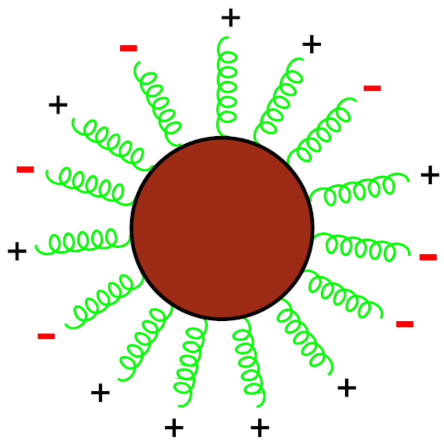
# Scattering Amplitudes in Gauge Theory and Gravity

## Lecture 3 – Gauge Theory at One Loop




Lance Dixon (CERN & SLAC)

CERN Winter School

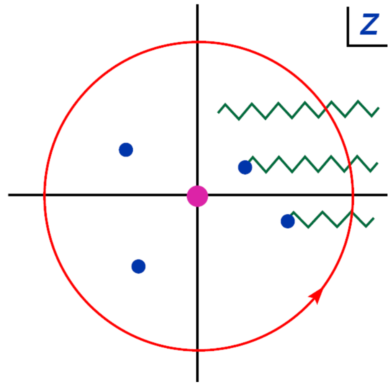
Jan. 24-28, 2011

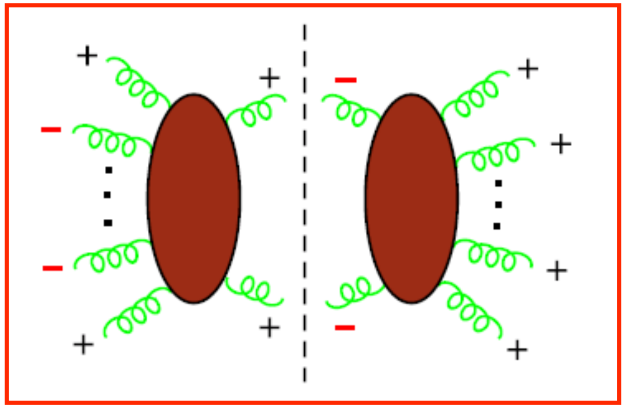
# On-Shell Recursion at One Loop

Bern, LD, Kosower, hep-th/0501240, hep-ph/0505055, hep-ph/0507005;
Berger, et al., hep-ph/0604195, hep-ph/0607014, 0803.4180

- **Same general techniques** work for one-loop QCD amplitudes

- **New features** compared with tree case, especially **branch cuts**
- Determine cut terms efficiently using (generalized) unitarity – to be discussed later

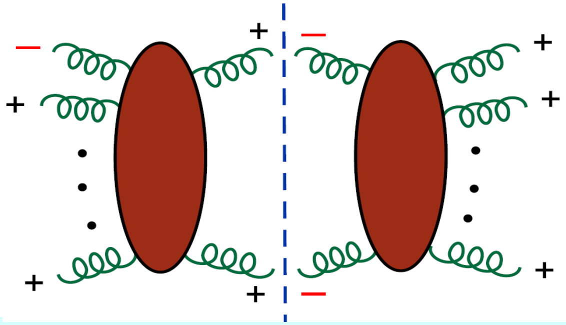


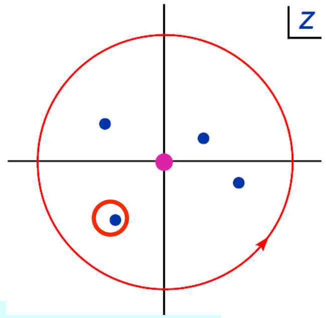
# Special one-loop amplitudes

Bern, LD, Kosower, hep-th/0501240, hep-th/0505055

- Some one-loop amplitudes are "tree-like" in that they have no cuts, only poles. For example, the "more-than-MHV" $n$-gluon amplitudes:

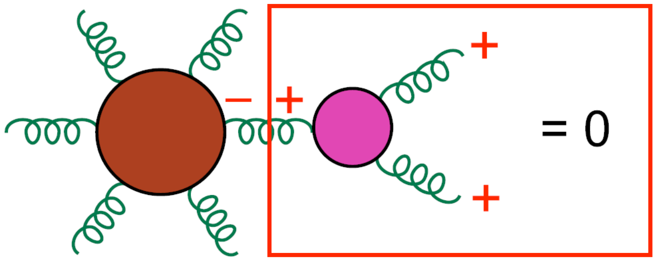
$A_n^{1-\text{loop}}(1^\pm, 2^+, 3^+, \ldots, n^+)$  $= 0$





