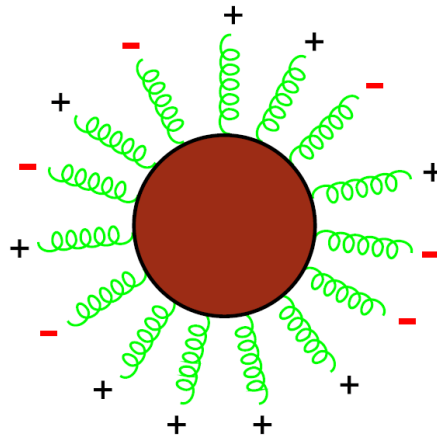# Scattering Amplitudes in Gauge Theory and Gravity
Lecture 1 – Gauge Theory Basics
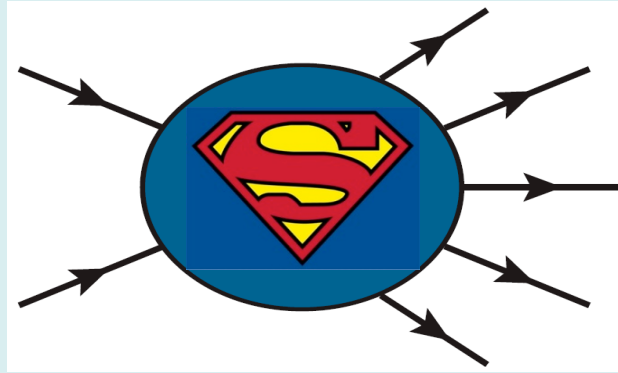
Lance Dixon (CERN & SLAC)

CERN Winter School
Jan. 24-28, 2011
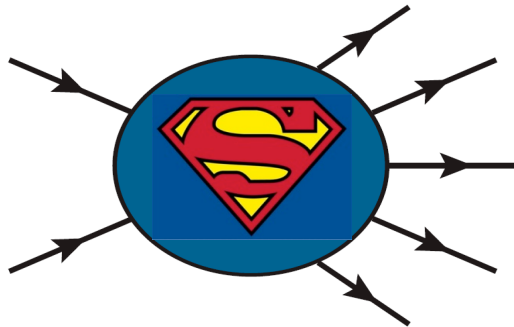
The *S* matrix reloaded

- Almost everything we know experimentally about gauge theory is based on **scattering processes** with asymptotic, on-shell states, evaluated in perturbation theory.

- Nonperturbative, off-shell information very useful, but at least in QCD it is often more qualitative (except for lattice gauge theory).

- All perturbative scattering amplitudes can be computed with Feynman diagrams – but that is not necessarily the best way, especially if there is hidden simplicity.

- On-shell methods can be much more efficient, and provide new insights.

- Use analytic properties of the *S* matrix directly, not Feynman diagrams.

Three Applications of On-Shell Methods

- **QCD**: a very practical application, needed for quantifying LHC backgrounds to new physics
lectures by Mangano

- **N=4 super-Yang-Mills theory**: lots of simplicity, both manifest and hidden. A particularly beautiful application of on-shell and related methods

- **N=8 supergravity**: amazingly good UV behavior, unveiled by on-shell methods
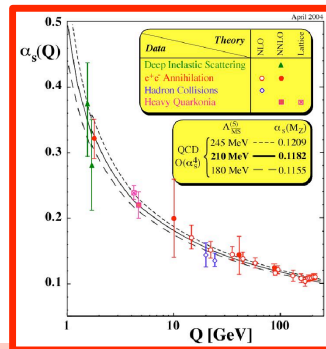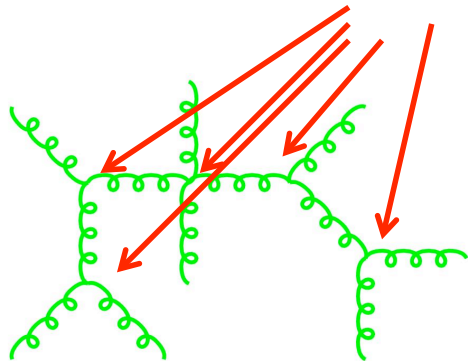
QCD

• At LHC, theoretical uncertainty for multi-jet final states - important for backgrounds to supersymmetry is very large, due to strong dependence on renormalization and fatorization scales.

$$\hat{\sigma}(\alpha_s, \mu_F, \mu_R) = [\alpha_s(\mu_R)]^{n_\alpha} \left[\hat{\sigma}^{(0)} + \frac{\alpha_s}{2\pi}\hat{\sigma}^{(1)}(\mu_F, \mu_R) + \left(\frac{\alpha_s}{2\pi}\right)^2 \hat{\sigma}^{(2)}(\mu_F, \mu_R) + \cdots \right]$$

LO   NLO   NNLO

• (At least) NLO QCD required → need (at least) one-loop QCD amplitudes with many external legs

N=4 SYM

- Dual to gravity/string theory on $AdS_5 \times S^5$

- Very similar in IR to QCD

- Planar (large N_c) theory is integrable   Bena, Polchinski, Roiban; Beisert, Eden, Staudacher; Berkovits, Maldacena; …

- Strong-coupling = minimal area problem (Wilson loop)   Alday, Maldacena

- Planar amplitudes possess dual conformal invariance
  Drummond, Henn, Smirnov, Sokatchev; Drummond, Henn, Korchemsky, Sokatchev;…

- Some planar amplitudes "known" to all orders in coupling
  Bern, LD, Smirnov + AM + DHKS

- Planar (MHV) amplitudes = expectation values of light-like Wilson loops
  DHKS; Brandhuber, Heslop, Travaligni; BDKRSVV; Alday et al; Eden, Korchemsky, Sokatchev

- N=8 supergravity closely linked by tree-level Kawai-Lewellen-Tye relation and more recent "duality" relations   Bern, Carrasco, Johansson;…

- More recent planar developments – momentum twistors, Grassmannians
  Hodges; Mason, Skinner; Arkani-Hamed et al.

