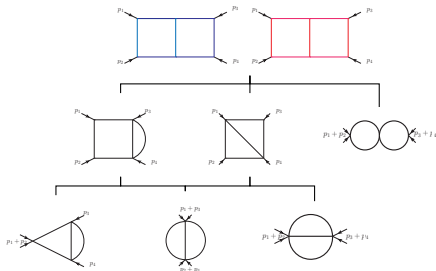


The diagrammatic coaction and cuts of the double box

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Why do we care about this work

The Diagrammatic Coaction

A conjectural statement on feynman integrals, interpreted diagrammatically through pairs of contracted and cut diagrams.



The coaction reveals the analytic structure of Feynman integrals

Encodes all the information about the basis of master integrands/contours.

Governs the space solutions of the differential equations.

Can be used to compute cuts of Feynman diagrams.

Encodes all the information about the space of solutions of differential equations. Can be used to compute Feynman diagrams.

The coaction's story so far

Feynman Integrals

A Feynman Integral

$$I(D) = \frac{e^{L\gamma_E \epsilon}}{(i\pi^{D/2})^L} \int_{-\infty}^{\infty} \prod_{l=1}^L d^D k_l \frac{1}{\prod_{i=1}^n D_i^{\alpha_i}}$$

Integration contour:

$$\langle \gamma | = \int_{\gamma}$$

- Prescribes the loop momentum integration path.
- Generates homology group of the integral.

Differential form:

$$|\omega\rangle = \frac{e^{L\gamma_E \epsilon}}{(i\pi^{D/2})^L} \prod_{l=1}^L d^D k_l \frac{1}{\prod_{i=1}^n D_i^{\alpha_i}}$$

- Provides the set of propagators & numerators to be integrated.
- Generates co-homology group of the integral.

Integral defined by pairing

$$I(D) = \langle \gamma | \omega \rangle$$

The co-homology group; IBP identities

The form of the propagators

$$D_i = \left(A_i^{m,n} k_m \cdot k_n + B_i^{m,n} k_m \cdot p_n + C_i^{m,n} p_m \cdot p_n + \sum_n (m_i^n)^2 \right)$$

Total Derivatives Vanish under the integral

$$\int \prod_{l=1}^L d^D k_l \frac{\partial}{\partial k_i^\mu} \left(\frac{\{k_j, p_j\}^\mu}{\prod_{i=1}^n D_i^{\alpha_i}} \right) = 0$$

Integration by parts (IBP) identities

Integrals with different integer powers of propagators and numerator insertions are related:

- 1 Loop: Natural basis with no non-unit powers or numerator insertions.
- 2 Loop and beyond: Basis requires non-unit propagator powers propagators/numerators.

The co-homology group; An example

The case of zero-mass 1-loop box

The integral with propagators raised to square powers is a linear combination of 1-loop integrals of unit power propagators:

The diagrammatic equation shows the decomposition of a box integral with four internal propagator dots into three terms:

$$\text{Box with 4 dots} = C_1 \text{Box} + C_2 \text{Bubble} + C_3 \text{Bubble}$$

The first term is a square box with external momenta p_1 (top-left), p_3 (top-right), p_2 (bottom-left), and p_4 (bottom-right). The second term is a bubble with external momenta $p_1 + p_2$ and $p_3 + p_4$. The third term is a bubble with external momenta $p_1 + p_3$ and $p_2 + p_4$.

The integrands of the RHS are the co-homology space basis

In terms of contour/ differential form notation:

$$\begin{aligned} \langle \gamma | \omega \rangle &= C_1 \langle \gamma | \omega_1 \rangle + C_2 \langle \gamma | \omega_2 \rangle + C_3 \langle \gamma | \omega_3 \rangle \Rightarrow \\ | \omega \rangle &= C_1 | \omega_1 \rangle + C_2 | \omega_2 \rangle + C_3 | \omega_3 \rangle \end{aligned}$$

where $| \omega_1 \rangle$, $| \omega_2 \rangle$ and $| \omega_3 \rangle$ are the basis vectors.

The homology group; The cut diagrams

The homology group is spanned by the independent cut diagrams

Generally, for an on-shell propagator $(k + q_i)^2$, we have:

$$\frac{1}{(k + q_i)^2} \rightarrow 2\pi i \delta((k + q_i)^2) = 2\pi i \text{Res}_{(k+q_i)^2=0} \left[\frac{1}{(k + q_i)^2} \dots \right]$$

Modify the integration contours to encircle the poles of the integral

$$\int_{\gamma} d^D k = \int_{-\infty}^{+\infty} d^D k \rightarrow \int_{\gamma_i} d^D k$$

where $\gamma_i \in (-\infty, +\infty)$ but now encircles the poles of the on-shell propagators.

