The diagrammatic coaction and cuts of the double box

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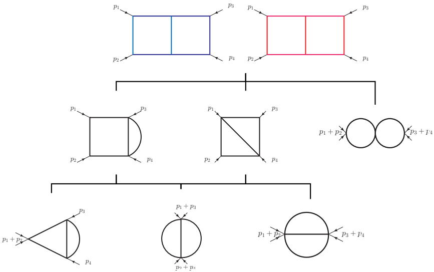


Why do we care about this work

The Diagrammatic Coaction

A conjectural statement on feynman integrals, interpreted diagrammatically through pairs of contracted and cut diagrams.

⇓

The coaction reveals the analytic structure of Feynman integrals

Encodes all the information about the basis of master integrands/contours.

Governs the space solutions of the differential equations.

Can be used to compute cuts of Feynman diagrams.

Encodes all the information about the [illegible] *of master* integrands|contours. Governs *the space of* solutions of differential *equations.* Can be used to compute *Feynman* diagrams.

The coaction's story so far

Feynman Integrals

A Feynman Integral

$$I(D) = \frac{e^{L\gamma_E \epsilon}}{(\mathrm{i}\pi^{D/2})^L} \int_{-\infty}^{\infty} \prod_{l=1}^{L} \mathrm{d}^D k_l \frac{1}{\prod_{i=1}^{n} D_i^{\alpha_i}}$$

Integration contour:
$\langle\gamma| = \int_\gamma$

- Prescribes the loop momentum integration path.
- Generates homology group of the integral.

Differential form:
$|\omega\rangle = \frac{e^{L\gamma_E \epsilon}}{(\mathrm{i}\pi^{D/2})^L} \prod_{l=1}^{L} \mathrm{d}^D k_l \frac{1}{\prod_{i=1}^{n} D_i^{\alpha_i}}$

- Provides the set of propagators & numerators to be integrated.
- Generates co-homology group of the integral.

Integral defined by pairing

$$I(D) = \langle\gamma|\omega\rangle$$

The co-homology group; IBP identities

The form of the propagators

$$D_i = \left(A_i^{m,n}\, k_m \cdot k_n + B_i^{m,n}\, k_m \cdot p_n + C_i^{m,n}\, p_m \cdot p_n + \sum_n (m_i^n)^2 \right)$$

Total Derivatives Vanish under the integral

$$\int \prod_{l=1}^{L} \mathrm{d}^D k_l \frac{\partial}{\partial k_i{}^{\mu}} \left(\frac{\{k_j, p_j\}^{\mu}}{\prod_{i=1}^{n} D_i^{\alpha_i}} \right) = 0$$

Integration by parts (IBP) identities

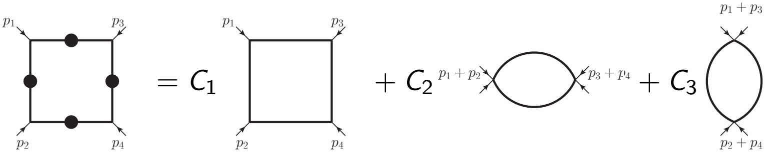
Integrals with different integer powers of propagators and numerator insertions are related:

- 1 Loop: Natural basis with no non-unit powers or numerator insertions.
- 2 Loop and beyond: Basis requires non-unit propagator powers propagators/numerators.

The co-homology group; An example

The case of zero-mass 1-loop box

The integral with propagators raised to square powers is a linear combination of 1-loop integrals of unit power propagators:



The integrands of the RHS are the co-homology space basis

In terms of contour/ differential form notation:

$$\langle \gamma | \omega \rangle = C_1 \langle \gamma | \omega_1 \rangle + C_2 \langle \gamma | \omega_2 \rangle + C_3 \langle \gamma | \omega_3 \rangle \Rightarrow$$
$$| \omega \rangle = C_1 | \omega_1 \rangle + C_2 | \omega_2 \rangle + C_3 | \omega_3 \rangle$$

where $|\omega_1\rangle, |\omega_2\rangle$ and $|\omega_3\rangle$ are the basis vectors.

The homology group; The cut diagrams

The homology group is spanned by the independent cut diagrams

Generally, for an on-shell propagator $(k+q_i)^2$, we have:

$$\frac{1}{(k+q_i)^2} \to 2\pi i\delta\left((k+q_i)^2\right) = 2\pi i Res_{(k+q_i)^2=0}\left[\frac{1}{(k+q_i)^2}\ ...\right]$$

Modify the integration contours to encircle the poles of the integral

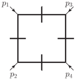
$$\int_\gamma d^D k = \int_{-\infty}^{+\infty} d^D k \to \int_{\gamma_i} d^D k$$

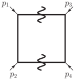
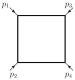
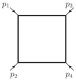
where $\gamma_i \in (-\infty, +\infty)$ but now encircles the poles of the on-shell propagators.

