Multiscale pentagon integrals to all orders

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based on work in collaboration with D. Canko and C. Papadopoulos

arXiv:2009.13917 [hep-ph] (JHEP)
arXiv:2010.06947 [hep-ph] (JHEP)
arXiv:2012.10635 [hep-ph]

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Table of Contents

Table of Contents

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Motivation

- Indications for the need of physics Beyond the Standard Model mostly from Cosmology (e.g. is dark matter a new particle/ particles?).
- Absence of a clear signal for BSM physics from the LHC $\rightarrow$ Collider Physics at the Precision Frontier[1].
- Measured cross sections are being compared to improved theoretical predictions looking for deviations from the SM $\rightarrow$ constraints on BSM physics.
- LHC Run 3 and HL-LHC Run will require at least NNLO corrections.
- $2 \rightarrow 3$ current precision frontier at NNLO.
- Numerical approaches not efficient (physical kinematics) $\rightarrow$ need for analytical results.

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[1] G. Heinrich, arXiv:2009.00516 [hep-ph]

Motivation

State-of-the-art for $2 \to 3$ NNLO calculations (focus on Master Integrals):

- All Master Integrals involving massless particles are known.
- All planar Master Integrals with up to one off-shell leg are known.
- Recent progress in non-planar two-loop Master Integrals with one off-shell leg (see talks from C. Papadopoulos and B. Page).

Motivation

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Looking ahead:

- At some point we will have to go beyond processes with one massive external particle and with internal masses.
- Starting point: one-loop five-point Master Integrals with $n \geq 1$ off-shell legs and $m \geq 1$ internal masses.
- This talk: Pentagons with up to three off-shell legs.

Table of Contents

Computing Master Integrals **analytically**

- IBP reduction identifies a minimal set of Master Integrals, called basis of MI **G**.
- Use DE to compute them.

$$\frac{\partial}{\partial s_{ij}}\mathbf{G} = \mathbf{A}(\{s_{ij}\},\epsilon)\mathbf{G} \tag{1}$$

- In general the matrix **A** can be very complicated.
- *Canonical revolution*[2]: instead of **G** use a special basis $\mathbf{g} = \mathbf{TG}$ for which

$$\mathrm{d}\mathbf{g} = \epsilon \sum_a \mathrm{d}\log\left(W_a\right)\tilde{\mathbf{M}}_a\mathbf{g} \tag{2}$$

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[2] J. M. Henn, Phys. Rev. Lett. **110** (2013), 251601

Computing Master Integrals **analytically**

- When W_a are *rational* functions of the differential variables, solved with recursive iterations in terms of Goncharov PolyLogarithms (GPLs)

$$\mathcal{G}(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{\mathrm{dt}}{t - a_1} \mathcal{G}(a_2, \ldots, a_n; t) \tag{3}$$

$$\mathcal{G}(0, \ldots, 0; x) = \frac{1}{n!} \log^n(x) \tag{4}$$

- For 5-point integrals going beyond weight 3 is a non-trivial task.
- Algebraic letters W_a prohibit a direct integration in terms of GPLs (roots arising from massive 3-point functions and Gramm determinant of external momenta).

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Computing Master Integrals **analytically**

- Simplified Differential Equations approach (SDE)[3]: introduce an external parameter x and differentiate wrt it.
- Combine with canonical (pure) basis $\mathbf{g}$:

$$\partial_x \mathbf{g} = \epsilon \sum_b \frac{1}{x - l_b} \mathbf{M}_b \, \mathbf{g} \tag{5}$$

- In many cases the new letters $W'_b = x - l_b$ are fully rationalised wrt x.
- Solution of canonical SDE trivially expressed in terms of GPLs.
- Boundary terms: compute $x \to 0$ limit with expansion-by-regions method.

[3] C. G. Papadopoulos, JHEP **07** (2014), 088

Table of Contents

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One-mass pentagon: Kinematics

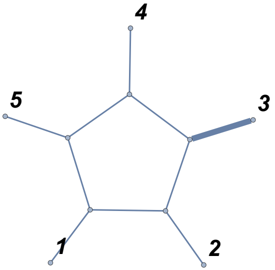

Figure: 13 MI.