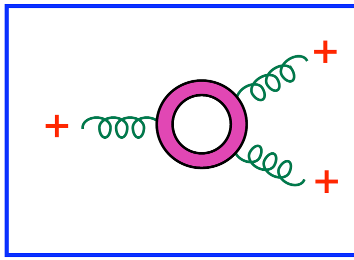


- New features **still arise** compared with BCFW for trees, due to different collinear behavior of loop amplitudes:





but

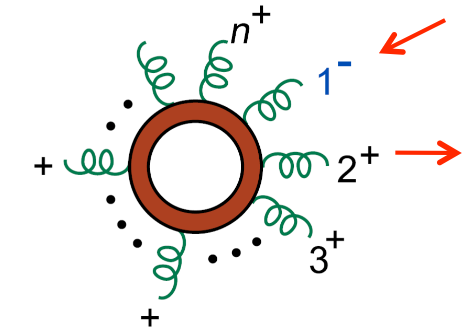




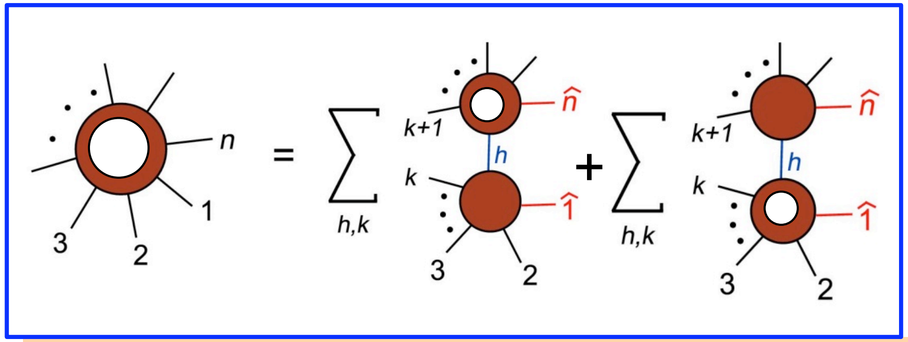
$\propto \dfrac{[i\,j]}{\langle i\,j \rangle^2}$

Leads to double poles in $z$ plane; residue from single pole needs some guesswork

# Sample one-loop all-$n$ recursion relation




$$
\begin{aligned}
A_n^{(1)}(1^-,2^+,\ldots,n^+) &= A_{n-1}^{(1)}(4^+,5^+,\ldots,n^+,\hat{1}^-,\hat{K}_{23}^+)\,\frac{i}{K_{23}^2}\,A_3^{(0)}(\hat{2}^+,3^+,-\hat{K}_{23}^-) \\
&+ \sum_{j=4}^{n-1} A_{n-j+2}^{(0)}((j+1)^+,5^+,\ldots,n^+,\hat{1}^-,\hat{K}_{2\ldots j}^-)\,\frac{i}{K_{2\ldots j}^2}\,A_j^{(1)}(\hat{2}^+,3^+,\ldots,j^+,-\hat{K}_{2\ldots j}^+) \\
&+ A_{n-1}^{(0)}(4^+,5^+,\ldots,n^+,\hat{1}^-,\hat{K}_{23}^-)\,\frac{i}{(K_{23}^2)^2}\,V_3^{(1)}(\hat{2}^+,3^+,-\hat{K}_{23}^+) \\
&\times \Big(1 + K_{23}^2\,\mathcal{S}^{(0)}(\hat{1},\hat{K}_{23}^+,4)\,\mathcal{S}^{(0)}(3,-\hat{K}_{23}^-,\hat{2})\Big)
\end{aligned}
$$

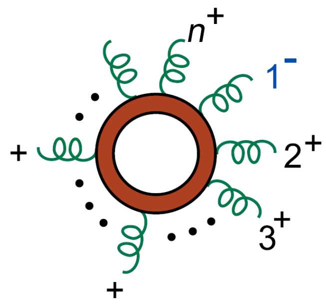



"soft factors" from single pole underneath double pole; later derived using space-cone gauge $A\cdot q = 0$, independence of $q$
Vaman, Yao, 0805.2645

Results agree with Mahlon, hep-ph/9312276 though much shorter formulae obtained here

# Solution to recursion relation

$$A_n^{(1)}(1^-,2^+,3^+,\ldots,n^+) = \frac{i}{3}\frac{T_1+T_2}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,,$$




where

$$T_1 = \sum_{l=2}^{n-1}\frac{\langle 1\,l\rangle\langle 1\,(l+1)\rangle\langle 1^-|\,\not{K}_{l,l+1}\not{K}_{(l+1)\cdots n}\,|1^+\rangle}{\langle l\,(l+1)\rangle}\,,$$

$$\begin{aligned} T_2 = \sum_{l=3}^{n-2}\sum_{p=l+1}^{n-1} & \frac{\langle (l-1)\,l\rangle}{\langle 1^-|\,\not{K}_{(p+1)\cdots n}\not{K}_{l\cdots p}\,|(l-1)^+\rangle\,\langle 1^-|\,\not{K}_{(p+1)\cdots n}\not{K}_{l\cdots p}\,|l^+\rangle} \\ & \times\frac{\langle p\,(p+1)\rangle}{\langle 1^-|\,\not{K}_{2\cdots(l-1)}\not{K}_{l\cdots p}\,|p^+\rangle\,\langle 1^-|\,\not{K}_{2\cdots(l-1)}\not{K}_{l\cdots p}\,|(p+1)^+\rangle} \\ & \times\langle 1^-|\,\not{K}_{l\cdots p}\not{K}_{(p+1)\cdots n}\,|1^+\rangle^3 \\ & \times\frac{\langle 1^-|\,\not{K}_{2\cdots(l-1)}[\mathcal{F}(l,p)]^2\not{K}_{(p+1)\cdots n}\,|1^+\rangle}{s_{l\cdots p}}\,. \end{aligned}$$

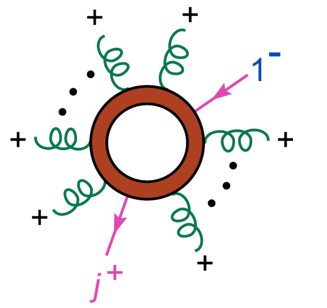
$$\mathcal{F}(l,p) = \sum_{i=l}^{p-1}\sum_{m=i+1}^{p}\not{k}_i\not{k}_m$$