The bases of homology/co-homology group can be made dual

$$\langle \gamma_i | \omega_j \rangle = \delta_{ij} + \mathcal{O}(\epsilon)$$

Not all cut results are correct for a dual basis choice:

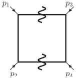
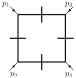
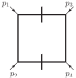
$\langle \gamma_1 | \omega_1 \rangle =$  $= 1 + \epsilon \log(\ldots) + \mathcal{O}(\epsilon^2)$

$\langle \tilde{\gamma_2} | \omega_1 \rangle =$  $= \mathrm{C}_{\mathrm{unitarity}} \Bigg($  $\Bigg) = Disc_s \Bigg($  $\Bigg)$

$$= -\frac{1}{\epsilon} + \epsilon \log(\ldots) + \mathcal{O}(\epsilon^2)$$

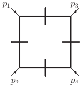
Define:

$$\langle \gamma_2 | = \left(\langle \tilde{\gamma_2} | + \frac{1}{\epsilon} \langle \gamma_1 | \right) \Rightarrow$$

$\langle \gamma_2 | \omega_1 \rangle = \langle \tilde{\gamma_2} | \omega_1 \rangle + \frac{1}{\epsilon} \langle \gamma_1 | \omega_1 \rangle =$  $+ \frac{1}{\epsilon}$  $=$  $=$

$\epsilon \log(\ldots) + \mathcal{O}(\epsilon^2)$

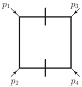
The period matrix for the box

$\langle \gamma_1 | \omega_1 \rangle =$ 

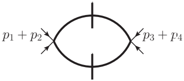
$= 1 + \mathcal{O}(\epsilon)$

$\langle \gamma_1 | \omega_2 \rangle = 0$

$\langle \gamma_1 | \omega_3 \rangle = 0$

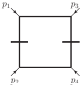
$\langle \gamma_2 | \omega_1 \rangle =$ 

$= \mathcal{O}(\epsilon)$

$\langle \gamma_2 | \omega_2 \rangle =$ 

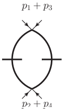
$= 1 + \mathcal{O}(\epsilon)$

$\langle \gamma_2 | \omega_3 \rangle = 0$

$\langle \gamma_3 | \omega_1 \rangle =$ 

$= \mathcal{O}(\epsilon)$

$\langle \gamma_3 | \omega_2 \rangle = 0$

$\langle \gamma_3 | \omega_3 \rangle =$ 

$= 1 + \mathcal{O}(\epsilon)$

The Diagrammatic Coaction

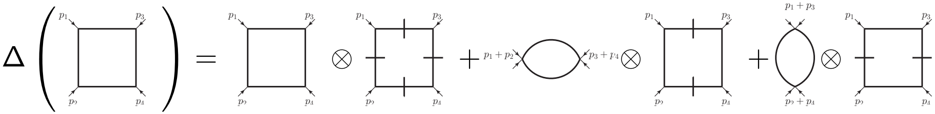
The master formula

$$\Delta \langle \gamma | \omega \rangle = \sum_i \langle \gamma | \omega_i \rangle \otimes \langle \gamma_i | \omega \rangle \text{ (Abreu et al., 2017)}$$

For the case of the zero-mass 1-loop box

$$\Delta \langle \gamma | \omega_1 \rangle = \langle \gamma | \omega_1 \rangle \otimes \langle \gamma_1 | \omega_1 \rangle + \langle \gamma | \omega_2 \rangle \otimes \langle \gamma_2 | \omega_1 \rangle + \langle \gamma | \omega_3 \rangle \otimes \langle \gamma_3 | \omega_1 \rangle$$

Diagrammatically:



Establish the Two-Loop Diagrammatic Coaction

Check the extent that the one-loop conjecture holds.

$\Downarrow$

Focus On the Case of the On-Shell Double Box

Prefer results in a closed form in ϵ to establish the coaction. Direct integration proved unsatisfactory.

$\Downarrow$

Turn to Differential Equations to compute cuts

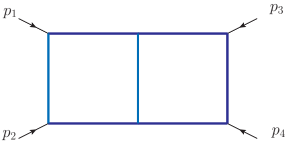
Use the homology theory of cuts and the cohomology theory of integrands to find solutions.

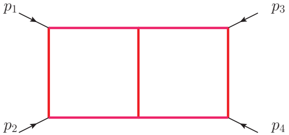
$\Downarrow$

Apply the coaction

See how the space of solutions of the differential equations are governed by the coaction.

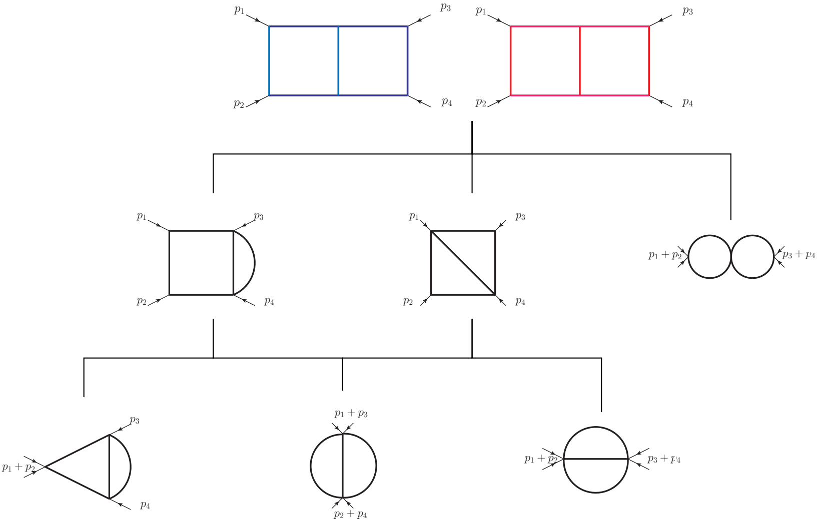
The two-loop frontier; The zero-mass double box