- Excellent arena for testing on-shell & related methods

N=8 Supergravity

- Quantum gravity nonrenormalizable by power counting since $G_N = 1/M_{Pl}^2$ is dimensionful
- String theory cures divergences of quantum gravity by introducing a new length scale (string tension), particles no longer pointlike
- Is this necessary? Or could enough symmetry, e.g. **N=8**, allow a point particle quantum gravity theory to be perturbatively UV finite?
- N=8 supergravity (ungauged)   DeWit, Freedman (1977); Cremmer, Julia, Scherk (1978); Cremmer, Julia (1978,1979)
- Many opinions expressed over the years, and lots of recent progress in using symmetries to constrain the set of potential counterterms   Bossard, Howe, Stelle, 1009.0743; Beisert et al. 1009.1643; …
- But the only way to know for sure that the theory diverges is to compute a divergence. Use on-shell methods & relation to N=4 SYM
- To date every computation that has been done – through L=4 loops – has found that N=8 SUGRA is "as finite as" N=4 SYM (in D>4). Predictions of a 7-loop divergence usually imply this will fail at L=5…

A few reviews

- Gauge theory tree amplitudes

Mangano, Parke, hep-th/0509223; LD, hep-ph/9601359

- Gauge theory loop amplitudes

Bern, LD, Kosower, hep-ph/9602280, 0704.2798 [hep-ph];

Berger, Forde, 0912.3534 [hep-ph]

- N=4 super-Yang-Mills amplitudes

Alday, Roiban, 0807.1889; Drummond, 1010.2418 (2010 CERN Winter School)

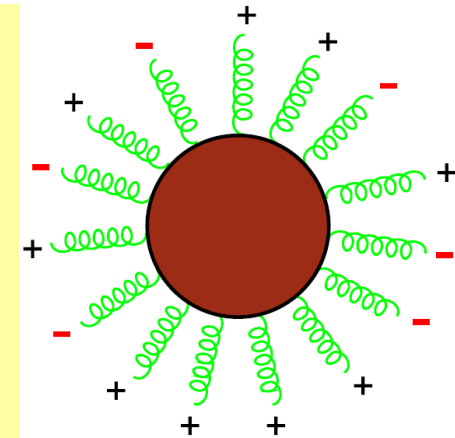- Supergravity tree and loop amplitudes and counterterms

Bern, gr-qc/0206071; Bern, Carrasco, Johansson, 0902.3765; LD, 1005.2703;

Elvang, Freedman, Kiermaier, 1012.3401

The Arena

- Flat Minkowski space-time

(or very, very close to it, for gravity case)

- D=4 (at least for the external momenta)
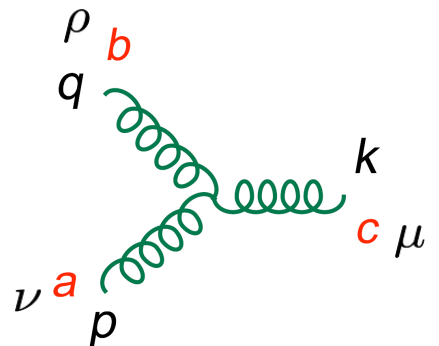
$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

- All particles with momenta $k_i{}^\mu$, massless, $k_i^2 = 0$

  (in QCD, masslessness not necessary, e.g. $m_t \neq 0$, but convenient)

- Assign momenta and helicities assuming particles are all outgoing, so $\Sigma_{i=1}^n \, k_i{}^\mu = 0$

- For incoming particles, let $k_i{}^\mu \rightarrow -k_i{}^\mu$ and helicity $h_i \rightarrow -h_i$

How to organize gauge theory amplitudes

- Avoid tangled algebra of color and Lorentz indices generated by Feynman rules

$$= ig f^{abc}[\eta_{\nu\rho}(p - q)_\mu + \eta_{\rho\mu}(q - k)_\nu + \eta_{\mu\nu}(k - p)_\rho]$$

structure constants

- Take advantage of physical properties of amplitudes

- Basic tools:
  - dual (trace-based) color decompositions
  - spinor helicity formalism

Color

Standard color factor for a QCD graph has lots of structure constants contracted in various orders; for example:

$$\propto f^{a_1 a_2 b} f^{a_3 a_4 c} f^{b c a_5}$$

We can write every *n*-gluon tree graph color factor as a sum of traces of matrices T^a in the fundamental (defining) representation of $SU(N_c)$:

$$\mathrm{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n})$$ + all non-cyclic permutations

Use definition: $[T^a, T^b] = i f^{abc} T^c$

+ normalization: $\mathrm{Tr}(T^a T^b) = \delta^{ab}$ → $f^{abc} = -i\, \mathrm{Tr}([T^a, T^b]\, T^c)$

Color in pictures

Insert $= f^{abc} = Tr([T,^a T^b] T^c) =$

and

where $= (T^a)^{\bar{j}}_i$ is color factor for *qqg* vertex

$-\dfrac{1}{N_c}$

into typical string of *f^{abc}* structure constants for a Feynman diagram:

 $=$ $+$ *permutations* $= \mathrm{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n})$ $+$ permutations

- Always single traces (at tree level)
- $\mathrm{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n})$ comes only from those planar diagrams with cyclic ordering of external legs fixed to 1,2,…,n

Trace-based (dual) color decomposition

Similarly

$$q\bar{q}gg \cdots g \text{ amplitudes} \Rightarrow (T^{a_1}T^{a_2} \cdots T^{a_n})_i^{\bar{j}}$$
$$+ \text{ permutations}$$