The bases of homology/co-homology group can be made dual

$$\langle \gamma_i | \omega_j \rangle = \delta_{ij} + \mathcal{O}(\epsilon)$$

The homology group; The contour choice

Not all cut results are correct for a dual basis choice:

$$\langle \gamma_1 | \omega_1 \rangle = \text{Diagram} = 1 + \epsilon \log(\dots) + \mathcal{O}(\epsilon^2)$$

$$\langle \tilde{\gamma}_2 | \omega_1 \rangle = \text{Diagram} = C_{\text{unitarity}} \left(\text{Diagram} \right) = Disc_S \left(\text{Diagram} \right)$$

$$= -\frac{1}{\epsilon} + \epsilon \log(\dots) + \mathcal{O}(\epsilon^2)$$

Define:

$$\langle \gamma_2 | = \left(\langle \tilde{\gamma}_2 | + \frac{1}{\epsilon} \langle \gamma_1 | \right) \Rightarrow$$

$$\langle \gamma_2 | \omega_1 \rangle = \langle \tilde{\gamma}_2 | \omega_1 \rangle + \frac{1}{\epsilon} \langle \gamma_1 | \omega_1 \rangle = \text{Diagram} + \frac{1}{\epsilon} \text{Diagram} = \text{Diagram} = \epsilon \log(\dots) + \mathcal{O}(\epsilon^2)$$

The period matrix for the box

$$\langle \gamma_1 | \omega_1 \rangle = \begin{array}{c} p_1 \quad p_2 \\ \diagdown \quad \diagup \\ | \quad | \\ \diagup \quad \diagdown \\ p_3 \quad p_4 \end{array} = 1 + \mathcal{O}(\epsilon)$$

$$\langle \gamma_1 | \omega_2 \rangle = 0$$

$$\langle \gamma_1 | \omega_3 \rangle = 0$$

$$\langle \gamma_2 | \omega_1 \rangle = \begin{array}{c} p_1 \quad p_2 \\ \diagdown \quad \diagup \\ | \quad | \\ \diagup \quad \diagdown \\ p_3 \quad p_4 \end{array} = \mathcal{O}(\epsilon)$$

$$\langle \gamma_2 | \omega_2 \rangle = \begin{array}{c} p_1 + p_2 \quad p_3 + p_4 \\ \diagdown \quad \diagup \\ | \quad | \\ \diagup \quad \diagdown \end{array} = 1 + \mathcal{O}(\epsilon)$$

$$\langle \gamma_2 | \omega_3 \rangle = 0$$

$$\langle \gamma_3 | \omega_1 \rangle = \begin{array}{c} p_1 \quad p_2 \\ \diagdown \quad \diagup \\ | \quad | \\ \diagup \quad \diagdown \\ p_3 \quad p_4 \end{array} = \mathcal{O}(\epsilon)$$

$$\langle \gamma_3 | \omega_2 \rangle = 0$$

$$\langle \gamma_3 | \omega_3 \rangle = \begin{array}{c} p_1 + p_2 \\ \diagdown \quad \diagup \\ | \quad | \\ \diagup \quad \diagdown \\ p_3 + p_4 \end{array} = 1 + \mathcal{O}(\epsilon)$$

The Diagrammatic Coaction

The master formula

$$\Delta \langle \gamma | \omega \rangle = \sum_i \langle \gamma | \omega_i \rangle \otimes \langle \gamma_i | \omega \rangle \quad (\text{Abreu et al., 2017})$$

For the case of the zero-mass 1-loop box

$$\Delta \langle \gamma | \omega_1 \rangle = \langle \gamma | \omega_1 \rangle \otimes \langle \gamma_1 | \omega_1 \rangle + \langle \gamma | \omega_2 \rangle \otimes \langle \gamma_2 | \omega_1 \rangle + \langle \gamma | \omega_3 \rangle \otimes \langle \gamma_3 | \omega_1 \rangle$$

Diagrammatically:

The diagrammatic equation shows the coaction Δ applied to a 1-loop box diagram. The left side is Δ applied to a square box with external momenta p_1, p_2 at the top and d_3, d_4 at the bottom. The right side is the sum of three terms, each representing a cut of the box diagram:

- 1. The original box diagram \otimes a box diagram with vertical lines cut (representing γ_1).
- 2. A bubble diagram with external momenta $p_1 + p_2$ and $p_3 + p_4$ \otimes a box diagram with horizontal lines cut (representing γ_2).
- 3. A bubble diagram with external momenta $p_1 + p_3$ and $p_2 + p_4$ \otimes a box diagram with diagonal lines cut (representing γ_3).

Establish the Two-Loop Diagrammatic Coaction

Check the extent that the one-loop conjecture holds.



Focus On the Case of the On-Shell Double Box

Prefer results in a closed form in ϵ to establish the coaction. Direct integration proved unsatisfactory.



Turn to Differential Equations to compute cuts

Use the homology theory of cuts and the cohomology theory of integrands to find solutions.