BDK, hep-ph/0505055

# Other infinite sequences known analytically

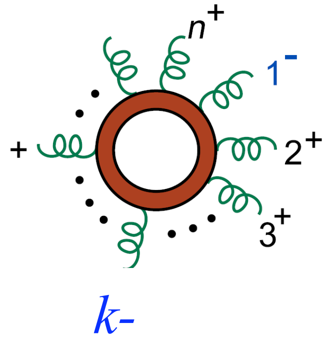
The cut-free amplitudes with 2 external fermions:

BDK, hep-th/0505055




The "one-loop Parke-Taylor" (MHV $n$-gluon) amplitudes:

Forde, Kosower, hep-ph/0509358;
Berger, Bern, LD, Forde, Kosower, hep-th/0607014




BBDFK, hep-ph/0604195

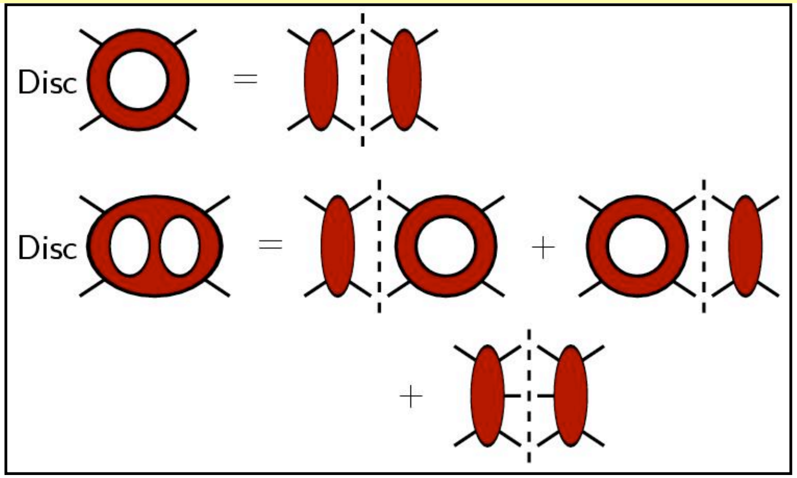
$n$-gluon "split" helicity $(- - \cdots - + + \cdots +)$
"Higgs" + $n$-gluon MHV $(\phi; - + \cdots + - + \cdots +)$

Badger, Glover, Risager, 0704.3194; Glover, Mastrolia, Williams, 0804.4149

Formulae get pretty involved. Fortunately, method can be implemented numerically as well, for generic helicity amplitudes (BlackHat) 0803.4180

# Generic Amplitudes Have Cuts – First determine cut parts using perturbative unitarity

• *S*-matrix a unitary operator between in and out states
→ unitarity relations (cutting rules) for amplitudes




• Reconstruction of full amplitudes from cuts very efficient, due to simple structure of tree and lower-loop helicity amplitudes

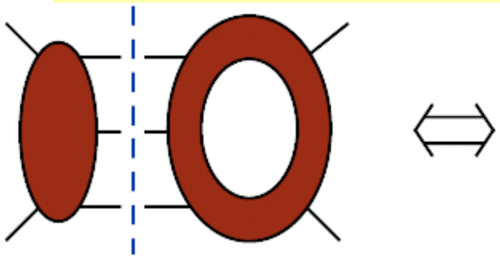
• Generalized unitarity (more propagators open) useful at both one loop and higher loops to reduce everything to trees

# Multi-loop generalized unitarity

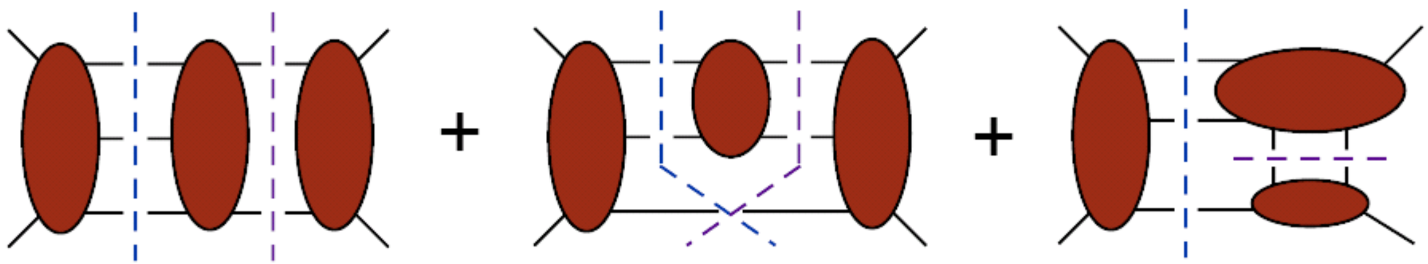
Bern, LD, Kosower, hep-ph/0001001; Bern, Czakon, LD, Kosower, Smirnov hep-th/0610248; Bern, Carrasco, LD, Johansson, Kosower, Roiban, hep-th/0702112; BCJK, 0705.1864; Cachazo, Skinner, 0801.4574; Cachazo, 0803.1988; Cachazo, Spradlin, Volovich, 0805.4832

Ordinary cuts of multi-loop amplitudes contain loop amplitudes.