In summary, for the n-gluon trees, the color decomposition is

$$\mathcal{A}_n^{\text{tree}}(\{k_i, a_i, h_i\}) = g^{n-2}\text{Tr}(T^{a_1}T^{a_2} \cdots T^{a_n}) A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n})$$
$$+ \text{ non-cyclic perm's}$$

momenta

color

helicities
$h_i = \pm 1$

color-ordered subamplitude only depends on momenta.
Compute separately for each cyclicly inequivalent
helicity configuration $(h_1, h_2, \ldots, h_n)$

- Because $A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n})$ comes from planar diagrams
with cyclic ordering of external legs fixed to 1,2,…,n,
it only has singularities in cyclicly-adjacent channels $s_{i,i+1}$, …

Color-ordered Feynman rules

In Feynman gauge, use these "color-stripped" rules

$$\frac{i}{\sqrt{2}}\left(\eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu + \eta_{\mu\nu}(k-p)_\rho\right)$$

$$= i\eta_{\mu\rho}\eta_{\nu\lambda} - \frac{i}{2}(\eta_{\mu\nu}\eta_{\rho\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho})$$

$$= \frac{i}{\sqrt{2}}\gamma_\mu \qquad = -i\frac{\eta_{\mu\nu}}{p^2}$$

$$= -\frac{i}{\sqrt{2}}\gamma_\mu \qquad = \frac{i}{\not{p}}$$

in the (1,2,…,n)-ordered planar diagrams, to compute

$$A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) \qquad A_n^{\text{tree}}(1_{\bar q}^-, 2_q^+, 3^{h_3}, \ldots, n^{h_n}) \qquad \text{etc.}$$

$$gg \cdots g \qquad\qquad q\bar{q}gg \cdots g$$

Strings and Color

Trace-based color decomposition can also be derived from string theory. $SU(N_c)$ generators in the fundamental representation, T^a, are known from the 1960s as "Chan-Paton factors" .
Today, $SU(N_c)$ gauge symmetry in open string theory is viewed as arising from a stack of N_c D-branes. T^a interpreted as transition factors for gluons which have ends on different branes.

$$\mathcal{A}_n^{\text{tree}}(\{k_i, a_i, h_i\}) = g^{n-2}\text{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n})\, A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n})$$

$$+ \text{ non-cyclic perm's}$$

Color sums

In the end, we want to sum/average over final/initial colors
(as well as helicities):

$$d\sigma^{\text{tree}} \propto \sum_{a_i} \sum_{h_i} |\mathcal{A}_n^{\text{tree}}(\{k_i, a_i, h_i\})|^2$$

Inserting:

$$\mathcal{A}_n^{\text{tree}}(\{k_i, a_i, h_i\}) = g^{n-2}\text{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n})\, A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n})$$
$$+ \text{ non-cyclic perm's}$$

and doing the color sums diagrammatically:

we get:

$$= N_c^n \qquad\qquad = N_c^n \times \frac{1}{N_c^2}$$

Exercise:
Convince
yourself
of this!

$$d\sigma^{\text{tree}} \propto N_c^n \sum_{\sigma \in S_n/Z_n} \sum_{h_i} |A_n^{\text{tree}}(\sigma(1^{h_1}), \sigma(2^{h_2}), \ldots, \sigma(n^{h_n}))|^2 + \mathcal{O}(N_c^{-2})$$

→ Up to $1/N_c^2$ suppressed effects, squared subamplitudes have definite color flow – important for development of parton shower

Spinor helicity formalism

Scattering amplitudes for massless
plane waves of definite momentum:
Lorentz 4-vectors k_i^μ $k_i^2=0$

Natural to use Lorentz-invariant products
(invariant masses): $s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$
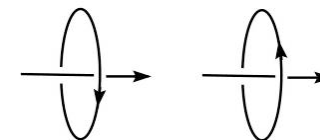
But for elementary particles with **spin** (*e.g.* all observed ones!)
there is a better way:

Take "square root" of 4-vectors k_i^μ (spin 1)
use Dirac (Weyl) spinors $u_\alpha(k_i)$ (spin ½)

right-handed: $(\lambda_i)_\alpha = u_+(k_i)$ left-handed: $(\tilde\lambda_i)_{\dot\alpha} = u_-(k_i)$

q,g,γ, all have 2 helicity states, $h = \pm$

Spinor products

Instead of Lorentz products: $\quad s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$

Use spinor products:

$$\bar{u}_-(k_i)u_+(k_j) = \varepsilon^{\alpha\beta}(\lambda_i)_\alpha(\lambda_j)_\beta = \langle i\,j \rangle$$

$$\bar{u}_+(k_i)u_-(k_j) = \varepsilon^{\dot\alpha\dot\beta}(\tilde\lambda_i)_{\dot\alpha}(\tilde\lambda_j)_{\dot\beta} = [i\,j]$$

Identity $\quad k_i^\mu(\sigma_\mu)_{\alpha\dot\alpha} = (\slashed{k}_i)_{\alpha\dot\alpha} = u_+(k_i)\bar{u}_+(k_i) = (\lambda_i)_\alpha(\tilde\lambda_i)_{\dot\alpha}$

$\Rightarrow$ These are **complex square roots** of Lorentz products (for real k_i):

$$\langle i\,j \rangle [j\,i] = \tfrac{1}{2}\mathsf{Tr}\left[\slashed{k}_i\,\slashed{k}_j\right] = 2k_i \cdot k_j = s_{ij}$$

$$\langle i\,j \rangle = \sqrt{s_{ij}}\, e^{i\phi_{ij}} \qquad\qquad [j\,i] = \sqrt{s_{ij}}\, e^{-i\phi_{ij}}$$