Apply the coaction

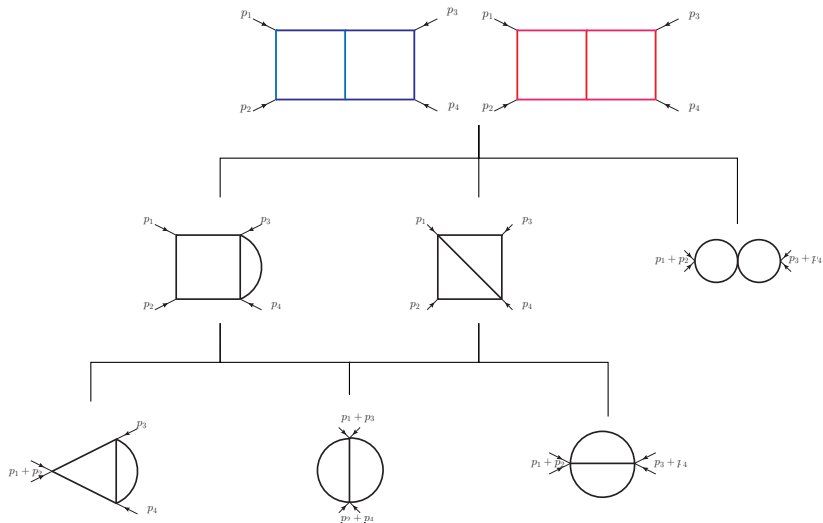
See how the space of solutions of the differential equations are governed by the coaction.

The two-loop frontier; The zero-mass double box

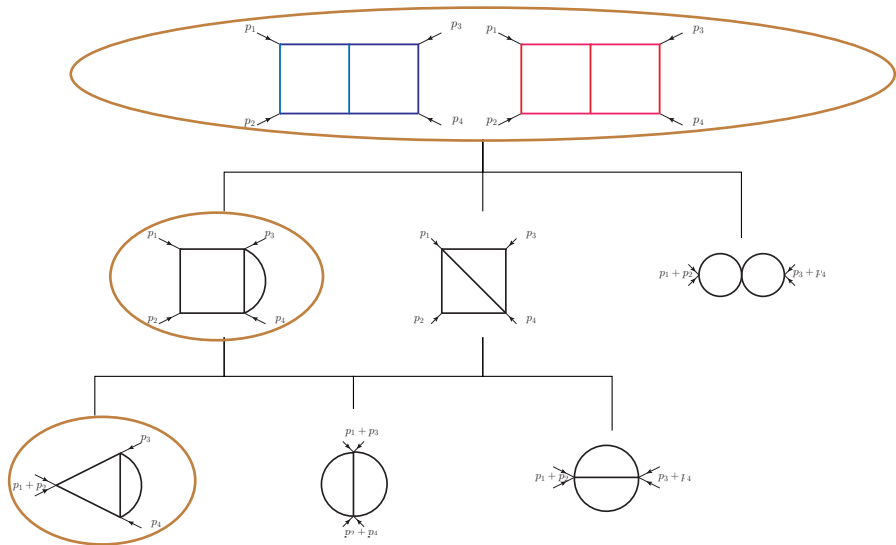
$$DB_1(s, t) = \text{Diagram 1} \quad DB_2(s, t) = \text{Diagram 2}$$

$$\langle \gamma | \omega_{a,b} \rangle = \frac{e^{2\gamma_E \epsilon}}{\left(i\pi^{\frac{D}{2}}\right)^2} \int_{-\infty}^{+\infty} \frac{d^D k d^D l \left((k-p_1)^2\right)^a \left((l-p_1)^2\right)^b}{k^2 l^2 (l+k)^2 (k+p_1)^2 (l+p_3)^2 (k+p_1+p_2)^2 (l-(p_1+p_2))^2}$$

