Karlsruhe Institute of Technology

The Kinetic Mass of Heavy Quarks at Three Loops

RADCOR-LoopFest 2021

Matteo Fael | May 18, 2021

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based on Fael, Schönwald, Steinhauser, PRL 125 (2020) 052003, JHEP 10 (2020) 087

Inclusive semileptonic B-meson decays

- $|V_{cb}^{\text{inc}}| = (42.19 \pm 0.78) \times 10^{-3}$

  HFLAV 2019
- Global fits in the **kinetic scheme**

  Bigi et al, PRD 56 (1997) 4017;
  Gambino, Schwanda, Phys.Rev.D 89 (2014) 1, 014022
  Alberti, Gambino, Healey, Nandi, Phys.Rev.Lett. 114 (2015) 6, 061802.
- Also 1S scheme

  Hoang, Ligeti, Manohar, Phys.Rev.Lett. 82 (1999) 277
  Bauer, Ligeti, Manohar, Trott, Phys.Rev.D 70 (2004) 094017

The heavy-quark expansion

$$_{sl} = C_0 + C \frac{^2}{m_b^2} + C_{G} \frac{^2_G}{m_b^2} + C_{D} \frac{^3_D}{m_b^3} + C_{LS} \frac{^3_{LS}}{m_b^3} + \ldots$$

- Wilson coefficients C_i are calculable in pQCD
- Non-perturbative HQE parameters $;\ _G;\ _D;\ _{LS}$ $\langle B|O_i|B\rangle$.
- Global fits determine $|V_{cb}|, m_b, m_c$ and the HQE parameters.

Why three-loop correction to m_b^{kin}?

- Precision predictions need short mass schemes for better convergence of $_s$ series.
- Precise conversion between $\overline{m}_b$ and m_b^{kin}.
- Improve the SM prediction by including ^3_s corrections in pQCD.

Gambino, Schwanda, Phys.Rev.D 89 (2014) 1

Belle coll. R. van Tonder, hep-ex/2105.08001
based on MF, Mannel, Vos, JHEP 02 (2019) 177

The heavy-quark expansion

	tree	s	s^2	s^3
1	3	3	3	NEW
$1=m_b^2$	3	3	!	
$1=m_b^3$	3	3		
$1=m_b^{4;5}$	3			
$\overline{m}_b$ m_b^{kin}		3	3	NEW

Jezabek, Kuhn, NPB 314 (1989) 1; Gambino et al., NPB 719 (2005) 77;
Melnikov, PLB 666 (2008) 336; Pak, Czarnecki, PRD 78 (2008) 114015.
MF, Schönwald, Steinhauser, hep-ph:2011.13654 ;

Alberti, Gambino, Nandi, JHEP 1401 (2014) 147;
Mannel, Pivovarov, Rosenthal, PRD 92 (2015) 054025.
Becher, Boos, Lunghi, JHEP 0712 (2007) 062.

Mannel, Pivovarov, PRD 100 (2019) 9.

Dassinger, Mannel, Turczyk, JHEP 0703 (2007) 087, JHEP 1011 (2010) 109
MF, Mannel, Vos, JHEP 02 (2019) 177, JHEP 12 (2019) 067.

Bigi et al, PRD 56 (1997) 4017; Czarnecki, Melnikov, Uraltsev, PRL 80 (1998) 3189.
MF, Steinhauser, Schönwald, PRL 125 (2020) 5; PRD 103 (2021) 014005