$DB_1(s,t) =$ 

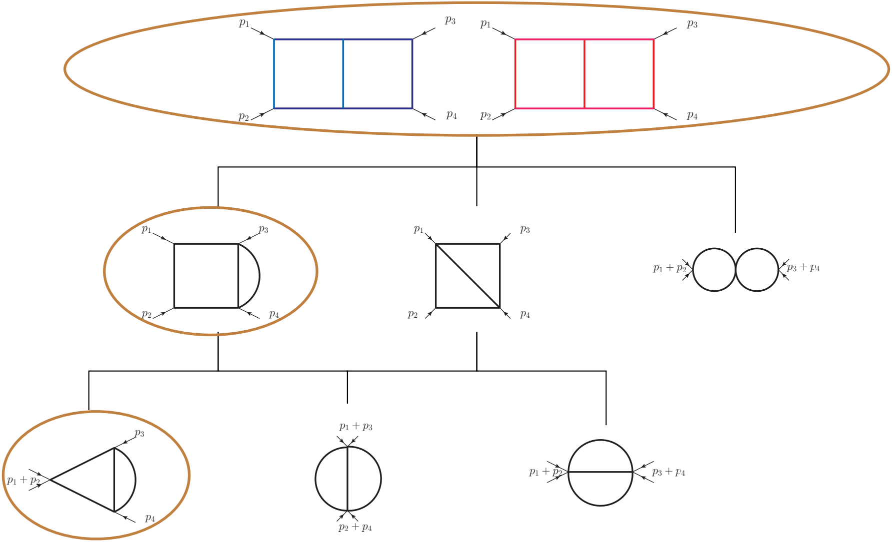
$DB_2(s,t) =$ 

$$\langle\gamma|\omega_{a,b}\rangle = \frac{e^{2\gamma_E\,\epsilon}}{\left(i\pi^{\frac{D}{2}}\right)^2}\int_{-\infty}^{+\infty}\frac{d^Dk\,d^Dl\,\left((k-p_1)^2\right)^a\left((l-p_1)^2\right)^b}{k^2l^2(l+k)^2(k+p_1)^2(l+p_3)^2(k+p_1+p_2)^2(l-(p_1+p_2))^2}$$

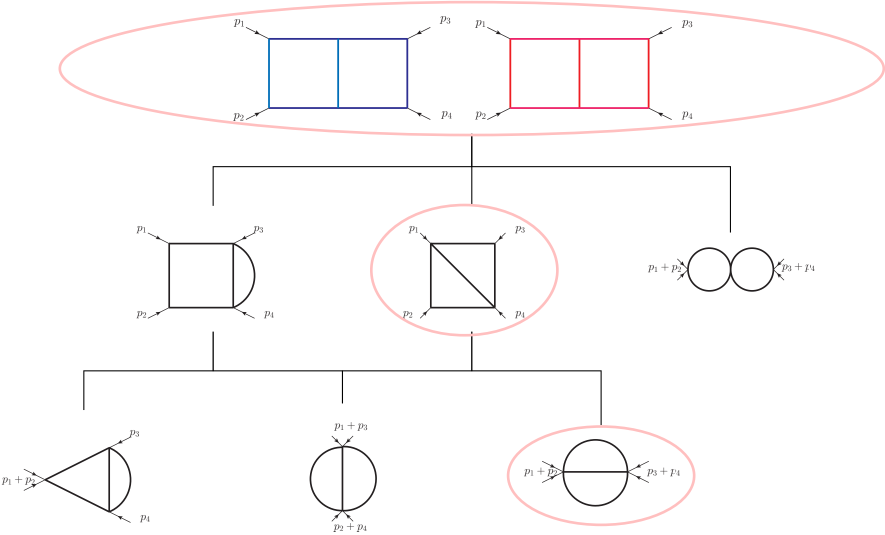
The hierarchy of the differential forms



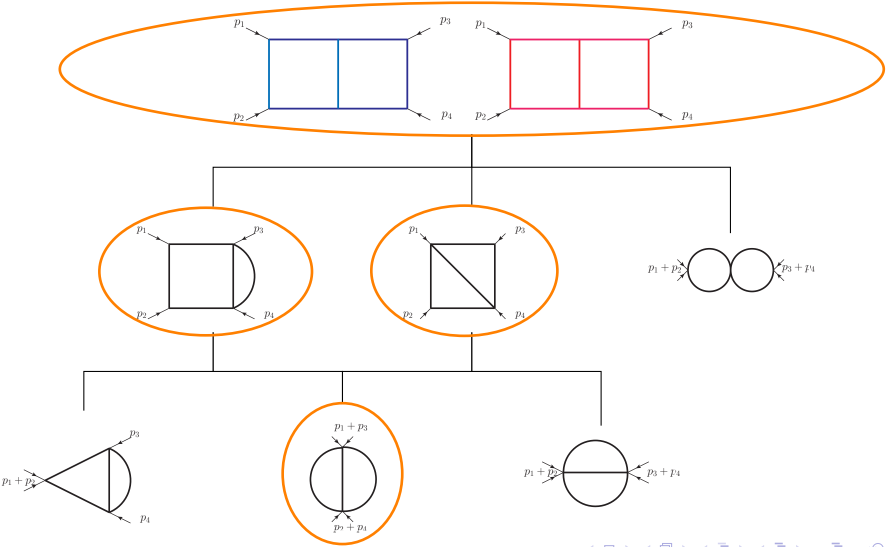
The hierarchy of the differential forms; 1st Family