Most famous (simplest) Feynman diagram

add helicity information, numeric labels

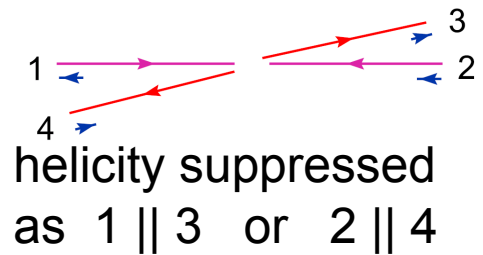$$\mathcal{A}_4 = 2ie^2 Q_e Q_q \delta_{i_3}^{\bar{i}_4} A_4$$

$$A_4 = \frac{1}{2\,s_{12}} \overline{v_-}(k_2)\gamma^\mu u_-(k_1)\ \overline{u_+}(k_3)\gamma_\mu v_+(k_4)$$

Fierz identity

$$(\sigma^\mu)_{\alpha\dot\alpha}(\sigma_\mu)^{\dot\beta\beta} = 2\,\delta_\alpha^\beta \delta_{\dot\alpha}^{\dot\beta}$$

$$= \frac{1}{2\,s_{12}} (\sigma^\mu)_{\alpha\dot\alpha}(\lambda_2)^\alpha(\tilde\lambda_1)^{\dot\alpha}\ (\sigma_\mu)^{\dot\beta\beta}(\tilde\lambda_3)_{\dot\beta}(\lambda_4)_\beta$$

$$= \frac{1}{s_{12}} (\lambda_2)^\alpha(\tilde\lambda_1)^{\dot\alpha}(\lambda_4)_\alpha(\tilde\lambda_3)_{\dot\alpha}$$
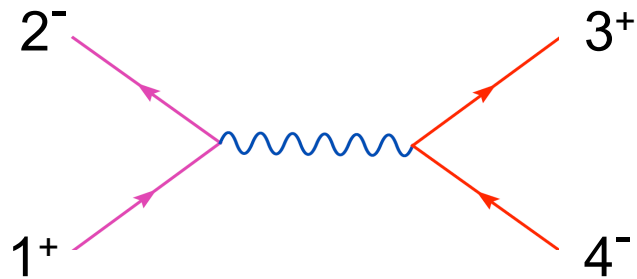
$$A_4 = \frac{\langle 2\,4\rangle\,[1\,3]}{s_{12}} = e^{i\phi}\frac{s_{13}}{s_{12}} = \frac{-e^{i\phi}}{2}(1-\cos\theta)$$

helicity suppressed
as 1 ‖ 3 or 2 ‖ 4

Useful to rewrite answer

2^- 3^+

1^+ 4^-

$$A_4 = \frac{\langle 2\,4\rangle [1\,3]}{s_{12}}$$

$$= \frac{\langle 2\,4\rangle [1\,3]\langle 1\,3\rangle}{\langle 1\,2\rangle [2\,1]\langle 1\,3\rangle}$$

$$= -\frac{\langle 2\,4\rangle [2\,4]\langle 2\,4\rangle}{\langle 1\,2\rangle [2\,4]\langle 4\,3\rangle}$$

$$\boxed{A_4 = \frac{\langle 2\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}}$$ "holomorphic"

or $$\boxed{A_4 = \frac{[1\,3]^2}{[1\,2][3\,4]}}$$ "antiholomorphic"

useful identities:

$$\langle i\,j\rangle = -\langle j\,i\rangle$$
$$[i\,j] = -[j\,i]$$
$$\langle i\,i\rangle = [i\,i] = 0$$
$$\langle i\,j\rangle [j\,i] = s_{ij}$$
$$\sum_{j=1}^{n} \langle i\,j\rangle [j\,k] = 0$$
$$s_{12} = s_{34}$$
$$s_{13} = s_{24}$$
$$\langle i\,j\rangle\langle k\,l\rangle - \langle i\,k\rangle\langle j\,l\rangle = \langle i\,l\rangle\langle k\,j\rangle$$

Schouten

Symmetries for all other helicity config's

$$2^-$$ $$3^+$$

$$C$$

$$1 \leftrightarrow 2$$

$$\Rightarrow$$

$$2^+$$ $$3^+$$

$$1^+$$ $$4^-$$

$$1^-$$ $$4^-$$

$$A_4 = \frac{\langle 2\,4 \rangle^2}{\langle 1\,2 \rangle \langle 3\,4 \rangle}$$

$$A_4 = -\frac{\langle 1\,4 \rangle^2}{\langle 1\,2 \rangle \langle 3\,4 \rangle}$$

$$P \quad \Downarrow \quad \langle \, \rangle \leftrightarrow [\,] \qquad \Downarrow$$

$$2^+$$ $$3^-$$

$$1^-$$ $$4^+$$

$$\Rightarrow$$

$$2^-$$ $$3^-$$

$$1^+$$ $$4^+$$

$$A_4 = \frac{[2\,4]^2}{[1\,2][3\,4]}$$

$$A_4 = -\frac{[1\,4]^2}{[1\,2][3\,4]}$$
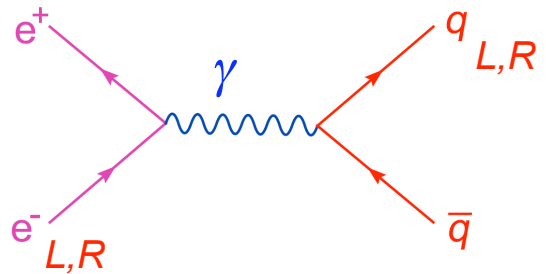
Unpolarized, helicity-summed cross sections