For example, at 3 loops, one encounters the product of a 5-point tree and a 5-point one-loop amplitude:



Cut 5-point loop amplitude further, into (4-point tree) x (5-point tree), in all 3 inequivalent ways:



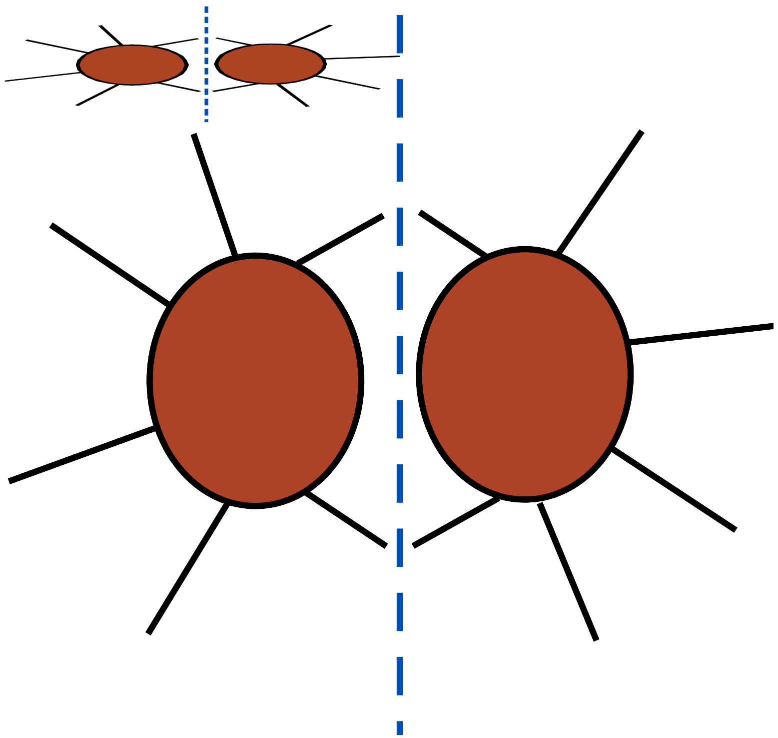
cut conditions satisfied by real momenta

# Generalized unitarity

**Ordinary unitarity:**

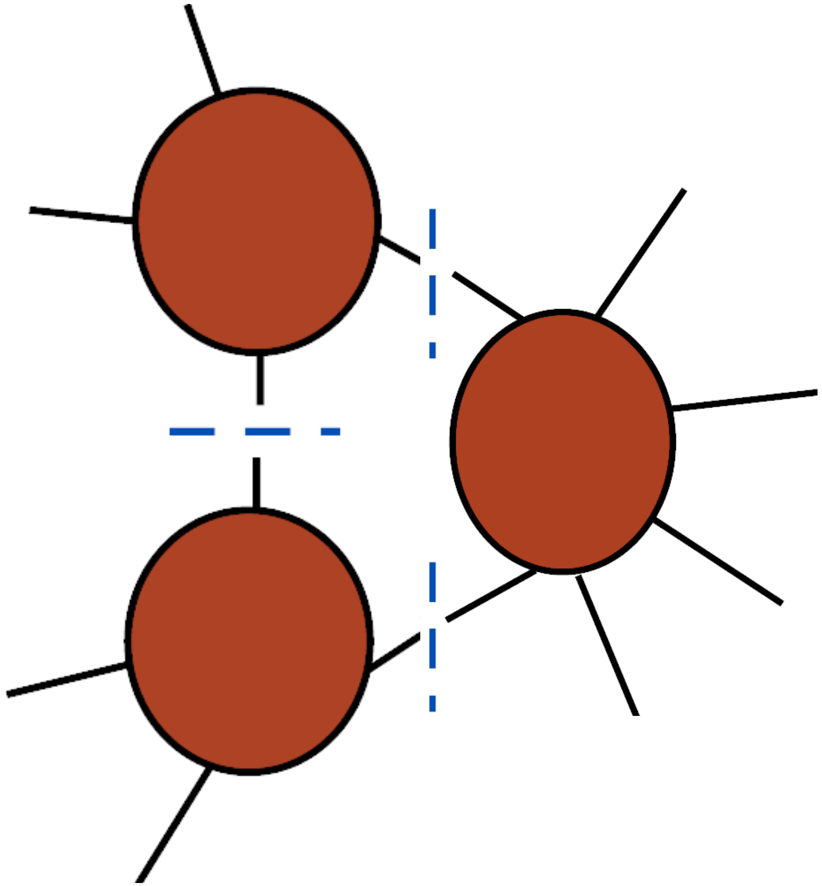
$\mathrm{Im}\ T = T^{\dagger}T$

put 2 particles on shell



**Generalized unitarity:**

put 3 or 4 particles on shell

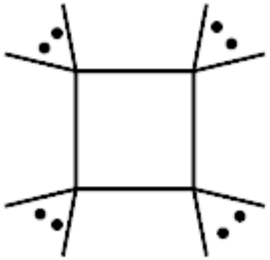
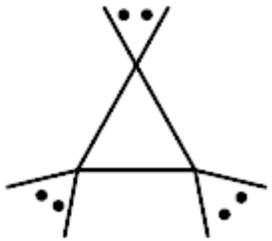
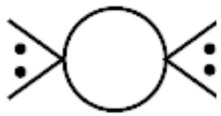


# One-loop amplitudes reduced to trees

When all external momenta are in $D = 4$, loop momenta in $D = 4\text{-}2\varepsilon$ (dimensional regularization), one can write:

Bern, LD, Dunbar, Kosower (1994)

coefficients are all rational functions – determine algebraically from products of trees using (generalized) unitarity

$$A^{1\text{-loop}} = \sum_i d_i$$  $$+ \sum_i c_i$$  $$+ \sum_i b_i$$ 

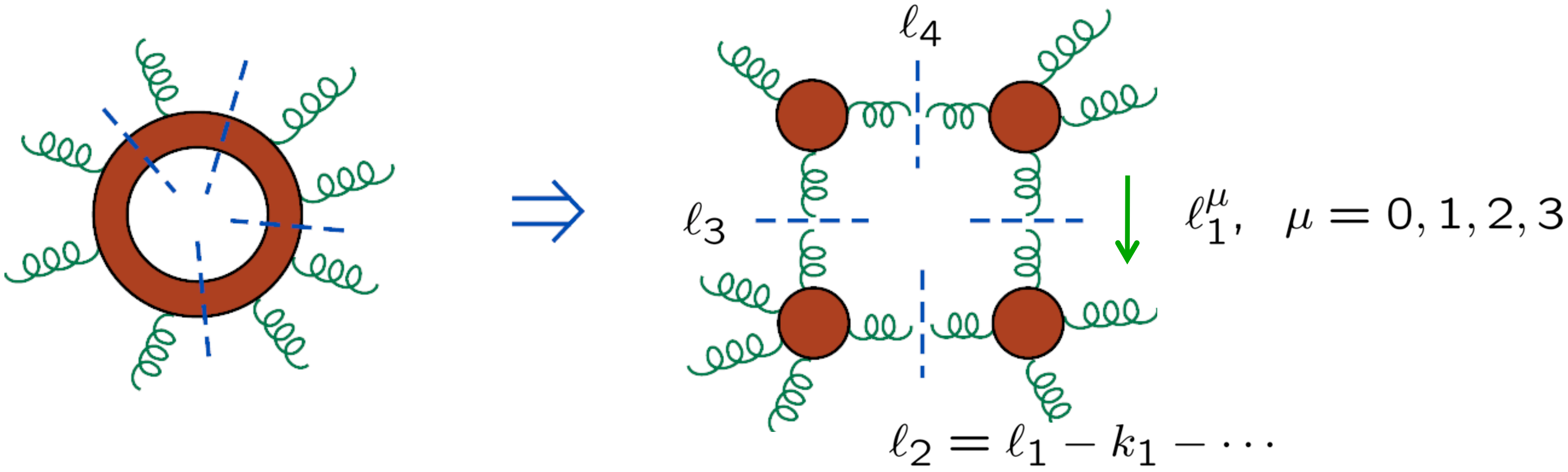
$$+ R + \mathcal{O}(\epsilon)$$

rational part

known **scalar** one-loop integrals, same for all amplitudes

# Generalized Unitarity for Box Coefficients $d_i$

Britto, Cachazo, Feng, hep-th/0412308



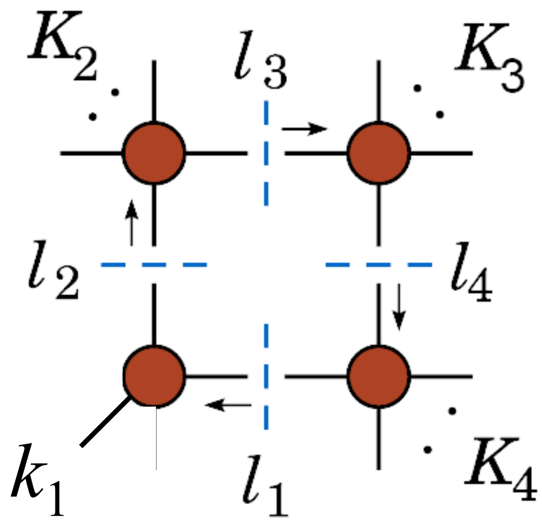

$$\int d^4\ell \; \delta(\ell_1^2 - m_1^2)\, \delta(\ell_2^2 - m_2^2) \times \delta(\ell_3^2 - m_3^2)\, \delta(\ell_4^2 - m_4^2) \times A^{1-\text{loop}}(\ell_i)$$
$$= \sum_{\pm} A_1^{\text{tree}}(\ell_0^\pm) A_2^{\text{tree}}(\ell_0^\pm) A_3^{\text{tree}}(\ell_0^\pm) A_4^{\text{tree}}(\ell_0^\pm)$$
$$= d_i^+ + d_i^-$$

No. of dimensions = 4 = no. of constraints → discrete solutions

Easy to code, numerically very stable

# Box coefficients $d_i$ (cont.)

Solutions simplify (and are more stable numerically) when all internal lines massless, at least one external line ($k_1$) massless:




$$(l_1^{(\pm)})^\mu = \frac{\langle 1^\mp | \not{K}_2 \not{K}_3 \not{K}_4 \gamma^\mu | 1^\pm \rangle}{2 \langle 1^\mp | \not{K}_2 \not{K}_4 | 1^\pm \rangle}, \qquad (l_2^{(\pm)})^\mu = -\frac{\langle 1^\mp | \gamma^\mu \not{K}_2 \not{K}_3 \not{K}_4 | 1^\pm \rangle}{2 \langle 1^\mp | \not{K}_2 \not{K}_4 | 1^\pm \rangle},$$

