Multiscale pentagon integrals to all orders

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based on work in collaboration with D. Canko and C. Papadopoulos


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Florida State University 17-21 May, 2021 (Virtual)
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Motivation

- Indications for the need of physics Beyond the Standard Model mostly from Cosmology (e.g. is dark matter a new particle/particles?).
- Absence of a clear signal for BSM physics from the LHC → Collider Physics at the Precision Frontier\(^1\).
- Measured cross sections are being compared to improved theoretical predictions looking for deviations from the SM → constraints on BSM physics.
- LHC Run 3 and HL-LHC Run will require at least NNLO corrections.
- 2 → 3 current precision frontier at NNLO.
- Numerical approaches not efficient (physical kinematics) → need for analytical results.

Motivation

State-of-the-art for $2 \to 3$ NNLO calculations (focus on Master Integrals):

- All Master Integrals involving massless particles are known.
- All planar Master Integrals with up to one off-shell leg are known.
- Recent progress in non-planar two-loop Master Integrals with one off-shell leg (see talks from C. Papadopoulos and B. Page).

Looking ahead: At some point we will have to go beyond processes with one massive external particle and with internal masses. Starting point: one-loop five-point Master Integrals with $n \geq 1$ off-shell legs and $m \geq 1$ internal masses.

This talk: Pentagons with up to three off-shell legs.
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- At some point we will have to go beyond processes with one massive external particle and with internal masses.
- Starting point: one-loop five-point Master Integrals with $n \geq 1$ off-shell legs and $m \geq 1$ internal masses.

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Computing Master Integrals \textit{analytically}

- IBP reduction identifies a minimal set of Master Integrals, called basis of MI $G$.
- Use DE to compute them.

\begin{equation}
\frac{\partial}{\partial s_{ij}} G = A(\{s_{ij}\}, \epsilon) G
\end{equation}

In general the matrix $A$ can be very complicated.

- \textit{Canonical revolution}²: instead of $G$ use a special basis $g = TG$ for which

\begin{equation}
dg = \epsilon \sum_a d \log (W_a) \tilde{M}_a g
\end{equation}

²J. M. Henn, Phys. Rev. Lett. 110 (2013), 251601
Computing Master Integrals analytically

- When $W_a$ are rational functions of the differential variables, solved with recursive iterations in terms of Goncharov PolyLogarithms (GPLs)

\[
G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t) \quad (3)
\]

\[
G(0, \ldots, 0; x) = \frac{1}{n!} \log^n(x) \quad (4)
\]

- For 5-point integrals going beyond weight 3 is a non-trivial task.

- Algebraic letters $W_a$ prohibit a direct integration in terms of GPLs (roots arising from massive 3-point functions and Gramm determinant of external momenta).
Computing Master Integrals \textit{analytically}

- Simplified Differential Equations approach (SDE)\textsuperscript{3}: introduce an external parameter $x$ and differentiate wrt it.
- Combine with canonical (pure) basis $g$:

$$\partial_x g = \epsilon \sum_b \frac{1}{x - l_b} M_b g$$

(5)

- In many cases the new letters $W'_b = x - l_b$ are fully rationalised wrt $x$.
- Solution of canonical SDE trivially expressed in terms of GPLs.
- Boundary terms: compute $x \rightarrow 0$ limit with expansion-by-regions method.

\textsuperscript{3}C. G. Papadopoulos, JHEP 07 (2014), 088
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One-mass pentagon: Kinematics

- External momenta $q_i$, $i = 1 \ldots 5$

  - $\sum_1^5 q_i = 0$, $q_3^2 \equiv m^2$, $q_i^2 = 0$, $i = 1, 2, 4, 5$

  - $\{ q_3^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15} \}$, with $s_{ij} := (q_i + q_j)^2$
One-mass pentagon: SDE parametrization

\[ q_1 = \lambda p_1, \quad q_2 = \lambda p_2, \quad q_3 = p_{123} - \lambda p_{12}, \quad q_4 = -p_{123}, \quad q_5 = -p_{1234} \quad (6) \]