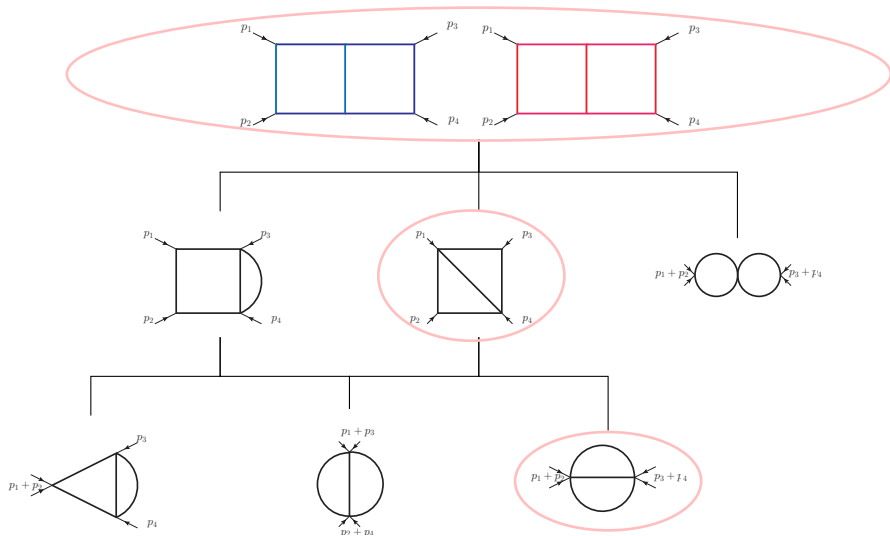
The hierarchy of the differential forms



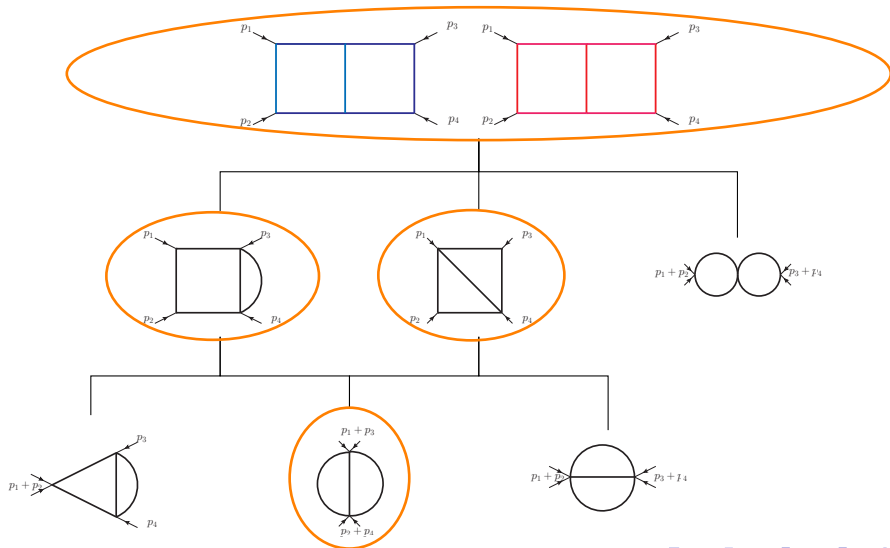
The hierarchy of the differential forms; 1st Family



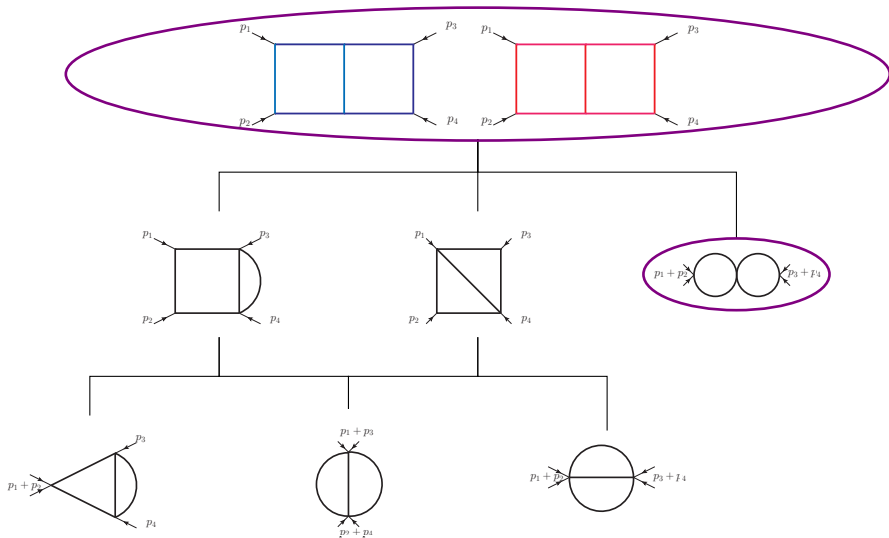
The hierarchy of the differential forms; 2nd Family



The hierarchy of the differential forms, The two families meet



The hierarchy of the differential forms; The 4th family



The diagrammatic coaction of the double box; a conjecture

$$\Delta \left(\text{Diagram 1} \right) = \text{Diagram 2} \otimes \text{Diagram 3} + \text{Diagram 4} \otimes \text{Diagram 5} + \text{Diagram 6} \otimes \text{Diagram 7} + \text{Diagram 8} \otimes \text{Diagram 9} + \text{Diagram 10} \otimes \text{Diagram 11} + \text{Diagram 12} \otimes \text{Diagram 13} + \text{Diagram 14} \otimes \text{Diagram 15} + \text{Diagram 16} \otimes \text{Diagram 17}$$

The diagrammatic coaction of the double box is given by the following sum of terms:

- Diagram 1:** A double box with external momenta p_1, p_2, p_3, p_4 .
- Diagram 2:** A double box with external momenta p_1, p_2, p_3, p_4 .
- Diagram 3:** A double box with external momenta p_1, p_2, p_3, p_4 and internal lines marked with ticks.
- Diagram 4:** A double box with external momenta p_1, p_2, p_3, p_4 and internal lines marked with ticks.
- Diagram 5:** A double box with external momenta p_1, p_2, p_3, p_4 and internal lines marked with ticks.
- Diagram 6:** A square with a diagonal line and external momenta p_1, p_2, p_3, p_4 .
- Diagram 7:** A double box with external momenta p_1, p_2, p_3, p_4 and internal lines marked with ticks.
- Diagram 8:** Two circles with external momenta $p_1 + p_2$ and $p_3 + p_4$.
- Diagram 9:** A double box with external momenta p_1, p_2, p_3, p_4 and internal lines marked with ticks.
- Diagram 10:** A triangle with a semi-circular arc and external momenta p_1, p_2, p_3, p_4 .
- Diagram 11:** A double box with external momenta p_1, p_2, p_3, p_4 and internal lines marked with ticks.
- Diagram 12:** A circle with a horizontal line and external momenta $p_1 + p_2$ and $p_3 + p_4$.
- Diagram 13:** A double box with external momenta p_1, p_2, p_3, p_4 and internal lines marked with ticks.
- Diagram 14:** A circle with a vertical line and external momenta $p_1 + p_3$ and $p_2 + p_4$.
- Diagram 15:** A double box with external momenta p_1, p_2, p_3, p_4 and internal lines marked with ticks.