The heavy-quark expansion

K. Schönwald's talk on Thursday
see also M. Steinhauser and A. Czarnecki talks

	tree	s	2 s	3 s	
1	3	3	3	NEW	Jezabek, Kuhn, NPB 314 (1989) 1; Gambino et al., NPB 719 (2005) 77; Melnikov, PLB 666 (2008) 336; Pak, Czarnecki, PRD 78 (2008) 114015. MF, Sch önwald, Steinhauser, hep-ph:2011.13654 ;
$1{=}m_b^2$	3	3	!		Alberti, Gambino, Nandi, JHEP 1401 (2014) 147; Mannel, Pivovarov, Rosenthal, PRD 92 (2015) 054025. Becher, Boos, Lunghi, JHEP 0712 (2007) 062.
$1{=}m_b^3$	3	3			Mannel, Pivovarov, PRD 100 (2019) 9.
$1{=}m_b^{4;5}$	3				Dassinger, Mannel, Turczyk, JHEP 0703 (2007) 087, JHEP 1011 (2010) 109 MF, Mannel, Vos, JHEP 02 (2019) 177, JHEP 12 (2019) 067.
$\overline{m}_b$ m_b^{kin}		3	3	NEW	Bigi et al, PRD 56 (1997) 4017; Czarnecki, Melnikov, Uraltsev, PRL 80 (1998) 3189. MF, Steinhauser, Sch önwald, PRL 125 (2020) 5; PRD 103 (2021) 014005

A short distance mass for $B \to X_c \ell \bar\nu$

$$\Gamma_{sl} = \frac{G_F^2 |V_{cb}|^2 (m_b^{OS})^5}{192\pi^3} f(0.25) \left[1 - 1.78 \frac{\alpha_s}{\pi} - 13.1 \left(\frac{\alpha_s}{\pi}\right)^2 - 163.3 \left(\frac{\alpha_s}{\pi}\right)^3 \right] + O\left(\frac{1}{m_b^2}\right)$$

See: Bigi, Shifman, Uraltsev, Vainshtein PRD 50 (1994) 2234; Beneke, Braun, NPB 426 (1994) 301; Ball, Beneke, Braun, PRD 52 (1995) 3929; Melnikov, van Ritbergen, PLB 482 (2000) 99.

- Mass scheme change: $m_b^{OS} \to \bar{m}_b \left(1 + c \frac{\alpha_s}{\pi}\right)$

$$\Gamma_{sl} \propto (\bar{m}_b)^n \left[1 + (nc + a_1) \frac{\alpha_s}{\pi} + \left(\frac{n(n+1)}{2} c^2 + nc\, a_1 + a_2 \right) \left(\frac{\alpha_s}{\pi}\right)^2 + \dots \right]$$

- Can we resum the power enhanced $(n\,\alpha_s)^k$ terms (with $n = 5$)?

Meson-quark mass relation

$$m_b = M_B \quad \bar{} \quad \frac{^{2}}{2m_b} + :::$$

- $\bar{}$: the B-meson binding energy.
- : the kinetic energy induced by the residual motion of the heavy quark.

The relevant parameter in $_{sl}$ is m_b^5, not M_B^5:

$$_{sl} \; ' \; \frac{G_F^2 |V_{cb}|^5}{192 \; ^3} (M_B \quad \bar{})^5$$

Bigi, Shifman, Uraltsev, Vainshtein, PRD 56 (1997) 4017

The kinetic mass

$$m_b^{kin}(\quad) = m_b^{OS} \quad [\overline{(}\quad)]_{pert} \quad \frac{[\quad {}^2(\quad)]_{pert}}{2m_b^{kin}(\quad)} \quad :::$$

Bigi, Shifman, Uraltsev, Vainshtein, PRD 56 (1997) 4017.
see also: Czarnecki, Melnikov, Uraltsev, PRL 80 (1998) 3189;
Gambino, JHEP 09 (2011) 055;

- In pQCD, we can peel off the IR renormalon from the on-shell mass identifying:

$$m_b(\quad)\; ! \quad m_b^{kin}(\quad) \qquad\qquad \overline{M}_B\; ! \quad m_b^{OS}$$

$$\overline{(}\quad)\; ! \quad [\overline{(}\quad)]_{pert} \qquad\qquad [\quad {}^2(\quad)]\; ! \quad [\quad {}^2(\quad)]_{pert}$$