- Underline momenta \( p_i, \ i = 1 \ldots 5 \)
- \( \sum_{1}^{5} p_i = 0, \ p_i^2 = 0, \ i = 1 \ldots 5, \) with \( p_{i \ldots j} := p_i + \ldots + p_j \)
- \( \{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}, \lambda \}, \) with \( S_{ij} := (p_i + p_j)^2 \)
- Mapping of kinematic invariants:
  \[ m^2 = (\lambda - 1) (S_{12} \lambda - S_{45}), \quad s_{12} = S_{12} \lambda^2, \quad s_{23} = S_{23} \lambda - S_{45} \lambda + S_{45}, \]
  \[ s_{34} = \lambda (S_{12}(\lambda - 1) + S_{34}), \quad s_{45} = S_{45}, \quad s_{51} = S_{51} \lambda \]
One-mass pentagon: Pure basis

Top-sector pure basis element:

\[ \epsilon^2 \frac{\mathcal{P}_{11111}^2}{\Delta_5} G_{11111} \]  

\[ G_{a_1 a_2 a_3 a_4 a_5} = \int \frac{d^d k_1}{i \pi^{(d/2)}} \frac{e^{i \epsilon \gamma E}}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4} D_5^{a_5}} \]

\[ D_1 = -(k_1)^2, \quad D_2 = -(k_1 + q_1)^2, \quad D_3 = -(k_1 + q_1 + q_2)^2 \]

\[ D_4 = -(k_1 + q_1 + q_2 + q_3)^2, \quad D_5 = -(k_1 + q_1 + q_2 + q_3 + q_4)^2 \]  

\[ \mathcal{P}_{11111} \text{ is the Baikov polynomial of } G_{11111}. \]

\[ \Delta_5 = \sqrt{\det[q_i \cdot q_j]} \]
One-mass pentagon: Canonical SDE & boundaries

\[ \partial_x g = \epsilon \left( \sum_{i=1}^{11} \frac{M_i}{x - l_i} \right) g \]  

(10)
One-mass pentagon: Canonical SDE & boundaries

\[ \partial_x g = \epsilon \left( \sum_{i=1}^{11} \frac{M_i}{x - l_i} \right) g \]  \hspace{1cm} (10)

- \( M_i \) independent of the kinematics.
- All kinematic dependence in \( l_i \).
- Resummation matrix for \( l_1 = 0, (M_1 = SDS^{-1}) \)

\[ R = S e^{\epsilon D \log(x)} S^{-1} \]  \hspace{1cm} (11)

- \( g = TG \)
- Expansion-by-regions: \( G_i = \sum_j x^{b_j + a_j \epsilon} G_i^{(b_j + a_j \epsilon)} \)

\[ Rb = \lim_{x \to 0} TG \bigg|_{\mathcal{O}(x^{0+a_j \epsilon})}, \quad b = \sum_{i=0}^{n} \epsilon^i b_0^{(i)} \]
One-mass pentagon: Solution up to weight four

\[
g = \epsilon^0 b_0^{(0)} + \epsilon \left( \sum G_a M_a b_0^{(0)} + b_0^{(1)} \right) \\
+ \epsilon^2 \left( \sum G_{ab} M_a M_b b_0^{(0)} + \sum G_a M_a b_0^{(1)} + b_0^{(2)} \right) \\
+ \epsilon^3 \left( \sum G_{abc} M_a M_b M_c b_0^{(0)} + \sum G_{ab} M_a M_b b_0^{(1)} + \sum G_a M_a b_0^{(2)} + b_0^{(3)} \right) \\
+ \epsilon^4 \left( \sum G_{abcd} M_a M_b M_c M_d b_0^{(0)} + \sum G_{abc} M_a M_b M_c b_0^{(1)} \\
+ \sum G_{ab} M_a M_b b_0^{(2)} + \sum G_a M_a b_0^{(3)} + b_0^{(4)} \right)
\]  

(13)

\(G_{ab...} := G(l_a, l_b, \ldots ; x)\)

Boundaries in closed form.