The Diagrammatic Coaction of the double box maximal cut

Restricting the differential equation to the maximal cut subspace

$$\text{For } j \geq 3: \langle \gamma_1 | \omega_j \rangle = \langle \gamma_2 | \omega_j \rangle = 0$$

Diagrams with less propagators than cut vanish in the differential equation.

Restrict the coaction to the maximal cut subspace

$$\Delta \langle \gamma_1 | \omega_1 \rangle = \langle \gamma_1 | \omega_1 \rangle \otimes \langle \gamma_1 | \omega_1 \rangle + \langle \gamma_1 | \omega_2 \rangle \otimes \langle \gamma_2 | \omega_1 \rangle$$

Diagrammatically:

$$\Delta \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right] = \begin{array}{c} \text{Diagram 1} \otimes \text{Diagram 1} \\ + \\ \text{Diagram 2} \otimes \text{Diagram 1} \end{array}$$

Finding the homology basis; The first contour

Start with a generic differential form

$$|\omega_{a,b}\rangle = \frac{e^{2\gamma E \epsilon}}{\left(i\pi \frac{D}{2}\right)^2} \frac{d^D k \wedge d^D l \left((k-p_1)^2\right)^a \left((l-p_1)^2\right)^b}{k^2 l^2 (l+k)^2 (k+p_1)^2 (l+p_3)^2 (k+p_1+p_2)^2 (l-(p_1+p_2))^2}$$

Compute a maximal cut

$$\begin{aligned} \langle \gamma_1 | \omega_{a,b} \rangle &= \frac{-2e^{2\gamma E \epsilon} \Gamma(a-\epsilon) \Gamma(-\epsilon)}{\Gamma(-2\epsilon) \Gamma(a-2\epsilon)} t^{-3-2\epsilon+a+b} x^{-2-2\epsilon+b} (1-x)^\epsilon {}_2F_1(1+2\epsilon, b-\epsilon; 1+b-a; x) \\ &= t^{-3-2\epsilon+a+b} f_{a,b}(x) {}_2F_1(b-\epsilon, 1+2\epsilon; 1+b-a; x), \text{ with } x = -\frac{s}{t} \text{ (Bosma et al., 2017)} \end{aligned}$$

The hypergeometric function ${}_2F_1$

$${}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\gamma - \beta)} \int_0^1 du u^{\beta-1} (1-u)^{\gamma-\beta-1} (1-ux)^{-\alpha}$$

The second contour and the choice of integrand

Restrict Integration Between Critical Points

$${}_2F_1(b - \epsilon, 1 + 2\epsilon; 1 + b - a; x) \equiv \frac{\Gamma(1 + b - a)}{\Gamma(1 + 2\epsilon)\Gamma(b - a - 2\epsilon)} \int_0^1 du u^{b-\epsilon-1}(1-u)^{-b+\epsilon}(1-ux)^{-1-2\epsilon}$$
$$\int_0^1 du u^{b-\epsilon-1}(1-u)^{-b+\epsilon}(1-ux)^{-1-2\epsilon} \rightarrow \int_0^{\frac{1}{x}} du u^{b-\epsilon-1}(1-u)^{-b+\epsilon}(1-ux)^{-1-2\epsilon} \text{ (Abreau et al., 2019)}$$

Obtain the second maximal cut

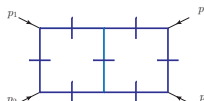
$$\langle \gamma_2 | \omega_{a,b} \rangle = t^{-3-2\epsilon+a+b} \tilde{f}_{a,b}(x) {}_2F_1\left(b - \epsilon, a - \epsilon; a + b - 3\epsilon; \frac{1}{x}\right), \text{ with } x = -\frac{s}{t}$$

Maximal Cuts of the Double Box

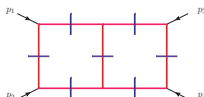
$$\langle \gamma_1 | \omega_{a=0,b=0} \rangle = \langle \gamma_1 | \omega_1 \rangle = 1 + \mathcal{O}(\epsilon), \quad \langle \gamma_1 | \omega_{a=0,b=1} \rangle = \langle \gamma_1 | \omega_2 \rangle = \mathcal{O}(\epsilon)$$
$$\langle \gamma_2 | \omega_{a=0,b=0} \rangle = \langle \gamma_2 | \omega_1 \rangle = \mathcal{O}(\epsilon), \quad \langle \gamma_2 | \omega_{a=0,b=1} \rangle = \langle \gamma_2 | \omega_2 \rangle = 1 + \mathcal{O}(\epsilon)$$