The Small Velocity Sum Rules

- How to give an operative de nition of ¯ and ^{2}?
- Moments of the excitation energy:

$$I_n(q^2) = Z \; d! \; !^n \, \frac{d}{d! \; dq^2}$$

with ! $= E_{X_c}$ M_D and q $=$ $p_{`}$ $+$ p

The Small Velocity Sum Rules

Take the limit where the X_c's velocity is small: $|\vec{v}| = |\vec{q}/m_c| \ll 1$:

$$I_0(\vec{q}^{\,2}) = |\vec{q}\,| \frac{G_F^2 |V_{cb}|^2}{8\pi^3} (m_b - m_c)^2 + O\left(|\vec{v}|^2, \frac{\Lambda_{QCD}}{m_b}\right)$$

$$I_1(\vec{q}^{\,2}) = I_0 \frac{\vec{v}^{\,2}}{2} + O\left(|\vec{v}|^3, \frac{\Lambda^2_{QCD}}{m_b^2}\right)$$

$$I_2(\vec{q}^{\,2}) = I_0 \frac{\vec{v}^{\,2}}{3}\, ^2 + O\left(|\vec{v}|^3, \frac{\Lambda^3_{QCD}}{m_b^3}\right)$$

The Small Velocity Sum Rules

$$[\bar{\ }(\ )]_{\text{pert}} = \lim_{v \to 0} \lim_{m_b \to \infty} \frac{2}{v^2} \frac{\int_0 d\omega\, \omega \ \ W(\omega; v)}{\int_0 d\omega\ W(\omega; v)}$$

$$[\ ^2(\ )]_{\text{pert}} = \lim_{v \to 0} \lim_{m_b \to \infty} \frac{3}{v^2} \frac{\int_0 d\omega\, \omega^2 W(\omega; v)}{\int_0 d\omega\ W(\omega; v)}$$

- Consider only soft radiation $_{\text{QCD}}$ m_b

The Small Velocity Sum Rules

$$[\bar{\Lambda}(\mu)]_{\text{pert}} = \lim_{v \to 0} \lim_{m_b \to 0} \frac{2}{v^2} \frac{\int_0^{\mu} d\omega \, \omega \, W(\omega; v)}{\int_0^{\mu} d\omega \, W(\omega; v)}$$

$$[\mu_\pi^2(\mu)]_{\text{pert}} = \lim_{v \to 0} \lim_{m_b \to 0} \frac{3}{v^2} \frac{\int_0^{\mu} d\omega \, \omega^2 \, W(\omega; v)}{\int_0^{\mu} d\omega \, W(\omega; v)}$$

SV Limit ↔ Threshold Limit

- Excite the heavy quark,
  but just a bit …

$$y = s - m_b^2 \simeq 2 m_b \omega \ll m_b^2$$

- SV Limit corresponds to <u>1 Particle</u> Threshold limit !

- Factorization:

$$W(\omega, \mu) \simeq H \, U(\omega, \mu)$$

Ingredients for m_b^{kin} at $O(\ {}^3_s)$

- $W(!;\ \forall)$ up to $O(\ {}^3_s)$
- Imaginary part of forward scattering amplitudes:

$$W(!;\ \forall) = W_{el}(\forall)\ \ (!\ ) + \frac{\forall^2}{!} W_{real}(!\ )\ \ (!\ ) + O\ \ v^4; \frac{!}{m_b}$$

- Threshold expansion via method of regions: $y = s \quad m_b^2$.