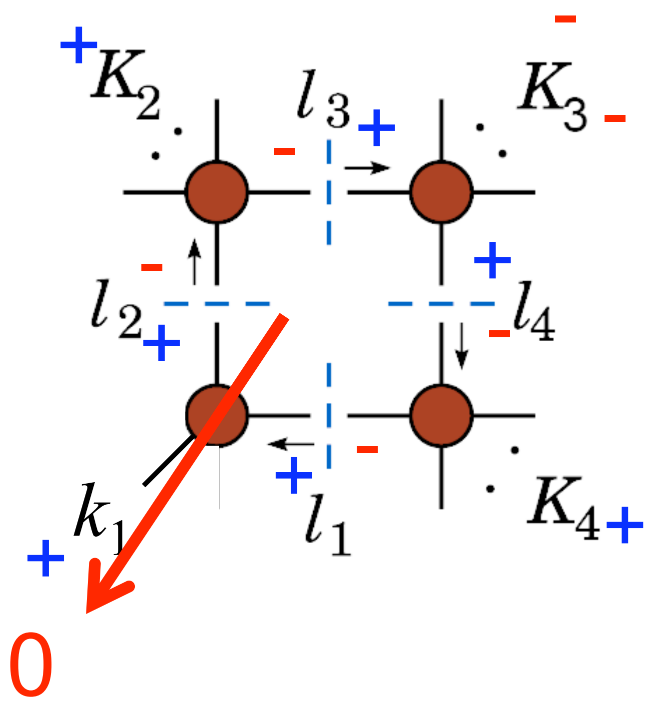
$$(l_3^{(\pm)})^\mu = \frac{\langle 1^\mp | \not{K}_2 \gamma^\mu \not{K}_3 \not{K}_4 | 1^\pm \rangle}{2 \langle 1^\mp | \not{K}_2 \not{K}_4 | 1^\pm \rangle}, \qquad (l_4^{(\pm)})^\mu = -\frac{\langle 1^\mp | \not{K}_2 \not{K}_3 \gamma^\mu \not{K}_4 | 1^\pm \rangle}{2 \langle 1^\mp | \not{K}_2 \not{K}_4 | 1^\pm \rangle}.$$

BH, 0803.4180; Risager 0804.3310

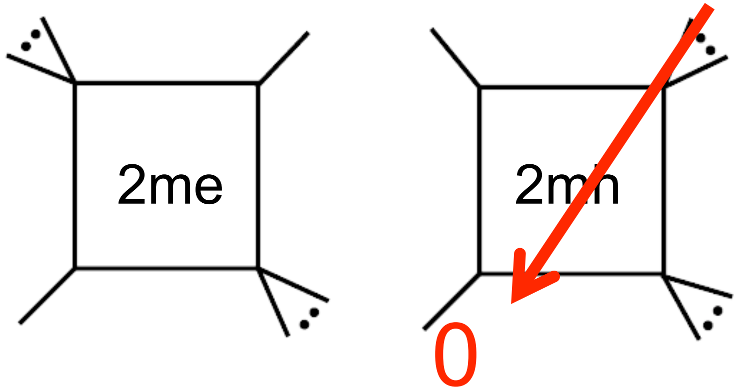
# MHV Box example

All 3-mass boxes vanish trivially
– not enough (-) helicities

Have 2 + 4 = 6 (-) helicities,
but need 2 + 2 + 2 + 1 = 7

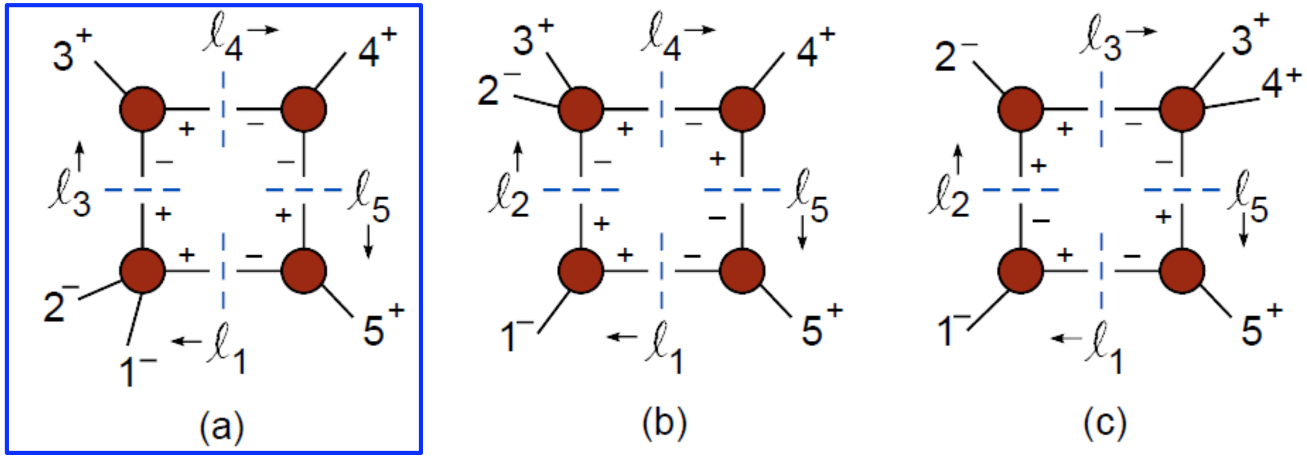



2-mass boxes come in two types:



# 5-point MHV Box example

For (--+++), 3 inequivalent boxes to consider




Look at this one. Corresponding integral is, in dim. reg.:

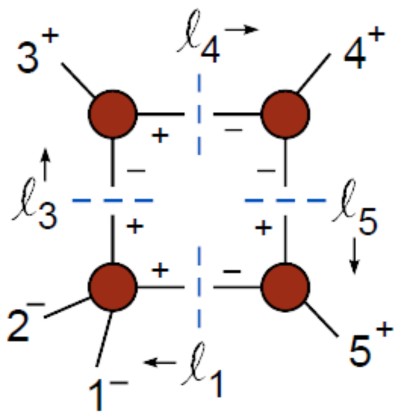
$$\mathcal{I}(K_{12}) = \mu^{2\epsilon} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{1}{\ell^2(\ell-K_{12})^2(\ell-K_{123})^2(\ell+k_5)^2}$$

$$= \frac{-2i\, c_\Gamma}{s_{34}s_{45}} \Bigg\{ -\frac{1}{\epsilon^2}\Bigg[\left(\frac{\mu^2}{-s_{34}}\right)^\epsilon + \left(\frac{\mu^2}{-s_{45}}\right)^\epsilon - \left(\frac{\mu^2}{-s_{12}}\right)^\epsilon\Bigg]$$

$$+ \mathrm{Li}_2\left(1-\frac{s_{12}}{s_{34}}\right) + \mathrm{Li}_2\left(1-\frac{s_{12}}{s_{45}}\right) + \frac{1}{2}\ln^2\left(\frac{-s_{34}}{-s_{45}}\right) + \frac{\pi^2}{6}\Bigg\}$$

$$+ \mathcal{O}(\epsilon)\,,$$

# 5-point MHV Box example




$$\ell_4^\mu = \tfrac{1}{2}\xi_4 \left\langle 3^- \right| \gamma^\mu \left| 4^- \right\rangle .$$

The constant $\xi_4$ is fixed by the last of the four on-shell equations,

$$\ell_1^2 = (\ell_4 - K_{45})^2 = -\xi_4 \left\langle 3^- \right| 5 \left| 4^- \right\rangle + s_{45} = 0\,,$$

to have the value $\xi_4 = \langle 4\,5\rangle \,/\, \langle 3\,5\rangle$.

$$c_{12} = \frac{1}{2} A_4^{\rm tree}(-\ell_1^+, 1^-, 2^-, \ell_3^+) A_3^{\rm tree}(-\ell_3^-, 3^+, \ell_4^+) A_3^{\rm tree}(-\ell_4^-, 4^+, \ell_5^-) A_3^{\rm tree}(-\ell_5^+, 5^+, \ell_1^-)$$

$$= \frac{1}{2} \frac{\langle 1\,2\rangle^3}{\langle 2\,\ell_3\rangle \langle \ell_3\,(-\ell_1)\rangle \langle (-\ell_1)\,1\rangle} \frac{[3\,\ell_4]^3}{[\ell_4\,(-\ell_3)]\,[(-\ell_3)\,3]} \frac{\langle \ell_5\,(-\ell_4)\rangle^3}{\langle 4\,\ell_5\rangle \langle (-\ell_4)\,4\rangle} \frac{[(-\ell_5)\,5]^3}{[5\,\ell_1]\,[\ell_1\,(-\ell_5)]}$$

$$= -\frac{1}{2} \frac{\langle 1\,2\rangle^3 \left\langle 3^+ \right| \ell_4 \ell_5 \left| 5^- \right\rangle^3}{\left\langle 2^- \right| \ell_3 \left| 3^- \right\rangle \left\langle 4^- \right| \ell_4 \ell_3 \ell_1 \left| 5^- \right\rangle \left\langle 1^- \right| \ell_1 \ell_5 \left| 4^+ \right\rangle} .$$

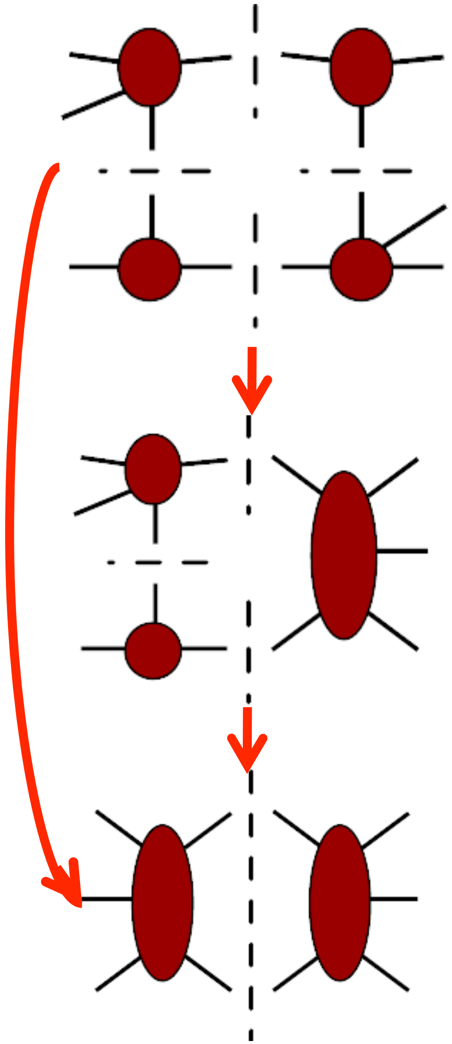
$$c_{12} = \frac{1}{2} \frac{\langle 1\,2\rangle^3 \left\langle 4^- \right| \ell_4 \left| 3^- \right\rangle^2 [4\,5]^3}{\left\langle 2^- \right| \ell_4 \left| 3^- \right\rangle \langle 3\,4\rangle\, [4\,5]\, \langle 1\,5\rangle \left\langle 4^- \right| \ell_4 \left| 5^- \right\rangle}$$