- External momenta q_i, $i = 1 \dots 5$
- $\sum_1^5 q_i = 0$, $q_3^2 \equiv m^2$, $q_i^2 = 0$, $i = 1, 2, 4, 5$
- $\{q_3^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}$, with $s_{ij} := (q_i + q_j)^2$

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One-mass pentagon: SDE parametrization

$$q_1 = xp_1,\ q_2 = xp_2,\ q_3 = p_{123} - xp_{12},\ q_4 = -p_{123},\ q_5 = -p_{1234} \tag{6}$$

- Underline momenta $p_i,\ i = 1\ldots5$
- $\sum_1^5 p_i = 0,\ p_i^2 = 0,\ i = 1\ldots5$, with $p_{i\ldots j} := p_i + \ldots + p_j$
- $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}, x\}$, with $S_{ij} := (p_i + p_j)^2$
- Mapping of kinematic invariants:

$$m^2 = (x-1)\left(S_{12}x - S_{45}\right),\ s_{12} = S_{12}x^2,\ s_{23} = S_{23}x - S_{45}x + S_{45},$$
$$s_{34} = x\left(S_{12}(x-1) + S_{34}\right),\ s_{45} = S_{45},\ s_{51} = S_{51}x$$

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One-mass pentagon: Pure basis

Top-sector pure basis element:

$$\epsilon^2 \frac{\mathcal{P}_{11111}}{\Delta_5} G_{11111} \tag{8}$$

- $G_{a_1 a_2 a_3 a_4 a_5} = \int \frac{\mathrm{d}^d k_1}{i\pi^{(d/2)}} \frac{\mathrm{e}^{\epsilon\gamma_E}}{\mathcal{D}_1^{a_1}\mathcal{D}_2^{a_2}\mathcal{D}_3^{a_3}\mathcal{D}_4^{a_4}\mathcal{D}_5^{a_5}}$

$$\begin{aligned} &\mathcal{D}_1 = -(k_1)^2,\ \mathcal{D}_2 = -(k_1+q_1)^2,\ \mathcal{D}_3 = -(k_1+q_1+q_2)^2 \\ &\mathcal{D}_4 = -(k_1+q_1+q_2+q_3)^2,\ \mathcal{D}_5 = -(k_1+q_1+q_2+q_3+q_4)^2 \end{aligned} \tag{9}$$

- $\mathcal{P}_{11111}$ is the Baikov polynomial of G_{11111}.
- $\Delta_5 = \sqrt{\det[q_i \cdot q_j]}$

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One-mass pentagon: Canonical SDE & boundaries

$$\partial_x \mathbf{g} = \epsilon \left(\sum_{i=1}^{11} \frac{\mathsf{M}_i}{x - l_i} \right) \mathbf{g} \tag{10}$$

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One-mass pentagon: Canonical SDE & boundaries

$$\partial_x \mathbf{g} = \epsilon \left(\sum_{i=1}^{11} \frac{\mathbf{M}_i}{x - l_i} \right) \mathbf{g} \tag{10}$$

- $\mathbf{M}_i$ independent of the kinematics.
- All kinematic dependence in l_i.
- Resummation matrix for $l_1 = 0$, $(\mathbf{M}_1 = \mathbf{SDS}^{-1})$

$$\mathbf{R} = \mathbf{S} e^{\epsilon \mathbf{D} \log(x)} \mathbf{S}^{-1} \tag{11}$$

- $\mathbf{g} = \mathbf{TG}$
- Expansion-by-regions: $G_i \underset{x \to 0}{=} \sum_j x^{b_j + a_j \epsilon} G_i^{(b_j + a_j \epsilon)}$

$$\mathbf{Rb} = \lim_{x \to 0} \mathbf{TG}\Big|_{\mathcal{O}\left(x^{0+a_j\epsilon}\right)}, \quad \mathbf{b} = \sum_{i=0}^{n} \epsilon^i \mathbf{b}_0^{(i)}$$

One-mass pentagon: Solution up to weight four

$$
\begin{aligned}
\mathbf{g} = {} & \epsilon^0 \mathbf{b}_0^{(0)} + \epsilon\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)} \right) \\
& + \epsilon^2 \left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)} \right) \\
& + \epsilon^3 \left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(2)} + \mathbf{b}_0^{(3)} \right) \\
& + \epsilon^4 \left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(1)} \right. \\
& \left. + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(3)} + \mathbf{b}_0^{(4)} \right) \qquad (13)
\end{aligned}
$$

$\mathcal{G}_{ab\ldots} := \mathcal{G}(l_a, l_b, \ldots; x)$

Boundaries in closed form.