Solution trivially extended to higher weights!
Two-mass pentagon: Kinematics

- External momenta $q_i$, $i = 1 \ldots 5$
- $\sum_{i=1}^{5} q_i = 0$, $\{q_3^2 \equiv m_3^2, q_5^2 \equiv m_5^2\}$, $q_i^2 = 0$, $i = 1, 2, 4$
- $\{m_3^2, m_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}$, with $s_{ij} := (q_i + q_j)^2$
Two-mass pentagon: SDE parametrization

\[ q_1 = x p_1, \quad q_2 = x p_2, \quad q_3 = p_{123} - x p_{12}, \quad q_4 = -p_{123}, \quad q_5 = -p_{1234} \quad (14) \]

- Underline momenta \( p_i, \quad i = 1 \ldots 5 \)
- \( \sum_{i=1}^{5} p_i = 0, \quad p_i^2 = 0, \quad i = 1, 2, 3, 4, \quad p_5^2 = m_5^2, \) with \( p_{i\ldots j} := p_i + \ldots + p_j \)
- \( \{ m_5^2, S_{12}, S_{23}, S_{34}, S_{45}, S_{51}, x \} \), with \( S_{ij} := (p_i + p_j)^2 \)
Two-mass pentagon: Canonical SDE

- Top-sector pure basis element as for one-mass pentagon.

\[ \partial_x g = \epsilon \left( \sum_{i=1}^{14} \frac{M_i}{x - l_i} \right) g \]  \hspace{1cm} (15)

- Alphabet rational in \( x \).
- Boundaries in closed form.
- Solution in terms of GLPs effectively to all orders in \( \epsilon \).
Three-mass pentagon: Kinematics

- External momenta $q_i$, $i = 1 \ldots 5$
- $\sum_{i=1}^{5} q_i = 0$, $\{q_1^2 \equiv m_1^2, q_3^2 \equiv m_3^2, q_5^2 \equiv m_5^2\}$, $q_i^2 = 0$, $i = 2, 4$
- $\{m_1^2, m_3^2, m_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}$, with $s_{ij} := (q_i + q_j)^2$
Three-mass pentagon: SDE parametrization

\[ q_1 = x p_1, \quad q_2 = x p_2, \quad q_3 = p_{123} - x p_{12}, \quad q_4 = -p_{123}, \quad q_5 = -p_{1234} \quad (16) \]

- Underline momenta \( p_i, \ i = 1 \ldots 5 \)
- \( \sum_{1}^{5} p_i = 0, \quad p_i^2 = 0, \quad i = 2, 3, 4, \quad p_1^2 = \hat{m}_1^2, \quad p_5^2 = m_5^2, \) with \( p_{i \ldots j} := p_i + \ldots + p_j \)
- \( \{ \hat{m}_1^2, \ m_5^2, \ S_{12}, \ S_{23}, \ S_{34}, \ S_{45}, \ S_{51}, \ x \} \), with \( S_{ij} := (p_i + p_j)^2 \)
Three-mass pentagon: Canonical SDE

- Top-sector pure basis element as for one-mass pentagon.

\[ \partial_x g = \epsilon \left( \sum_{i=1}^{19} \frac{M_i}{x - l_i} \right) g \]  \hspace{1cm} (17)

- Alphabet rational in \( x \).
- Boundaries in closed form.
- Solution in terms of GLPs effectively to all orders in \( \epsilon \).
## Analysis of resulting functions

<table>
<thead>
<tr>
<th>Family</th>
<th>$W=1$</th>
<th>$W=2$</th>
<th>$W=3$</th>
<th>$W=4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{1m}$</td>
<td>10 (2)</td>
<td>50 (21)</td>
<td>170 (99)</td>
<td>496 (339)</td>
</tr>
<tr>
<td>$P_{2m}$</td>
<td>9 (0)</td>
<td>54 (16)</td>
<td>204 (106)</td>
<td>628 (406)</td>
</tr>
<tr>
<td>$P_{3m}$</td>
<td>13 (0)</td>
<td>87 (24)</td>
<td>349 (172)</td>
<td>1115 (696)</td>
</tr>
</tbody>
</table>

**Table:** Number of GPLs entering in the solution. Top-sector b.e. in parenthesis.

<table>
<thead>
<tr>
<th>Family</th>
<th>$W=5$</th>
<th>$W=6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{1m}$</td>
<td>1322 (959)</td>
<td>2983 (1924)</td>
</tr>
<tr>
<td>$P_{2m}$</td>
<td>1728 (1254)</td>
<td>4341 (2656)</td>
</tr>
<tr>
<td>$P_{3m}$</td>
<td>3145 (2228)</td>
<td>7849 (4656)</td>
</tr>
</tbody>
</table>