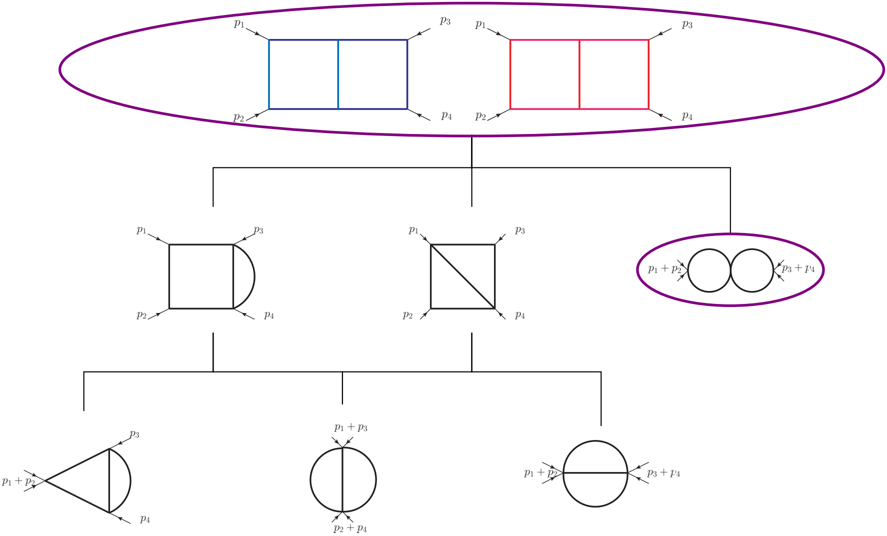
The hierarchy of the differential forms; 2nd Family



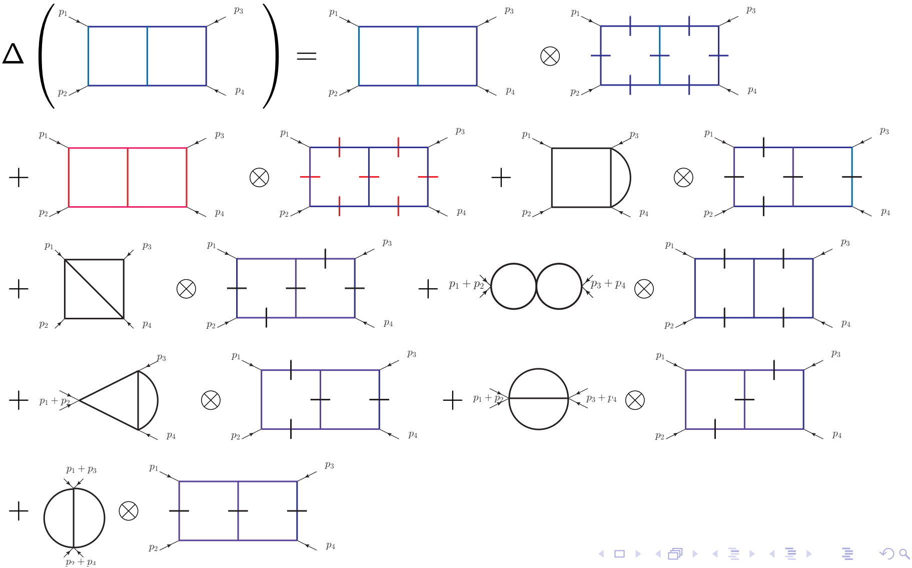
The hierarchy of the differential forms, The two families meet



The hierarchy of the differential forms; The 4th family



The diagrammatic coaction of the double box; a conjecture



The Diagrammatic Coaction of the double box maximal cut

Restricting the differential equation to the maximal cut subspace

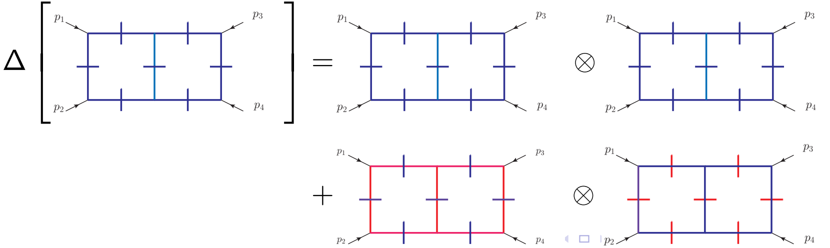
For $j \geq 3$: $\langle \gamma_1 | \omega_j \rangle = \langle \gamma_2 | \omega_j \rangle = 0$

Diagrams with less propagators than cut vanish in the differential equation.