Beneke, Smirnov, NPB 522 (1998) 321; Smirnov Springer Tracts Mod. Phys. 250 (2010)

Expansion by Regions

- For one heavy particle threshold, there are two regions:

  see also: Smirnov Springer Tracts Mod. Phys. 250 (2010)

hard (h): k_i m_b

ultra-soft (us): k_i $y\,m_b$

(u) (h)

(uu) (uh) (hh)

(uuu) (uuh) (uhh) (hhh)

Expansion by regions

- For one heavy particle threshold, there are two regions:

see also: Smirnov Springer Tracts Mod. Phys. 250 (2010)

hard (h): k_i m_b

ultra-soft (us): k_i $y = m_b$

(u) (h) (uu) (uh) (hh) (uuu) (uuh) (uhh) (hhh)

- All hard regions don't contribute: no imaginary part.
- After renormalization and decoupling $^{(n_l+n_h)}_{s}$! $^{(n_l)}_{s}$, only all ultra-soft part remains

Expansion by regions

- For one heavy particle threshold, there are two regions:

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$$\text{hard (h)}: \; k_i \qquad m_b$$

$$\text{ultra-soft (us)}: \; k_i \qquad y{=}m_b$$

(u) (h)

(uu) (uh) (hh)

(uuu) (uuh) (uhh) (hhh)

- All hard regions don't contribute: no imaginary part.
- After renormalization and decoupling $^{(n_l+n_h)}_{s}$! $^{(n_l)}_{s}$, only all ultra-soft part remains

For the statistics . . .

- 4, 66 and 1586 diagrams at one, two and three loops
- Partial fractioning and mapping between families implemented in FORM thanks to code LIMIT.

  Herren, PhD thesis, KIT, 2020
- FIRE and LiteRed reduction of integral families:

  Smirnov, Chuharev, hep-ph/1901.07808; Lee, hep-ph/1212.2685.
  - $\{2; 2\}$ in the $\{(uu); (uh)\}$ regions at two loops
  - $\{14; 4; 3\}$ in the $\{(uuu); (uuh); (uhh)\}$ regions at three loops
- New master integrals:
  - 3 in the (uu) region
  - 20 in the (uuu) region
- Heavy quark form factors up to $O(\alpha_s^2)$ (static limit)

  Lee, Smirnov, Smirnov, Steinhauser, JHEP 1805 (2018) 187; Blümlein, Marquard, Rana, PRD 99 (2019) 016013

I_1^{2l} I_2^{2l} I_3^{3l}

- The new master integrals contains linear-massive propagators:

$$I_2^{2l} = \int d^dk_1 d^dk_2 \frac{1}{k_1^2(k_1 - k_2)^2(2k_1 \cdot p - y)(2k_2 \cdot p)}$$

- Mellin-Barnes method

  MBpackage
  Czakon, Comput. Phys. Commun. 175 (2006) 559;
  Smirnov[2], EPJC 62 (2009) 445.

- PSLQ
- Analytic summation of residues

  HarmincSums
  www3.risc.jku.at/research/combinat/software/HarmonicSums/

- Differential equations in auxiliary variable
  Kotikov, PLB 254 (1991), 158
  Gehrmann, Remiddi, NPB 580 (2000) 485
  Henn, PRL 110 (2013), 251601.

The kinetic mass

MF, Schönwald, Steinhauser, PRL 125 (2020) 052003

- The mass relation is written in terms of $_s^{(n_l)}$.
- n_l = number of massless quarks , $l = \log(2 = \ _s)$.

Charm mass effects: $0 < m_c, m_c \neq m_b$

- Charm mass effects to the $\overline{\text{MS}}$-on-shell mass relation known up to $O(\alpha_s^3)$.
  Davydychev, Grozin, PRD 59 (1999) 054024; Broadhurst, Gray, Schlicher, Z. Phys. C 52 (1991) 111;
  Bekavac, Grozin, Seidel, Steinhauser, JHEP 10 (2007) 006; MF, Schönwald, Steinhauser, JHEP 10 (2020) 087.
- No charm mass effects known for the kinetic-on-shell mass relation.
- We assume $|y|$ $m_c^2; m_b^2$, i.e. no cut through a charm loop