(the norm in QCD)

$$\frac{d\sigma(e^+ e^- \to q\bar{q})}{d\cos\theta} \propto \sum_{\text{hel.}} |A_4|^2 = 2\left\{\left|\frac{\langle 2\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}\right|^2 + \left|\frac{\langle 1\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}\right|^2\right\}$$

$$= 2\frac{s_{24}^2 + s_{14}^2}{s_{12}^2}$$

$$= \frac{1}{2}[(1 - \cos\theta)^2 + (1 + \cos\theta)^2]$$

$$= 1 + \cos^2\theta$$

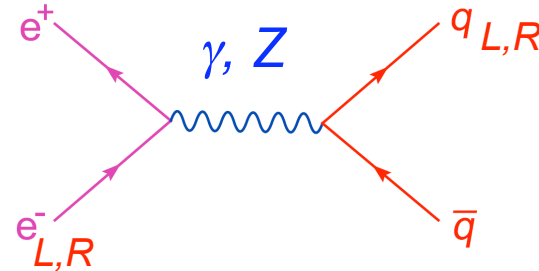# Reweight helicity amplitudes → electroweak/ QCD processes
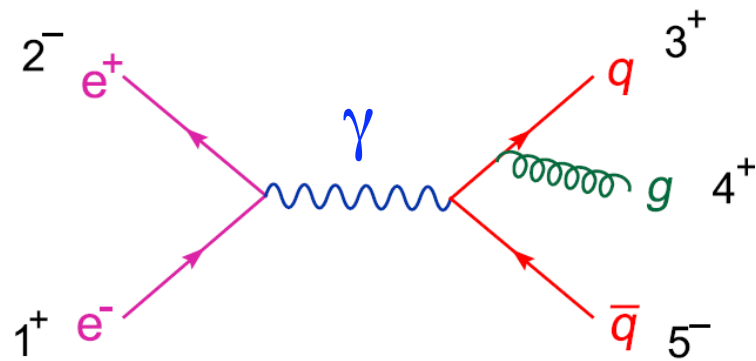
For example, *Z* exchange

$$ Q_e Q_q \qquad \Rightarrow \qquad Q_e Q_q + \frac{v_{L,R}^e v_{L,R}^q \; s}{s - M_Z^2 + i\Gamma_Z M_Z} $$

$$ v_L^f = \frac{2I_3^f - 2Q_f \sin^2 \theta_W}{\sin 2\theta_W} \qquad v_R^f = -\frac{2Q_f \sin^2 \theta_W}{\sin 2\theta_W} $$
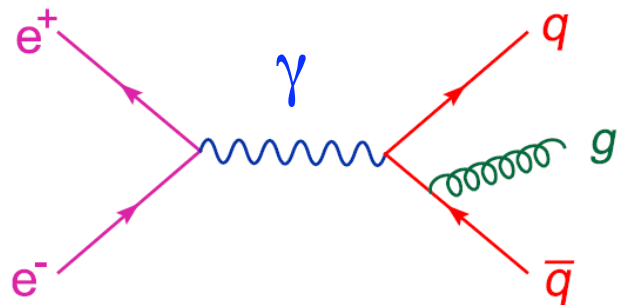
Next most famous pair of Feynman diagrams

(to a higher-order QCD person)

$$\mathcal{A}_5 = 2ie^2 g Q_e Q_q (T^{a_4})^{\bar{i}_5}_{i_3} A_5$$

$$A_5 = \frac{\langle 2\,5 \rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)\not{\epsilon}_4^+|3^-\rangle}{\sqrt{2}\,s_{34}}$$

$$+ \frac{[1\,3]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)\not{\epsilon}_4^+|5^+\rangle}{\sqrt{2}\,s_{45}}$$

Helicity formalism for massless vectors

Berends, Kleiss, De Causmaecker, Gastmans, Wu (1981); De Causmaecker, Gastmans, Troost, Wu (1982); Xu, Zhang, Chang (1984); Kleiss, Stirling (1985); Gunion, Kunszt (1985)

$$(\varepsilon_i^+)_\mu = \varepsilon_\mu^+(k_i, q) = \frac{\langle i^+|\gamma_\mu|q^+\rangle}{\sqrt{2}\,\langle i\,q\rangle}$$

$$(\not{\varepsilon}_i^+)_{\alpha\dot\alpha} = \not{\varepsilon}_{\alpha\dot\alpha}^+(k_i, q) = \frac{\sqrt{2}\,\tilde\lambda_i^{\dot\alpha}\lambda_q^\alpha}{\langle i\,q\rangle}$$

reference vector q^μ
is null, $q^2 = 0$

$$\not{q}|q^\pm\rangle = 0$$

obeys $\varepsilon_i^+ \cdot k_i = 0$ (required transversality)

$\varepsilon_i^+ \cdot q = 0$ (bonus)

under azimuthal rotation about k_i axis, helicity +1/2 $\tilde\lambda_i^{\dot\alpha} \to e^{i\phi/2}\tilde\lambda_i^{\dot\alpha}$

helicity -1/2 $\lambda_i^\alpha \to e^{-i\phi/2}\lambda_i^\alpha$

so $\not{\varepsilon}_i^+ \propto \dfrac{\tilde\lambda_i^{\dot\alpha}}{\lambda_i^\alpha} \to e^{i\phi}\,\not{\varepsilon}_i^+$ as required for helicity +1

$e^+e^- \rightarrow qg\bar{q}$ (cont.)

$$A_5 = \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)\not{\epsilon}_4^+|3^-\rangle}{\sqrt{2}\,s_{34}}$$

$$+ \frac{[1\,3]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)\not{\epsilon}_4^+|5^+\rangle}{\sqrt{2}\,s_{45}}$$

$$= \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)|q^+\rangle[4\,3]}{s_{34}\langle 4\,5\rangle}$$

$$+ \frac{[1\,3]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)|4^-\rangle\langle q\,5\rangle}{s_{45}\langle 4\,5\rangle}$$

$$= \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)|5^+\rangle[4\,3]}{s_{34}\langle 4\,5\rangle}$$

$$= -\frac{\langle 2\,5\rangle[1\,2]\langle 2\,5\rangle[4\,3]}{\langle 1\,2\rangle[2\,1]\langle 3\,4\rangle[4\,3]\langle 4\,5\rangle}$$