Restrict the coaction to the maximal cut subspace

$$\Delta \langle \gamma_1 | \omega_1 \rangle = \langle \gamma_1 | \omega_1 \rangle \otimes \langle \gamma_1 | \omega_1 \rangle + \langle \gamma_1 | \omega_2 \rangle \otimes \langle \gamma_2 | \omega_1 \rangle$$

Diagrammatically:



Finding the homology basis; The first contour

Start with a generic differential form

$$|\omega_{a,b}\rangle = \frac{e^{2\gamma_E\,\epsilon}}{\left(i\pi^{\frac{D}{2}}\right)^2}\frac{d^Dk\wedge d^Dl\,\left((k-p_1)^2\right)^a\left((l-p_1)^2\right)^b}{k^2l^2(l+k)^2(k+p_1)^2(l+p_3)^2(k+p_1+p_2)^2(l-(p_1+p_2))^2}$$

Compute a maximal cut

$$\langle\gamma_1|\omega_{a,b}\rangle = \frac{-2e^{2\gamma_E\,\epsilon}\Gamma(a-\epsilon)\Gamma(-\epsilon)}{\Gamma(-2\,\epsilon)\Gamma(a-2\,\epsilon)}t^{-3-2\,\epsilon+a+b}x^{-2-2\,\epsilon+b}(1-x)^{\epsilon}\,{}_2\mathrm{F}_1\left(1+2\,\epsilon,b-\epsilon;1+b-a;x\right)$$

$$= t^{-3-2\,\epsilon+a+b}f_{a,b}\left(x\right){}_2\mathrm{F}_1\left(b-\epsilon,1+2\,\epsilon;1+b-a;x\right),\ \text{with}\ x=-\frac{s}{t}$$ (Bosma et al., 2017)

The hypergeometric function ${}_2\mathrm{F}_1$

$${}_2\mathrm{F}_1\left(\alpha,\beta;\gamma;x\right) = \frac{\Gamma\left(\gamma\right)}{\Gamma\left(\beta\right)\Gamma\left(\gamma-\beta\right)}\int_0^1 du\,u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-u\,x)^{-\alpha}$$

The second contour and the choice of integrand

Restrict Integration Between Critical Points

$${}_2\mathrm{F}_1\left(b-\epsilon, 1+2\,\epsilon; 1+b-a; x\right) \equiv \frac{\Gamma\left(1+b-a\right)}{\Gamma\left(1+2\,\epsilon\right)\Gamma\left(b-a-2\,\epsilon\right)} \int_0^{1} du\, u^{b-\epsilon-1}(1-u)^{-b+\epsilon}\left(1-u\,x\right)^{-1-2\,\epsilon}$$

$$\int_0^{1} du u^{b-\epsilon-1}(1-u)^{-b+\epsilon}\left(1-u\,x\right)^{-1-2\,\epsilon} \to \int_0^{\frac{1}{x}} du u^{b-\epsilon-1}(1-u)^{-b+\epsilon}\left(1-u\,x\right)^{-1-2\,\epsilon}$$ (Abreau et al., 2019)

Obtain the second maximal cut

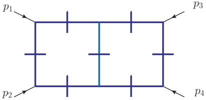
$$\langle\gamma_2|\omega_{a,b}\rangle = t^{-3-2\,\epsilon+a+b}\tilde{f}_{a,b}\left(x\right){}_2\mathrm{F}_1\left(b-\epsilon, a-\epsilon; a+b-3\,\epsilon; \frac{1}{x}\right), \text{ with } x = -\frac{s}{t}$$

Maximal Cuts of the Double Box

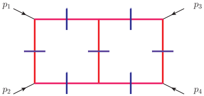
$$\langle\gamma_1|\omega_{a=0,b=0}\rangle = \langle\gamma_1|\omega_1\rangle = 1+\mathcal{O}\left(\epsilon\right), \quad \langle\gamma_1|\omega_{a=0,b=1}\rangle = \langle\gamma_1|\omega_2\rangle = \mathcal{O}(\epsilon)$$
$$\langle\gamma_2|\omega_{a=0,b=0}\rangle = \langle\gamma_2|\omega_1\rangle = \mathcal{O}\left(\epsilon\right), \quad \langle\gamma_2|\omega_{a=0,b=1}\rangle = \langle\gamma_2|\omega_2\rangle = 1+\mathcal{O}\left(\epsilon\right)$$

(Smirnov, 1999), (Anastasiou et al., 2000)

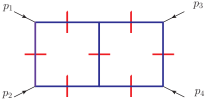
The maximal cut subspace

$\langle \gamma_1 | \omega_1 \rangle =$ 

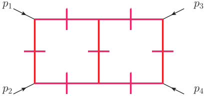
$= t^{-3-2\epsilon} f_{0,0}(x)\, {}_2\mathrm{F}_1\left(-\epsilon, 1+2\epsilon; 1; x\right)$