The kinetic-$\overline{\text{MS}}$ mass relation for bottom

- Input values:
  $_s^{(5)}(M_Z) = 0{:}1179$ $\quad\overline{m}_c(3\text{ GeV}) = 0{:}993\text{ GeV}$ $\quad\overline{m}_b(\overline{m}_b) = 4{:}163\text{ GeV}$
- The charm quark wants to be treated as heavy .

scheme	$_s^{(n_r)}$	m_c in $\overline{\text{MS}}$-OS	m_c in kin-OS	
(A)	3	3	-	$m_b^{kin}(1\text{ GeV}) = 4163 + 248 + 80 + 30 = 4520\text{ MeV}$
(B)	4	3	3	$4163 + 259 + 78 + 26 = 4526\text{ MeV}$
(C)	4	3	7	$4163 + 259 + 84 + 41 = 4547\text{ MeV}$
(D)	3	7	7	$4163 + 248 + 81 + 30 = 4521\text{ MeV}$

MF, Schönwald, Steinhauser, Phys.Rev.D 103 (2021) 014005

Scheme conversion uncertainty

- Half of $^{3}_{s}$ correction

  m_b^{kin} ' 15 MeV

- Large- $_0$ term at $^{4}_{s}$

  m_b^{kin} ' 8 MeV

Compared with:

- scheme conversion uncertainty at two-loops: m_b^{kin} = 30 MeV Gambino, JHEP 09 (2011) 055
- $m_b^{kin}(1\,\mathrm{GeV})$ = 4554 18 MeV from B ! X_c ` ` global t HFLAV 2019

Conclusions

- We computed the $O(\alpha_s^3)$ relation between the kinetic and the on-shell mass.
- We studied finite charm mass effects in m_b^{kin}: the charm wants to be heavy!
- Scheme conversion uncertainty is reduced by a factor of 2.
- Mass formula is necessary to improve $B \to X_c \ell \nu$ prediction in the SM.
- Our results are crucial for future extractions of $|V_{cb}|$ and m_b from Belle-II data.

Spare

Let's include radiative corrections . . .

$$I_n(\vec{q}^{\,2}) = \int_{|\vec{q}|}^{q_0^{max}} dq_0\, \omega^n \frac{d\,{}_{\text{tree}}}{dq_0\, d\vec{q}^{\,2}} + \int_{q_0^{max}}^{q_0^{max}} dq_0\, \omega^n \frac{d\,{}_{s}}{dq_0\, d\vec{q}^{\,2}} + \int_{|\vec{q}|} dq_0\, \omega^n \frac{d\,{}_{s}}{dq_0\, d\vec{q}^{\,2}}$$

- We introduce a Wilsonian cutoff , with $_{QCD}$ M_B, to separate gluons with
  - $\omega <$ that belong to the non-perturbative regime,
  - $\omega >$ that can be described in pQCD.

Let's include radiative corrections . . .

$$I_1(\vec q^{\,2}) = \underbrace{\int_{|\vec q|}^{q_0^{max}} dq_0\, \omega \frac{d\Gamma_{tree}}{dq_0\, d\vec q^{\,2}} + \int_{q_0^{max}}^{q_0^{max}} dq_0\, \omega \frac{d\Gamma_{\alpha_s}}{dq_0\, d\vec q^{\,2}}}_{\text{use this to define in the SV limit } \bar\Gamma(\mu)} + \int_{|\vec q|}^{q_0^{max}} dq_0\, \omega \frac{d\Gamma_{\alpha_s}}{dq_0\, d\vec q^{\,2}}$$

- We introduce a Wilsonian cutoff μ, with $\Lambda_{QCD} \ll \mu \ll M_B$, to separate gluons with
  - $\omega < \mu$ that belong to the non-perturbative regime,
  - $\omega > \mu$ that can be described in pQCD.