$$= -\frac{1}{2} \frac{\langle 1\,2\rangle^3\, s_{34} s_{45}}{\langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,5\rangle \langle 5\,1\rangle}$$

$$= \frac{i}{2} s_{34} s_{45}\, A_5^{\rm tree}(1^-, 2^-, 3^+, 4^+, 5^+) .$$

In fact, all 2me boxes in the 1-loop MHV amplitude in N=4 SYM (and many in QCD) are also proportional to $A_n^{\rm tree}$
BDDK (1994)

# Unitarity method – numerical implementation



Each box coefficient uniquely isolated by a “quadruple cut” given simply by a product of 4 tree amplitudes

Britto, Cachazo, Feng, hep-th/0412103

Ossola, Papadopolous, Pittau, hep-ph/0609007; Mastrolia, hep-th/0611091; Forde, 0704.1835; Ellis, Giele, Kunszt, 0708.2398; Berger et al., 0803.4180;…

triangle coefficients come from triple cuts, product of 3 tree amplitudes, but these are also “contaminated” by boxes

bubble coefficients come from ordinary double cuts, after removing contributions of boxes and triangles

# Triangle coefficients

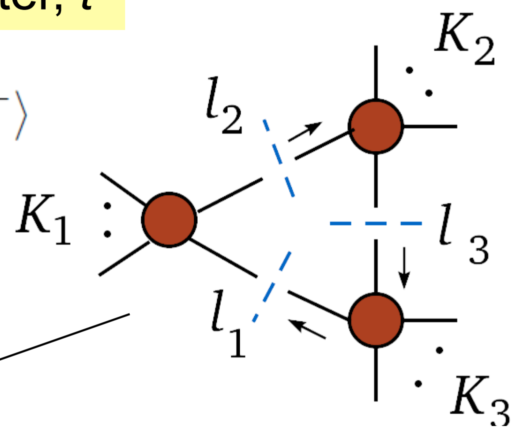
Forde, 0704.1835; BH, 0803.4180

Triple cut solution depends on one complex parameter, $t$

$$l_1^\mu(t) = \tilde{K}_1^\mu + \tilde{K}_3^\mu + \frac{t}{2}\langle \tilde{K}_1^- | \gamma^\mu | \tilde{K}_3^- \rangle + \frac{1}{2t}\langle \tilde{K}_3^- | \gamma^\mu | \tilde{K}_1^- \rangle$$

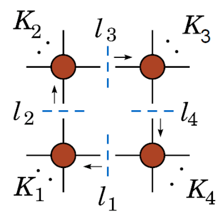
Solves $\quad l_1^2(t) = l_2^2(t) = l_3^2(t) = 0$

for suitable definitions of $\quad \tilde{K}_1^\mu \ \tilde{K}_3^\mu$



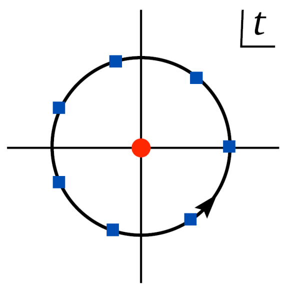

Box-subtracted triple cut has poles only at $t = 0, \infty$

$$T_3(t) \equiv C_3(t) - \sum_{\sigma=\pm} \sum_i \frac{d_i^\sigma}{\xi_i^\sigma (t - t_i^\sigma)}$$




$$T_3(t) = \sum_{j=-p}^{p} c_j t^j$$

Triangle coefficient $c_0$ plus all other coefficients $c_j$ obtained by discrete Fourier projection, sampling at $(2p+1)^{\text{th}}$ roots of unity




Bubble similar

# Automated On-Shell Programs at One Loop

**CutTools:** Ossola, Papadopolous, Pittau, 0711.3596

NLO *WWW, WWZ*, ... Binoth+OPP, 0804.0350

NLO $t\bar{t}b\bar{b}$, $t\bar{t}$ + 2 jets,… Bevilacqua, Czakon, Papadopoulos, Pittau, Worek, 0907.4723; 1002.4009; now going into MadGraph (Frederix,Frixione,...)

**Blackhat:** Berger, Bern, LD, Febres Cordero, Forde, H. Ita, D. Kosower, D. Maître; T. Gleisberg, 0803.4180, 0808.0941, 0907.1984, 1004.1659, 1009.2338

+ **Sherpa** → NLO *W,Z* + 3,4 jets

**Rocket:** Giele, Zanderighi, 0805.2152

Ellis, Giele, Kunszt, Melnikov, Zanderighi, 0810.2762

NLO *W* + 3 jets (large $N_c$), $W^+W^+$ + 2 jets

EMZ, 0901.4101, 0906.1445; Melia, Melnikov, Rontsch, Zanderighi, 1007.5313

**SAMURAI:** Mastrolia, Ossola, Reiter, Tramontano, 1006.0710

**NGluon:** Badger, Biedermann, Uwer, 1011.2900

# Bottom Line:

**Trees recycled into loops!**

Similar methods work for multiple loops
– especially in theories with lots of supersymmetry
like N=4 super-Yang-Mills and N=8 supergravity

# End of Lecture 3