Solution trivially extended to higher weights!

Two-mass pentagon: Kinematics

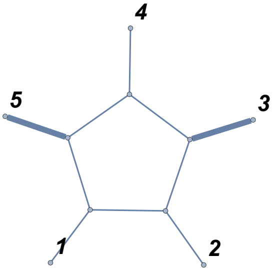

Figure: 15 MI.

- External momenta $q_i,\ i = 1\ldots 5$
- $\sum_1^5 q_i = 0,\ \{q_3^2 \equiv m_3^2, q_5^2 \equiv m_5^2\},\ q_i^2 = 0,\ i = 1,2,4$
- $\{m_3^2, m_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}$, with $s_{ij} := (q_i + q_j)^2$

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Two-mass pentagon: SDE parametrization

$$q_1 = xp_1,\ q_2 = xp_2,\ q_3 = p_{123} - xp_{12},\ q_4 = -p_{123},\ q_5 = -p_{1234} \quad (14)$$

- Underline momenta $p_i,\ i = 1 \ldots 5$
- $\sum_1^5 p_i = 0,\ p_i^2 = 0,\ i = 1, 2, 3, 4, p_5^2 = m_5^2$, with $p_{i\ldots j} := p_i + \ldots + p_j$
- $\{m_5^2, S_{12}, S_{23}, S_{34}, S_{45}, S_{51}, x\}$, with $S_{ij} := (p_i + p_j)^2$

Two-mass pentagon: Canonical SDE

- Top-sector pure basis element as for one-mass pentagon.
-
$$\partial_x \mathbf{g} = \epsilon \left(\sum_{i=1}^{14} \frac{\mathbf{M}_i}{x - l_i} \right) \mathbf{g} \tag{15}$$
- Alphabet rational in x.
- Boundaries in closed form.
- Solution in terms of GLPs effectively to all orders in ϵ.

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Three-mass pentagon: Kinematics

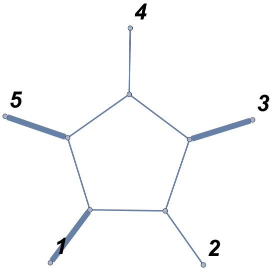

Figure: 18 MI.

- External momenta $q_i,\ i = 1 \ldots 5$
- $\sum_1^5 q_i = 0$, $\{q_1^2 \equiv m_1^2, q_3^2 \equiv m_3^2, q_5^2 \equiv m_5^2\}$, $q_i^2 = 0,\ i = 2, 4$
- $\{m_1^2, m_3^2, m_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}$, with $s_{ij} := (q_i + q_j)^2$

Three-mass pentagon: SDE parametrization

$$q_1 = xp_1,\ q_2 = xp_2,\ q_3 = p_{123} - xp_{12},\ q_4 = -p_{123},\ q_5 = -p_{1234} \quad (16)$$

- Underline momenta $p_i,\ i = 1 \dots 5$
- $\sum_1^5 p_i = 0,\ p_i^2 = 0,\ i = 2, 3, 4, p_1^2 = \hat{m}_1^2, p_5^2 = m_5^2$, with $p_{i\dots j} := p_i + \dots + p_j$
- $\{\hat{m}_1^2, m_5^2, S_{12}, S_{23}, S_{34}, S_{45}, S_{51}, x\}$, with $S_{ij} := (p_i + p_j)^2$

Three-mass pentagon: Canonical SDE

- Top-sector pure basis element as for one-mass pentagon.
- $$\partial_x \mathbf{g} = \epsilon \left(\sum_{i=1}^{19} \frac{\mathbf{M}_i}{x - l_i} \right) \mathbf{g} \tag{17}$$
- Alphabet rational in x.
- Boundaries in closed form.
- Solution in terms of GLPs effectively to all orders in ϵ.