$\langle \gamma_1 | \omega_2 \rangle =$ 

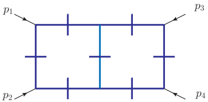
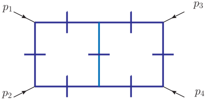
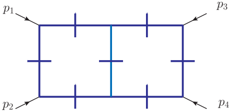
$= t^{-2-2\epsilon} f_{0,1}(x)\, {}_2\mathrm{F}_1\left(1-\epsilon, 1+2\epsilon; 2; x\right)$

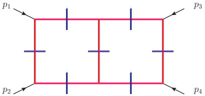
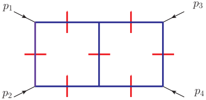
$\langle \gamma_2 | \omega_1 \rangle =$ 

$t^{-3-2\epsilon} \tilde{f}_{0,0}(x)\, {}_2\mathrm{F}_1\left(-\epsilon, a-\epsilon; a+b-3\epsilon; \frac{1}{x}\right)$

$\langle \gamma_2 | \omega_2 \rangle =$ 

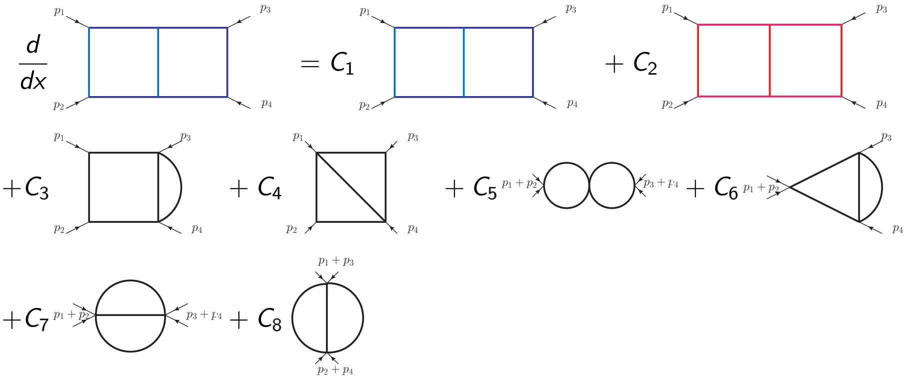
$= t^{-2-2\epsilon} \tilde{f}_{0,1}(x)\, {}_2\mathrm{F}_1\left(1-\epsilon, -\epsilon; 1-3\epsilon; \frac{1}{x}\right)$

$\Delta \Big[$  $\Big] =$  $\otimes$ 

$+$  $\otimes$ 

The Double Box differential equation

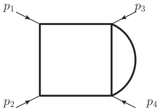
Apply a differential operator on $\langle\gamma|\omega_1\rangle$:



where $C_n = C_n\,(x, D)\,, n \in [1, 8]\,.$

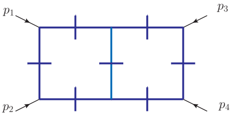
The maximal cut of the Double Box differential equation

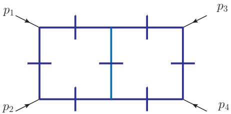
For diagrams with less than seven propagators (Eg: Double Edged Box)

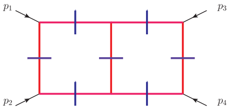


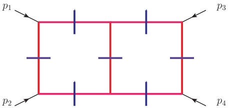
$\rightarrow 0$

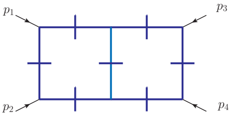
The maximal cut differential equations:

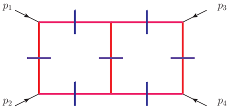
$\frac{d}{dx}$ 

$= C_1$ 

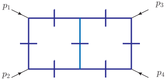
$+ C_2$ 

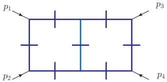
$\frac{d}{dx}$ 

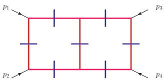
$= \tilde{C}_1$ 

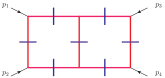
$+ \tilde{C}_2$ 

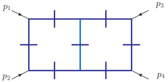
The homogeneous maximal cut differential equation

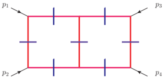
$\frac{d}{dx}$ 

$= C_1(x, D)$ 

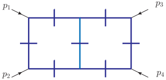
$+ C_2(x, D)$ 

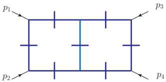
$\frac{d}{dx}$ 

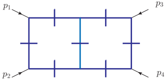
$= \tilde{C}_1(x, D)$ 

$+ \tilde{C}_2(x, D)$ 

Using the fact that $\langle \gamma_1 | \omega_2 \rangle$ obeys a similar differential equation we obtain a second order homogeneous differential equation:

$\frac{d^2}{dx^2}$ 

$+ A(x, D)\frac{d}{dx}$ 

$+ B(x, D)$ 

$= 0$

The differential equation defines the maximal cut homology group subspace

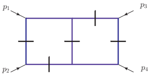
The function $\langle \gamma_1 | \omega_1 \rangle$ is a solution to this equation. So is $\langle \gamma_2 | \omega_1 \rangle$, all cuts in the subspace obey the same differential equation.