$$\boxed{A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle}}$$

Choose $q = k_5$
to remove 2nd graph

Properties of $\mathcal{A}_5(e^+e^- \to qg\bar{q})$

1. **Soft gluon behavior** $\quad k_4 \to 0$

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} = \frac{\langle 3\,5\rangle}{\langle 3\,4\rangle\langle 4\,5\rangle} \times \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,5\rangle}$$

$$\to \; \mathcal{S}(3, 4^+, 5) \times A_4(1^+, 2^-, 3^+, 5^-)$$

Universal "eikonal" factors
for emission of soft gluon s
between two hard partons a and b

$$\mathcal{S}(a, s^+, b) = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\langle s\,b\rangle}$$

$$\mathcal{S}(a, s^-, b) = -\frac{[a\,b]}{[a\,s][s\,b]}$$

Soft emission is from the classical chromoelectric current:
independent of parton type (q vs. g) and helicity
– only depends on momenta of a,b, and color charge:

$$\frac{\varepsilon_s^+(q)\cdot k_a}{k_a\cdot k_s} - \frac{\varepsilon_s^+(q)\cdot k_b}{k_b\cdot k_s} \propto \frac{\langle a\,q\rangle}{\langle s\,q\rangle\langle a\,s\rangle} - \frac{\langle b\,q\rangle}{\langle s\,q\rangle\langle b\,s\rangle} = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\langle s\,b\rangle}$$

Properties of $\mathcal{A}_5(e^+e^- \to qg\bar{q})$ (cont.)

2. Collinear behavior

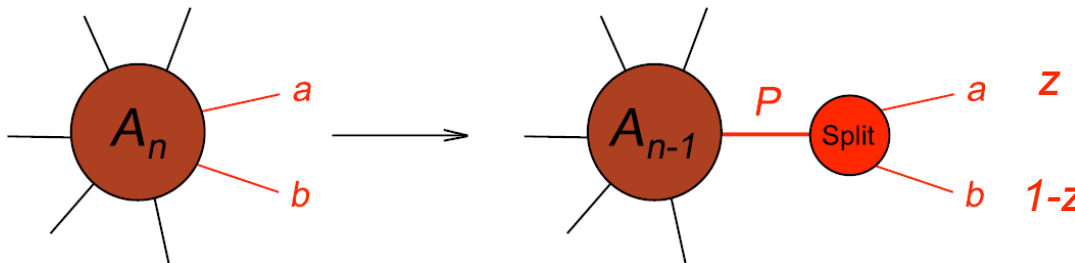$$k_3 \| k_4: \qquad k_3 = z\,k_P, \quad k_4 = (1-z)\,k_P$$
$$k_P \equiv k_3 + k_4, \quad k_P^2 \to 0$$

$$\lambda_3 \approx \sqrt{z}\lambda_P, \quad \lambda_4 \approx \sqrt{1-z}\lambda_P, \quad \text{etc.}$$

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} \approx \frac{1}{\sqrt{1-z}\,\langle 3\,4\rangle} \times \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle P\,5\rangle}$$

$$\to \text{Split}_-(3_q^+, 4_g^+) \times A_4(1^+, 2^-, P^+, 5^-)$$

Time-like kinematics (fragmentation).
Space-like (parton evolution) related by crossing

Universal collinear factors, or splitting amplitudes $\text{Split}_{-h_P}(a^{h_a}, b^{h_b})$ depend on parton type and helicity h

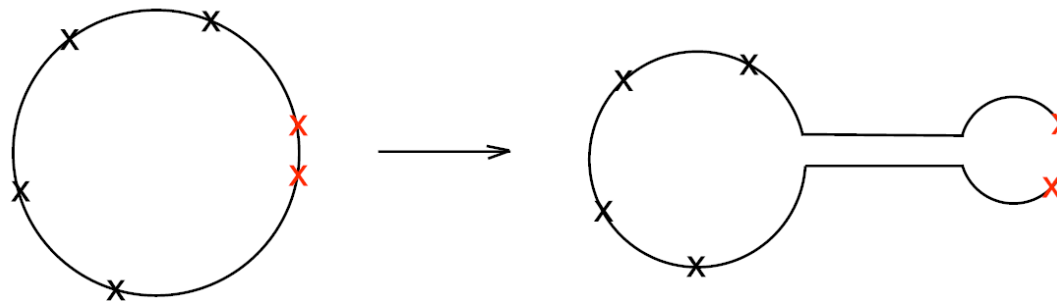# Collinear limits (cont.)

We found, from $k_3 \,||\, k_4$:

$$\mathrm{Split}_-(a_q^+, b_g^+) = \frac{1}{\sqrt{1-z}\,\langle a\,b\rangle}$$

Similarly, from $k_4 \,||\, k_5$:

$$\mathrm{Split}_+(a_g^+, b_{\bar{q}}^-) = \frac{1-z}{\sqrt{z}\,\langle a\,b\rangle}$$

$$\Downarrow$$

Applying C and P:

$$\mathrm{Split}_-(a_q^+, b_g^-) = -\frac{z}{\sqrt{1-z}\,[a\,b]}$$

Universality can be argued various ways, including from factorization + operator product expansion in string theory:

Mangano, Parke, Phys. Rept. 200, 301 (1991)

Simplest pure-gluonic amplitudes

Strikingly, many vanish:

$$A_n^{\text{tree}}(1^\pm, 2^+, \ldots, n^+) = \quad = \quad = 0$$

Maximally helicity-violating (MHV) amplitudes:

$$A_n^{ij,\ \text{MHV}} = A_n^{\text{tree}}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)$$

$$= \quad = \quad \frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$
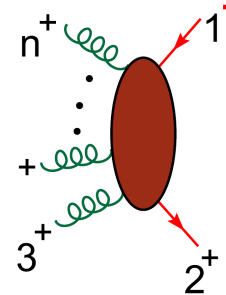

Parke-Taylor formula (1986)