Analysis of resulting functions

Family	W=1	W=2	W=3	W=4
P_{1m}	10 (2)	50 (21)	170 (99)	496 (339)
P_{2m}	9 (0)	54 (16)	204 (106)	628 (406)
P_{3m}	13 (0)	87 (24)	349 (172)	1115 (696)

Table: Number of GPLs entering in the solution. Top-sector b.e. in parenthesis.

Family	W=5	W=6
P_{1m}	1322 (959)	2983 (1924)
P_{2m}	1728 (1254)	4341 (2656)
P_{3m}	3145 (2228)	7849 (4656)

Table: Number of GPLs entering in the solution. Top-sector b.e. in parenthesis.

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Numerical results in Euclidean points

Top-Sec	Time (sec)	Result
g_{13}	1.90759	$0.0944261\epsilon^3 + 0.31615\epsilon^4 + 0.666923\epsilon^5 + 1.09948\epsilon^6$
g_{15}	3.75112	$-0.120811\epsilon^3 - 0.314547\epsilon^4 - 0.616424\epsilon^5 - 0.985647\epsilon^6$
g_{18}	9.27125	$-0.0215131\epsilon^3 - 0.0332408\epsilon^4 - 0.0501992\epsilon^5 - 0.057848\epsilon^6$

Table: Numerical computation of GPLs using GiNaC.

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Table of Contents

Conclusions

- Pure basis + SDE + Exp-by-reg. efficient computational framework for analytical computation of multiscale MI.
- Application to multiloop problems, see talks by C. Papadopoulos and D. Canko.
- Rational alphabet in $x \to$ need to study different parametrizations.
- Analytic continuation of GPLs for fast and stable numerical results in physical phase-space points.
- Next step: introduce internal masses.

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Thank you for your attention!

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Table of Contents

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Automated tools

- Ginac[4] for the numerical calculation of GPLs.
- PolyLogTools[5] for the algebraic manipulation of GPLs.
- FIRE6[6] and KIRA2[7] for the IBP reduction.
- FIESTA4[8] for Expansion-By-Regions.
- pySecDec[9] and FIESTA4 for numerical computation of FI, used for cross-checking our results.

[4] J. Vollinga and S. Weinzierl, Comput. Phys. Commun. **167** (2005), 177

[5] C. Duhr and F. Dulat, JHEP **08** (2019), 135

[6] A. V. Smirnov and F. S. Chuharev, arXiv:1901.07808 [hep-ph]

[7] J. Klappert, F. Lange, P. Maierhöfer and J. Usovitsch, arXiv:2008.06494 [hep-ph]

[8] A. V. Smirnov, Comput. Phys. Commun. **204** (2016), 189-199

[9] S. Borowka, G. Heinrich, S. Jahn, S. P. Jones, M. Kerner, J. Schlenk and T. Zirke, Comput. Phys. Commun. **222** (2018), 313-326

Scattering kinematics

- Results in Euclidean region (all initial kinematic invariants are negative).
- GPLs and solutions are real there.
- Analytic continuation to get results in physical regions for phenomenology.
- Fibration basis techniques (exploit symbol algebra and coproduct to analytically continue GPLs).
- Numerically: $\{S_{ij}, x\} \rightarrow \{S_{ij} + i\delta_{ij}\eta, x + i\delta\eta\}$ for $\eta \rightarrow 0$.
- Constraints on $\delta_{ij},\ \delta_x$ from one-scale integrals and second graph polynomial of top sector FI.

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Accumulated results

- Fully analytical results for all MI for $2 \to 2$ with up to one mass in all kinematic regions.
- Analytical results for all MI for $2 \to 2$ with two masses in Euclidean region (numerical analytic continuation for physical regions)[10][11][12].
- All planar 2-loop MI for $2 \to 3$ with up to one mass.
- Two planar families for $2 \to 2$ with up to one mass at three loops[13].
- Pentagon with up to one massive leg to all orders[14].

[10] C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **01** (2015), 072
[11] J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP **05** (2014), 090
[12] F. Caola, J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP **09** (2014), 043
[13] D. D. Canko and NS, [arXiv:2010.06947 [hep-ph]]
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