**Table:** Number of GPLs entering in the solution. Top-sector b.e. in parenthesis.
Numerical results in Euclidean points

<table>
<thead>
<tr>
<th>Top-Sec</th>
<th>Time (sec)</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{13}$</td>
<td>1.90759</td>
<td>$0.0944261 \epsilon^3 + 0.31615 \epsilon^4 + 0.666923 \epsilon^5 + 1.09948 \epsilon^6$</td>
</tr>
<tr>
<td>$g_{15}$</td>
<td>3.75112</td>
<td>$-0.120811 \epsilon^3 - 0.314547 \epsilon^4 - 0.616424 \epsilon^5 - 0.985647 \epsilon^6$</td>
</tr>
<tr>
<td>$g_{18}$</td>
<td>9.27125</td>
<td>$-0.0215131 \epsilon^3 - 0.0332408 \epsilon^4 - 0.0501992 \epsilon^5 - 0.057848 \epsilon^6$</td>
</tr>
</tbody>
</table>

**Table:** Numerical computation of GPLs using GiNaC.
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Conclusions

- Pure basis + SDE + Exp-by-reg. efficient computational framework for analytical computation of multiscale MI.
- Application to multiloop problems, see talks by C. Papadopoulos and D. Canko.
- Rational alphabet in $x \rightarrow$ need to study different parametrizations.
- Analytic continuation of GPLs for fast and stable numerical results in physical phase-space points.
- Next step: introduce internal masses.
Acknowledgement

This research is co-financed by Greece and the European Union (European Social Fund- ESF) through the Operational Program Human Resources Development, Education and Lifelong Learning 2014 - 2020 in the context of the project ”Higher order corrections in QCD with applications to High Energy experiments at LHC” -MIS 5047812.

Thank you for your attention!
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Automated tools

- Ginac\textsuperscript{4} for the numerical calculation of GPLs.
- PolyLogTools\textsuperscript{5} for the algebraic manipulation of GPLs.
- FIRE\textsuperscript{6} and KIRA\textsuperscript{7} for the IBP reduction.
- FIESTA\textsuperscript{4} for Expansion-By-Regions.
- pySecDec\textsuperscript{9} and FIESTA\textsuperscript{4} for numerical computation of FI, used for cross-checking our results.

\textsuperscript{4} J. Vollinga and S. Weinzierl, Comput. Phys. Commun. 167 (2005), 177
\textsuperscript{5} C. Duhr and F. Dulat, JHEP 08 (2019), 135
\textsuperscript{8} A. V. Smirnov, Comput. Phys. Commun. 204 (2016), 189-199
Scattering kinematics

- Results in Euclidean region (all initial kinematic invariants are negative).
- GPLs and solutions are real there.
- Analytic continuation to get results in physical regions for phenomenology.
- Fibration basis techniques (exploit symbol algebra and coproduct to analytically continue GPLs).
- Numerically: \( \{ S_{ij}, \chi \} \rightarrow \{ S_{ij} + i\delta_{ij}\eta, \chi + i\delta\eta \} \) for \( \eta \rightarrow 0 \).
- Constraints on \( \delta_{ij}, \delta_{\chi} \) from one-scale integrals and second graph polynomial of top sector FI.
Accumulated results

- Fully analytical results for all MI for $2 \rightarrow 2$ with up to one mass in all kinematic regions.
- Analytical results for all MI for $2 \rightarrow 2$ with two masses in Euclidean region (numerical analytic continuation for physical regions)\(^{10}\)\(^{11}\)\(^{12}\).
- All planar 2-loop MI for $2 \rightarrow 3$ with up to one mass.
- Two planar families for $2 \rightarrow 2$ with up to one mass at three loops\(^{13}\).
- Pentagon with up to one massive leg to all orders\(^{14}\).

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\(^{10}\) C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP 01 (2015), 072

\(^{11}\) J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP 05 (2014), 090

\(^{12}\) F. Caola, J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP 09 (2014), 042

\(^{13}\) D. D. Canko and NS, [arXiv:2010.06947 [hep-ph]]

\(^{14}\) NS, [arXiv:2012.10635 [hep-ph]]