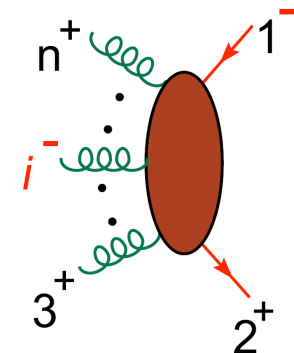Remarkable simplicity – has inspired many formal developments

MHV amplitudes with massless quarks

Helicity conservation on fermion line → $A_n^{\text{tree}}(1_{\bar{q}}^{\pm}, 2_q^{\pm}, 3^{h_3}, \ldots, n^{h_n}) \equiv 0$

more vanishing ones: $A_n^{\text{tree}}(1_{\bar{q}}^-, 2_q^+, 3^+, \ldots, n^+) =$ $= 0$

the MHV amplitudes:

$A_n^{\text{tree}}(1_{\bar{q}}^-, 2_q^+, \ldots, i^-, \ldots, n^+) =$ $= \dfrac{\langle 1\, i \rangle^3 \langle 2\, i \rangle}{\langle 1\, 2 \rangle \langle 2\, 3 \rangle \cdots \langle n\, 1 \rangle}$

Related to pure-gluon MHV amplitudes by a secret supersymmetry:
after stripping off color factors, massless quarks ~ gluinos

Grisaru, Pendleton, van Nieuwenhuizen (1977);
Parke, Taylor (1985); Kunszt (1986); Nair (1988)

Properties of MHV amplitudes

1. Verify soft limit

$$k_s \to 0$$

$$\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle \cdots \langle a\,s\rangle\langle s\,b\rangle \cdots \langle n\,1\rangle} = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\langle s\,b\rangle}\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle \cdots \langle a\,b\rangle \cdots \langle n\,1\rangle}$$

$$\to \mathsf{Soft}(a, s^+, b) \times A_{n-1}^{ij,\ \mathsf{MHV}}$$

2. Extract gluonic collinear limits: $\quad k_a \,\|\, k_b \quad (b = a+1)$

$$\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle \cdots \langle a-1,a\rangle\langle a\,b\rangle\langle b,b+1\rangle \cdots \langle n\,1\rangle} = \frac{1}{\sqrt{z(1-z)}\,\langle a\,b\rangle}\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle \cdots \langle a-1,P\rangle\langle P,b+1\rangle \cdots \langle n\,1\rangle}$$

$$\to \mathsf{Split}_-(a^+, b^+) \times A_{n-1}^{ij,\ \mathsf{MHV}}$$

So

$$\mathsf{Split}_-(a^+, b^+) = \frac{1}{\sqrt{z(1-z)}\,\langle a\,b\rangle}$$

and

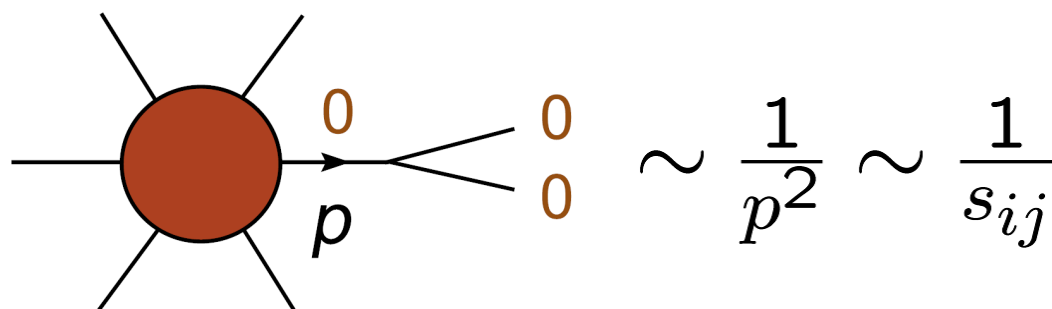$$\mathsf{Split}_+(a^-, b^+) = \frac{z^2}{\sqrt{z(1-z)}\,\langle a\,b\rangle}$$

plus parity conjugates

$$\mathsf{Split}_+(a^+, b^-) = \frac{(1-z)^2}{\sqrt{z(1-z)}\,\langle a\,b\rangle}$$
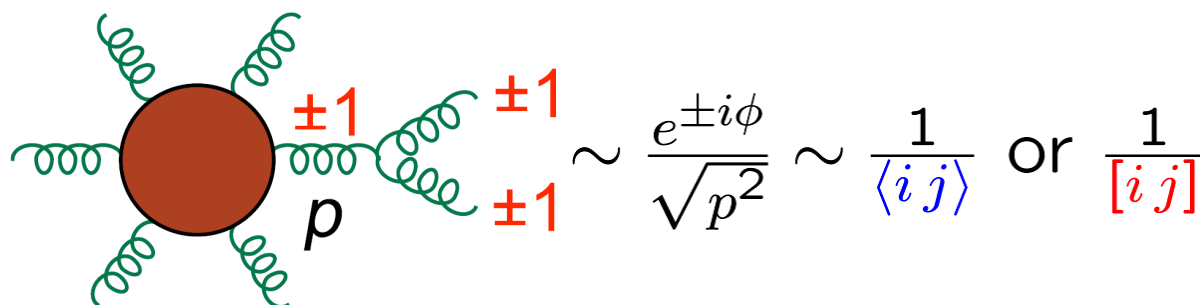
Spinor Magic