$m_b^{kin}(1\ \text{GeV})$ from $\overline{m}_b(\overline{m}_b)$

in MeV

scheme	(n_r) s	m_c in $\overline{\text{MS}}$-OS	m_c in kin-OS	
(A)	3	3	–	$4163 + 248 + (81 + 7\ _{m_c} + 12_{dec}\ \ 20_{n_c})$ $+ (30 + 14\ _{m_c} + 16_{dec}\ \ 30_{n_c}\ \ 1_{n_c}\ \ {dec} + 0{:}4\ _{m_c}\ \ {dec})$ $= 4163 + 248 + 80 + 30 = 4520$
(B)	4	3	3	$4163 + 259 + (88 + 7\ _{m_c} + 5\ ^{kin}_{m_c}\ \ 22_{n_c})$ $+ (34 + 16\ _{m_c} + 10\ ^{kin}_{m_c}\ \ 34_{n_c})$ $= 4163 + 259 + 78 + 26 = 4526$

$m_b^{kin}(1\text{ GeV})$ from $\overline{m}_b(\overline{m}_b)$

in MeV

	$\alpha_s^{(n_l)}$	m_c in $\overline{\text{MS}}$-OS	m_c in kin-OS	
(C)	4	3	7	$4163 + 259 + (99 + 7_{m_c} \; 22_{n_c}) + (59 + 16_{m_c} \; 34_{n_c}) = 4163 + 259 + 84 + 41 = 4547$
(D)	3	7	7	$4163 + 248 + 81 + 30 = 4521$

Charm quark mass

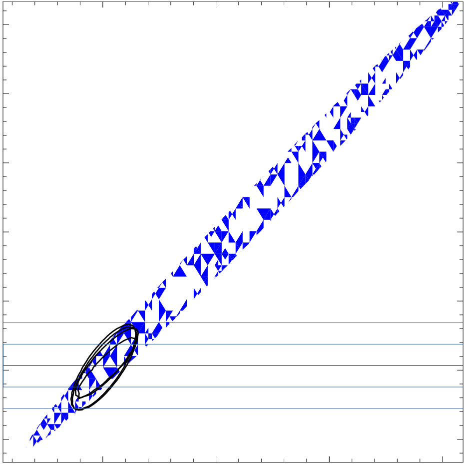
- $m_c^{kin}(0{:}5\ \mathrm{GeV})$:

$$
\begin{aligned}
m_c^{kin}(0{:}5\ \mathrm{GeV}) &= 993 + 191 + 100 + 52\ \mathrm{MeV} = 1336\ \mathrm{MeV}\,;\\
m_c^{kin}(0{:}5\ \mathrm{GeV}) &= 1099 + 163 + 76 + 34\ \mathrm{MeV} = 1372\ \mathrm{MeV}\,;\\
m_c^{kin}(0{:}5\ \mathrm{GeV}) &= 1279 + 84 + 30 + 11\ \mathrm{MeV} = 1404\ \mathrm{MeV}\,:
\end{aligned}
$$

- $m_c^{kin}(1\ \mathrm{GeV})$:

$$
\begin{aligned}
m_c^{kin}(1\ \mathrm{GeV}) &= 993 + 83 + 35 + 14\ \mathrm{MeV} = 1125\ \mathrm{MeV}\,;\\
m_c^{kin}(1\ \mathrm{GeV}) &= 1099 + 37 + 2 \quad 3\ \mathrm{MeV} = 1135\ \mathrm{MeV}\,;\\
m_c^{kin}(1\ \mathrm{GeV}) &= 1279 \quad 73 \quad 61 \quad 17\ \mathrm{MeV} = 1128\ \mathrm{MeV}\,;
\end{aligned}
$$

where from top to bottom $_s$ = 3 GeV; 2 GeV and $\overline{m}_c$ for $\overline{m}_c(\ _s)$ and $^{(3)}_s(\ _s)$.



Original plot from

Gambino, Schwanda, PRD 89 (2014) 014022

Horizontal error bands superimposed by MF