(Smirnov, 1999), (Anastasiou et al., 2000)

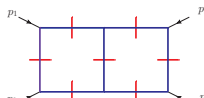
The maximal cut subspace

$$\langle \gamma_1 | \omega_1 \rangle =$$



$$= t^{-3-2\epsilon} f_{0,0}(x) {}_2F_1(-\epsilon, 1+2\epsilon; 1; x)$$

$$\langle \gamma_1 | \omega_2 \rangle =$$


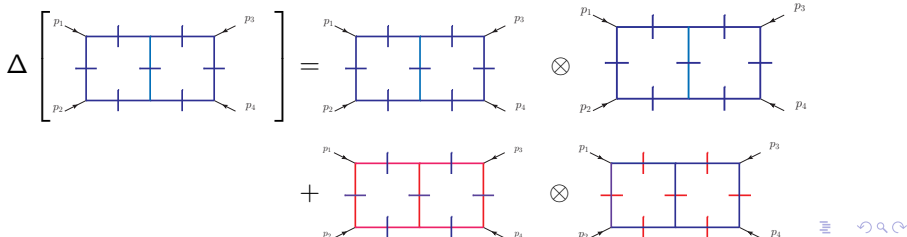
$$= t^{-2-2\epsilon} f_{0,1}(x) {}_2F_1(1-\epsilon, 1+2\epsilon; 2; x)$$

$$\langle \gamma_2 | \omega_1 \rangle =$$


$$t^{-3-2\epsilon} \tilde{f}_{0,0}(x) {}_2F_1(-\epsilon, a-\epsilon; a+b-3\epsilon; \frac{1}{x})$$

$$\langle \gamma_2 | \omega_2 \rangle =$$


$$= t^{-2-2\epsilon} \tilde{f}_{0,1}(x) {}_2F_1(1-\epsilon, -\epsilon; 1-3\epsilon; \frac{1}{x})$$

$$\Delta \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right] = \text{Diagram 1} \otimes \text{Diagram 3} + \text{Diagram 2} \otimes \text{Diagram 4}$$


The Double Box differential equation

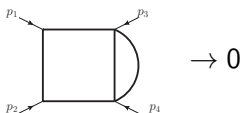
Apply a differential operator on $\langle \gamma | \omega_1 \rangle$:

$$\begin{aligned}
 \frac{d}{dx} & \left[\text{Diagram 1} \right] = C_1 \left[\text{Diagram 2} \right] + C_2 \left[\text{Diagram 3} \right] \\
 & + C_3 \left[\text{Diagram 4} \right] + C_4 \left[\text{Diagram 5} \right] + C_5 \left[\text{Diagram 6} \right] + C_6 \left[\text{Diagram 7} \right] \\
 & + C_7 \left[\text{Diagram 8} \right] + C_8 \left[\text{Diagram 9} \right]
 \end{aligned}$$

where $C_n = C_n(x, D)$, $n \in [1, 8]$.

The maximal cut of the Double Box differential equation

For diagrams with less than seven propagators (Eg: Double Edged Box)



The maximal cut differential equations:

$$\begin{aligned} \frac{d}{dx} & \left[\text{Diagram 1: Double Edged Box with blue lines and vertical cuts} \right] = C_1 \left[\text{Diagram 2: Double Edged Box with blue lines and vertical cuts} \right] + C_2 \left[\text{Diagram 3: Double Edged Box with red lines and vertical cuts} \right] \\ \frac{d}{dx} & \left[\text{Diagram 4: Double Edged Box with red lines and vertical cuts} \right] = \tilde{C}_1 \left[\text{Diagram 5: Double Edged Box with blue lines and vertical cuts} \right] + \tilde{C}_2 \left[\text{Diagram 6: Double Edged Box with red lines and vertical cuts} \right] \end{aligned}$$

The homogeneous maximal cut differential equation

$$\begin{aligned}
 \frac{d}{dx} \text{[Diagram 1]} &= C_1(x, D) \text{[Diagram 2]} + C_2(x, D) \text{[Diagram 3]} \\
 \frac{d}{dx} \text{[Diagram 4]} &= \tilde{C}_1(x, D) \text{[Diagram 5]} + \tilde{C}_2(x, D) \text{[Diagram 6]}
 \end{aligned}$$

The diagrams are maximal cut graphs with four external legs labeled p_1, p_2, p_3, p_4 . Diagram 1 is a blue square with two vertical internal lines. Diagram 2 is a blue square with two vertical internal lines. Diagram 3 is a red square with two vertical internal lines. Diagram 4 is a red square with two vertical internal lines. Diagram 5 is a blue square with two vertical internal lines. Diagram 6 is a red square with two vertical internal lines.

Using the fact that $\langle \gamma_1 | \omega_2 \rangle$ obeys a similar differential equation we obtain a second order homogeneous differential equation:

$$\frac{d^2}{dx^2} \text{[Diagram 1]} + A(x, D) \frac{d}{dx} \text{[Diagram 2]} + B(x, D) \text{[Diagram 3]} = 0$$

The diagrams are the same as in the previous block.