Spinor products precisely capture
square-root + phase behavior in **collinear limit**.
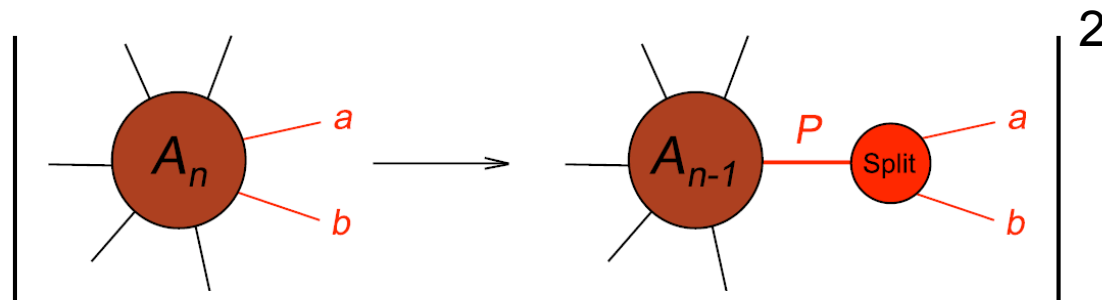Excellent variables for **helicity amplitudes**

scalars

$$\sim \frac{1}{p^2} \sim \frac{1}{s_{ij}}$$

gauge theory

angular momentum
mismatch

$$\sim \frac{e^{\pm i\phi}}{\sqrt{p^2}} \sim \frac{1}{\langle i\,j\rangle} \text{ or } \frac{1}{[i\,j]}$$

From splitting amplitudes to probabilities

$$d\sigma_n \sim d\sigma_{n-1} \times \frac{1}{s_{ab}} \times P(z)$$

$$P(z) \propto \sum_{h_P, h_a, h_b} |\mathsf{Split}_{-h_P}(a^{h_a}, b^{h_b})|^2 \, s_{ab}$$

$q \to qg$:

$$P_{qq}(z) \propto C_F \left\{ \left| \frac{1}{\sqrt{1-z}} \right|^2 + \left| \frac{z}{\sqrt{1-z}} \right|^2 \right\}$$

$$= C_F \frac{1+z^2}{1-z} \qquad z < 1$$

$$C_F = \frac{N_c^2 - 1}{2N_c}$$

Note soft-gluon singularity as $z_g = 1 - z \to 0$

Space-like splitting

- The case relevant for parton evolution
- Related by crossing to time-like case
- Have to watch out for flux factor, however
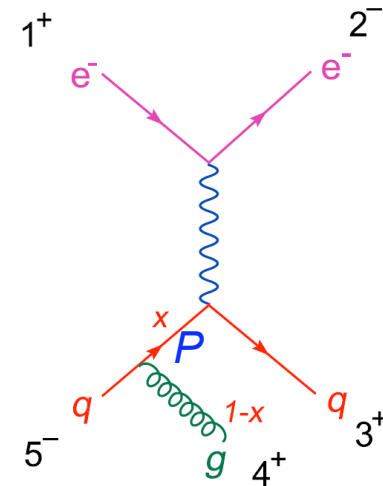
$$q \to qg: \quad k_P = x\,k_5, \quad k_4 = (1-x)\,k_5$$

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} \approx \frac{\frac{1}{x}}{\sqrt{\frac{1-x}{x}}\,\langle 4\,5\rangle} \times \frac{\langle 2\,P\rangle^2}{\langle 1\,2\rangle\langle 3\,P\rangle}$$

$$= \frac{1}{\sqrt{x}}\frac{1}{\sqrt{1-x}\,\langle 4\,5\rangle} \times \frac{\langle 2\,P\rangle^2}{\langle 1\,2\rangle\langle 3\,P\rangle}$$

absorb into flux factor:

$$d\sigma_5 \propto \frac{1}{s_{15}}$$

$$d\sigma_4 \propto \frac{1}{s_{1P}} = \frac{1}{x\,s_{15}}$$

When dust settles, get exactly the same splitting kernels (at LO)

Similarly for gluons

$g \to gg$:

$$P_{gg}(z) \propto C_A \left\{ \left| \frac{1}{\sqrt{z(1-z)}} \right|^2 + \left| \frac{z^2}{\sqrt{z(1-z)}} \right|^2 + \left| \frac{(1-z)^2}{\sqrt{z(1-z)}} \right|^2 \right\}$$

$$= C_A \frac{1 + z^4 + (1-z)^4}{z(1-z)} \qquad\qquad C_A = N_c$$

$$= 2C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right] \qquad z < 1$$

Again a soft-gluon singularity. Gluon number not conserved.
But momentum is. Momentum conservation mixes $g \to gg$ with

$g \to q\bar{q}$:

$$\boxed{P_{qg}(z) = T_R \left[z^2 + (1-z)^2 \right]} \qquad T_R = \tfrac{1}{2}$$

(can deduce, up to color factors, by taking
$e^+ \,||\, e^-$ in $\mathcal{A}_5(e^+ e^- \to q g \bar{q})$)

Exercise

Gluon splitting (cont.)

$g \to gg$:

Applying momentum conservation,

Exercise: Work out b_0

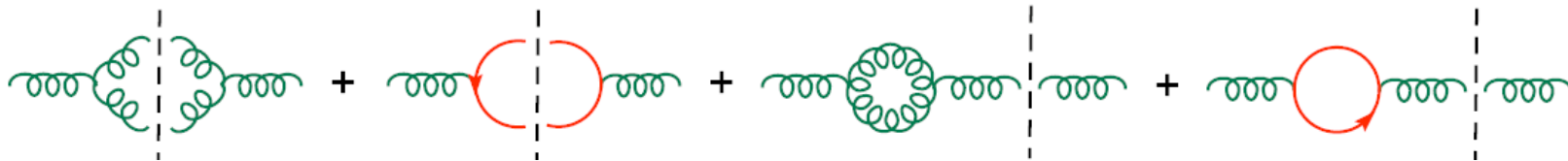$$\int_0^1 dz\, z\, \left[P_{gg}(z) + 2n_f P_{qg}(z) \right] = 0$$

gives

$$P_{gg}(z) = 2C_A \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + b_0\, \delta(1-z)$$

$$b_0 = \frac{11C_A - 4n_f T_R}{6}$$

Amusing that first β-function coefficient enters,
since no loops were done, except implicitly via unitarity:

End of Lecture 1