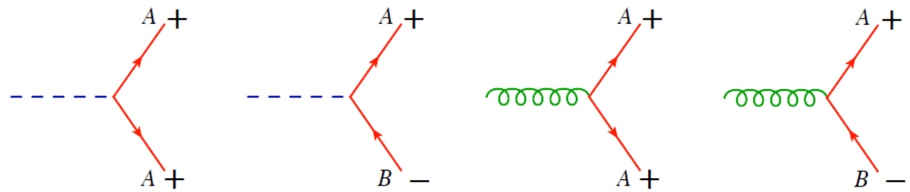


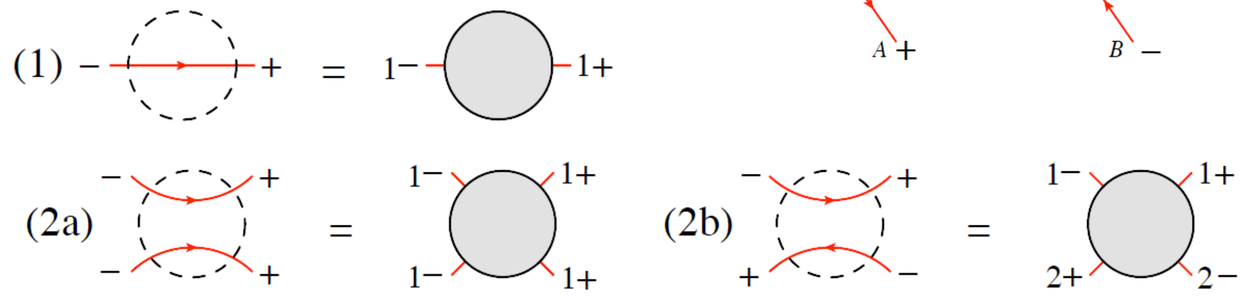
N=4 → QCD at tree level

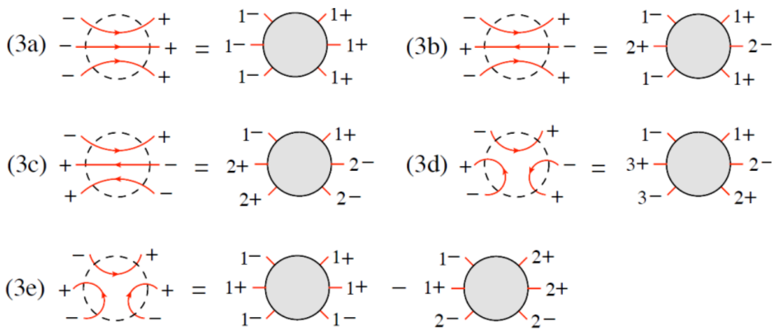
Can avoid unwanted scalar exchange between different fermion lines



Using helicity + color + vanishing of the following vertices:







N=4 trees → QCD trees through at least 4 fermion lines

LD, Henn, Plefka, Schuster, 1010.3991

End of Lecture 2