A non-maximal cut differential equation

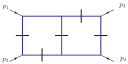
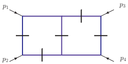
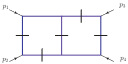
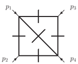
Non-maximal cut differential equations

Each equation features cut subtpologies as inhomogeneous terms of the maximal cut homogeneous differential eqaution

An unknown cut diagram:



A differential equation with an inhomogeneous term:

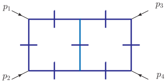
$\frac{d^2}{dx^2}$  $+ A(x, D) \frac{d}{dx}$  $+ B(x, D)$  $= C(x, D)$ 

The maximal cut is part of the solution space

Finding the particular solution amounts to finding the new element of the space.

How to solve the differential equations

Recall the form of the homogeneous solution:



$$= \langle \gamma_{cut} | \omega_1 \rangle = t^{-3-2\epsilon} f_{0,0}(x) \, {}_2F_1(-\epsilon, 1+2\epsilon; 1; x)$$

Divide all cut diagrams by the maximal cut scale:

$$g_{cut}(x) = \frac{\langle \gamma_{cut} | \omega_1 \rangle}{t^{-3-2\epsilon} f_{0,0}(x)}$$

The choice is motivated by:

$$g_1(x) = {}_2F_1(-\epsilon, 1+2\epsilon; 1; x)$$

Obtain a third order differential equation for the normalised form:

$$\frac{d^3 g_{cut}(x)}{dx^3} + C_1' \frac{d^2 g_{cut}(x)}{dx^2} + C_2' \frac{d g_{cut}(x)}{dx} + C' g_{cut}(x) = 0$$

The differential equation form

The differential equation becomes the $3F_2$ hypergeometric function defining equation

$$(1-x)x^2\frac{d^3 g_{cut}(x)}{dx^3}+2x(1-2x)\frac{d^2 g_{cut}(x)}{dx^2}$$
$$+\left(x(3\,\epsilon^2+2\,\epsilon-2\,\epsilon)+4\,\epsilon(1+\epsilon)\right)\frac{dg_{cut}(x)}{dx}-\epsilon^2(1+2\,\epsilon)g_{cut}(x)=0$$

Read off the three independent solutions of the differential equation and restore the scale

- $\langle\gamma_1|\omega_1\rangle=S_1=c_1(\epsilon)\;t^{-3-2\,\epsilon}f_{0,0}(x)\,{}_2\mathrm{F}_1\,(-\,\epsilon,1+2\,\epsilon;1;x)$
- $\langle\gamma_2|\omega_1\rangle=S_2=c_2(\epsilon)\;t^{-3-2\,\epsilon}f_{0,0}(x)\;(1-x)^{-\,\epsilon}\,{}_2\mathrm{F}_1\left(1+2\,\epsilon,1+2\,\epsilon;2+3\,\epsilon;\frac{1}{x}\right)$
- $\langle\gamma_4|\omega_1\rangle=S_3=c_3(\epsilon)\;t^{-3-2\,\epsilon}f_{0,0}(x)\;(1-x)^{1+2\,\epsilon}\;{}_3F_2\,(1,1,2+3\,\epsilon;2+\epsilon,2+2\,\epsilon;1-x)$

We can always guarantee an orthogonal basis

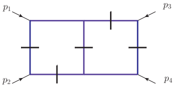
$\langle\gamma_4|\omega_2\rangle=\frac{3}{2}+\mathcal{O}(\epsilon)$, Modify the contour : $\langle\tilde{\gamma}_4|=\langle\gamma_4|-\frac{3}{2}\langle\gamma_2|\Rightarrow\langle\tilde{\gamma}_4|\omega_2\rangle=\mathcal{O}(\epsilon)$

Orthogonality not affected for the rest of the comohomology:

$\langle\tilde{\gamma}_4|\omega_1\rangle=\mathcal{O}(\epsilon),\;\langle\tilde{\gamma}_4|\omega_4\rangle=\langle\gamma_4|\omega_4\rangle=1+\mathcal{O}(\epsilon)$

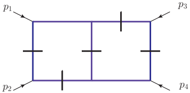
Determining the Coefficents

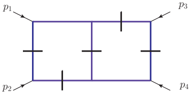
Use the already known solutions:

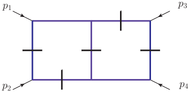


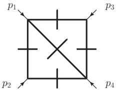
$$= c_1(\epsilon)\ t^{-3-2\epsilon} f_{0,0}(x)\, {}_2\mathrm{F}_1(-\epsilon, 1+2\epsilon; 1; x) + c_2(\epsilon)\ t^{-3-2\epsilon} f_{0,0}(x)\ (1-x)^{-\epsilon}\, {}_2\mathrm{F}_1\left(1+2\epsilon, 1+2\epsilon; 2+3\epsilon; \frac{1}{x}\right)$$

$$+ c_3(\epsilon)\ t^{-3-2\epsilon} f_{0,0}(x)\ (1-x)^{1+2\epsilon}\ {}_3F_2(1, 1, 2+3\epsilon; 2+\epsilon, 2+2\epsilon; 1-x)$$