The differential equation defines the maximal cut homology group subspace

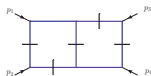
The function $\langle \gamma_1 | \omega_1 \rangle$ is a solution to this equation. So is $\langle \gamma_2 | \omega_1 \rangle$, all cuts in the subspace obey the same differential equation.

A non-maximal cut differential equation

Non-maximal cut differential equations

Each equation features cut subtologies as inhomogeneous terms of the maximal cut homogeneous differential equation

An unknown cut diagram:



A differential equation with an inhomogeneous term:

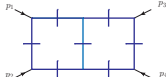
$$\frac{d^2}{dx^2} \text{[Diagram]} + A(x, D) \frac{d}{dx} \text{[Diagram]} + B(x, D) \text{[Diagram]} = C(x, D) \text{[Diagram]}$$

The maximal cut is part of the solution space

Finding the particular solution amounts to finding the new element of the space.

How to solve the differential equations

Recall the form of the homogeneous solution:


$$= \langle \gamma_{cut} | \omega_1 \rangle = t^{-3-2\epsilon} f_{0,0}(x) {}_2F_1(-\epsilon, 1+2\epsilon; 1; x)$$

Divide all cut diagrams by the maximal cut scale:

$$g_{cut}(x) = \frac{\langle \gamma_{cut} | \omega_1 \rangle}{t^{-3-2\epsilon} f_{0,0}(x)}$$

The choice is motivated by:

$$g_1(x) = {}_2F_1(-\epsilon, 1+2\epsilon; 1; x)$$

Obtain a third order differential equation for the normalised form:

$$\frac{d^3 g_{cut}(x)}{dx^3} + C_1' \frac{d^2 g_{cut}(x)}{dx^2} + C_2' \frac{d g_{cut}(x)}{dx} + C_3' g_{cut}(x) = 0$$

The differential equation form

The differential equation becomes the ${}_3F_2$ hypergeometric function defining equation

$$(1-x)x^2 \frac{d^3 g_{cut}(x)}{dx^3} + 2x(1-2x) \frac{d^2 g_{cut}(x)}{dx^2} + (x(3\epsilon^2 + 2\epsilon - 2\epsilon) + 4\epsilon(1+\epsilon)) \frac{dg_{cut}(x)}{dx} - \epsilon^2(1+2\epsilon)g_{cut}(x) = 0$$

Read off the three independent solutions of the differential equation and restore the scale

- $\langle \gamma_1 | \omega_1 \rangle = S_1 = c_1(\epsilon) t^{-3-2\epsilon} f_{0,0}(x) {}_2F_1(-\epsilon, 1+2\epsilon; 1; x)$
- $\langle \gamma_2 | \omega_1 \rangle = S_2 = c_2(\epsilon) t^{-3-2\epsilon} f_{0,0}(x) (1-x)^{-\epsilon} {}_2F_1(1+2\epsilon, 1+2\epsilon; 2+3\epsilon; \frac{1}{x})$
- $\langle \gamma_4 | \omega_1 \rangle = S_3 = c_3(\epsilon) t^{-3-2\epsilon} f_{0,0}(x) (1-x)^{1+2\epsilon} {}_3F_2(1, 1, 2+3\epsilon; 2+\epsilon, 2+2\epsilon; 1-x)$

We can always guarantee an orthogonal basis

$\langle \gamma_4 | \omega_2 \rangle = \frac{3}{2} + \mathcal{O}(\epsilon)$, Modify the contour : $\langle \tilde{\gamma}_4 | = \langle \gamma_4 | - \frac{3}{2} \langle \gamma_2 | \Rightarrow \langle \tilde{\gamma}_4 | \omega_2 \rangle = \mathcal{O}(\epsilon)$

Orthogonality not affected for the rest of the comohomology:

$$\langle \tilde{\gamma}_4 | \omega_1 \rangle = \mathcal{O}(\epsilon), \langle \tilde{\gamma}_4 | \omega_4 \rangle = \langle \gamma_4 | \omega_4 \rangle = 1 + \mathcal{O}(\epsilon)$$

Determining the Coefficients

Use the already known solutions:

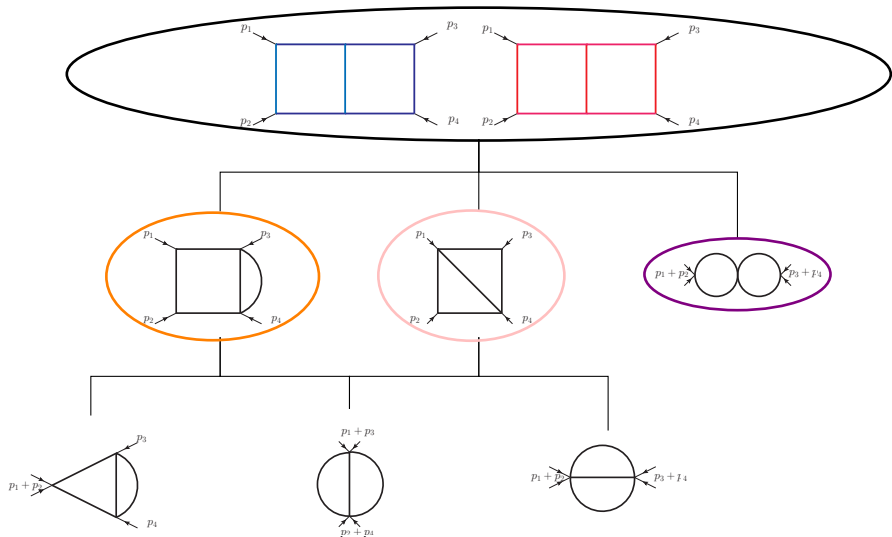
$$\begin{aligned}
 & \text{Diagram 1} = c_1(\epsilon) t^{-3-2\epsilon} f_{0,0}(x) {}_2F_1(-\epsilon, 1+2\epsilon; 1; x) + c_2(\epsilon) t^{-3-2\epsilon} f_{0,0}(x) (1-x)^{-\epsilon} {}_2F_1\left(1+2\epsilon, 1+2\epsilon; 2+3\epsilon; \frac{1}{x}\right) \\
 & + c_3(\epsilon) t^{-3-2\epsilon} f_{0,0}(x) (1-x)^{1+2\epsilon} {}_3F_2(1, 1, 2+3\epsilon; 2+\epsilon, 2+2\epsilon; 1-x)
 \end{aligned}$$

$$\frac{d^3}{dx^3} \text{Diagram 1} + C_1' \frac{d^2}{dx^2} \text{Diagram 2}$$

$$\frac{d}{dx} C_2' \text{Diagram 3} + C_3' \text{Diagram 4} = 0 \Rightarrow$$

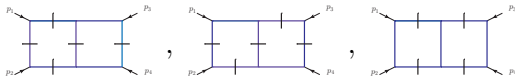
$$C_3 = \frac{12\epsilon(1+3\epsilon)}{(1+\epsilon)(1+2\epsilon)} \frac{x^{2+2\epsilon}(1-x)^{1-2\epsilon}}{t^{-3-2\epsilon}} \text{Diagram 5} = \frac{12\epsilon(1+3\epsilon)}{(1+\epsilon)(1+2\epsilon)} \frac{\Gamma(1+3\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)^3}$$

Focus on the second graph of each family



The Next level of Cut Differential Equations

Three unknown cut diagrams:



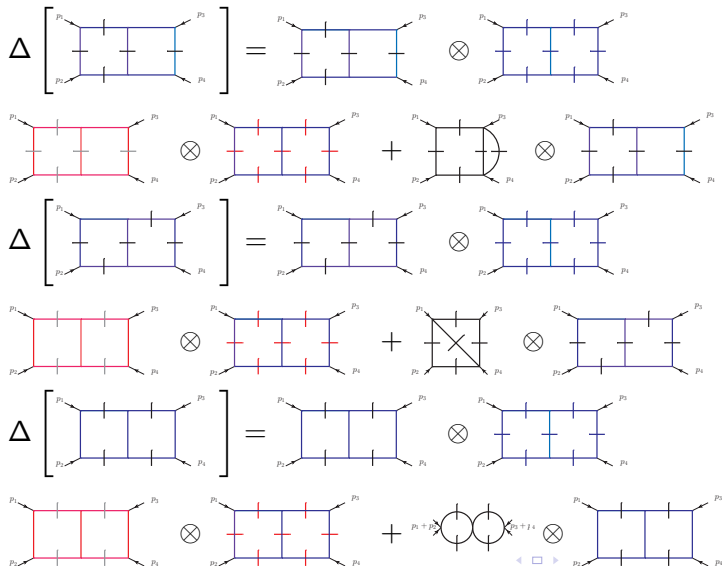
Three differential equations with different inhomogeneous terms:

$$\frac{d^2}{dx^2} \text{[Diagram 1]} + A(x, D) \frac{d}{dx} \text{[Diagram 2]} + B(x, D) \text{[Diagram 3]} = C_3(x, D) \text{[Diagram 4]}$$

$$\frac{d^2}{dx^2} \text{[Diagram 1]} + A(x, D) \frac{d}{dx} \text{[Diagram 2]} + B(x, D) \text{[Diagram 3]} = C_4(x, D) \text{[Diagram 5]}$$

$$\frac{d^2}{dx^2} \text{[Diagram 1]} + A(x, D) \frac{d}{dx} \text{[Diagram 2]} + B(x, D) \text{[Diagram 3]} = C_5(x, D) \text{[Diagram 6]}$$

The cut diagram coactions



The homology theory of Hypergeometric Functions

Integrating between different branch points provides inequivalent contours.



The differential equations

The maximal cut space are defined by the homogenous differential equation. Non maximal cuts appear as inhomogeneous terms.



The differential equations solutions

The non-maximal cut differential equation always features the maximal cut solution. Using this solves the higher order differential equation.



The coaction governs the contour/integrand relations

The coaction reveals the duality between the two bases and knowing the form that the coaction should take can reveal the results integrals.