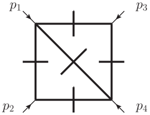
$$\frac{d^3}{dx^3}$$


$$+ C_1' \frac{d^2}{dx^2}$$


$$\frac{d}{dx} C_2'$$


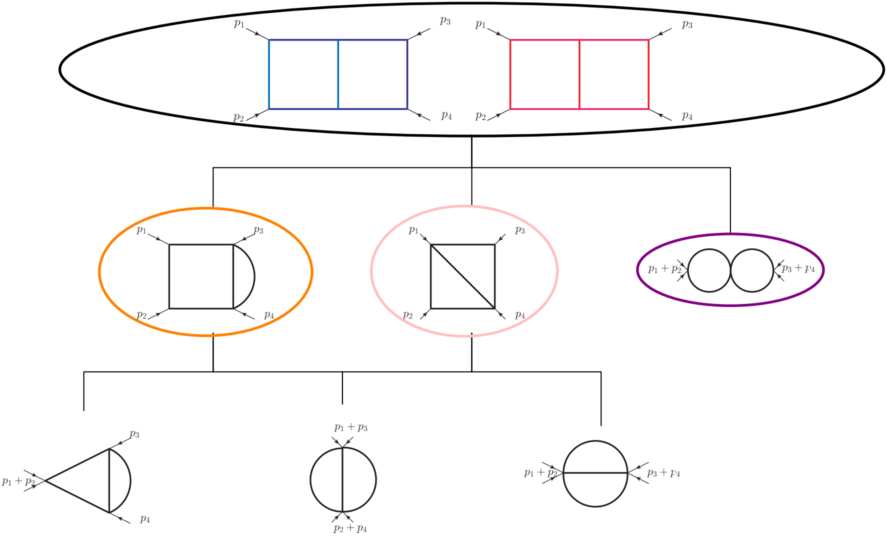
$$+ C_3'$$


$$= 0 \Rightarrow$$

$$c_3 = \frac{12\,\epsilon(1+3\,\epsilon)}{(1+\epsilon)(1+2\,\epsilon)} \frac{x^{2+2\,\epsilon}(1-x)^{1-2\,\epsilon}}{t^{-3-2\,\epsilon}}$$


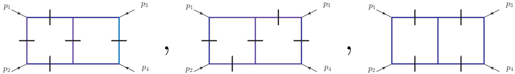
$$= \frac{12\,\epsilon(1+3\,\epsilon)}{(1+\epsilon)(1+2\,\epsilon)} \frac{\Gamma(1+3\,\epsilon)\Gamma(1+2\,\epsilon)}{\Gamma(1+\epsilon)^3}$$

Focus on the second graph of each family

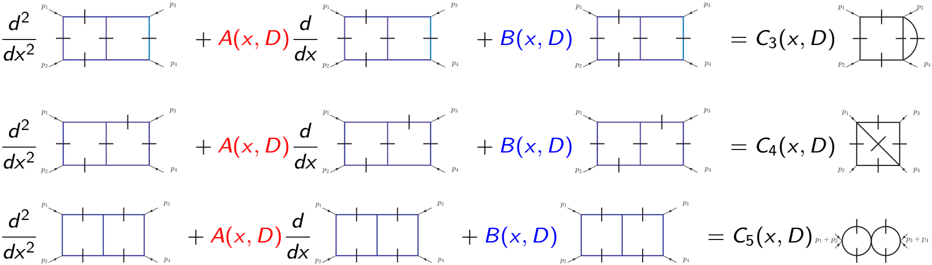


The Next level of Cut Differential Equations

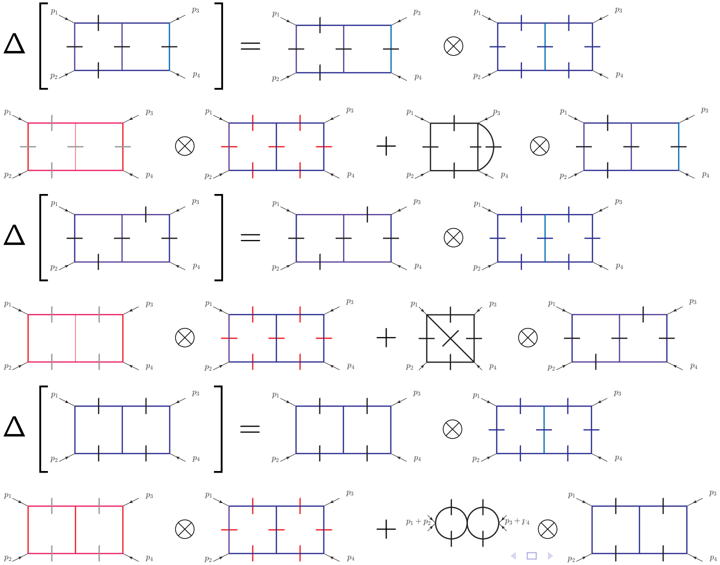
Three unknown cut diagrams:



Three differential equations with different inhomogeneous terms:



The cut diagram coactions



The homology theory of Hypergeometric Functions

Integrating between different branch points provides inequivalent contours.

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The differential equations

The maximal cut space are defined by the homogenous differential equation. Non maximal cuts appear as inhomogeneous terms.

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The differential equations solutions

The non-maximal cut differential equation always features the maximal cut solution. Using this solves the higher order differential equation.

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The coaction governs the contour/integrand relations

The coaction reveals the duality between the two bases and knowing the form that the coaction should take can reveal the results integrals.