Evaluating planar master integrals for Bhabha scattering

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in collaboration with Claude Duhr and Lorenzo Tancredi
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Two-loop Bhabha scattering in QED: four-point diagrams with all the external points on the mass shell, $p_i^2 = m^2$. Three variables, $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, $m^2$. 
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Now: analytic evaluation of master integrals for graph (b). Evaluating integrals for graph (a) with two different masses [M. Heller’21].
$$F_{a_1,a_2,\ldots,a_9} = \int \int \frac{d^D k_1 \, d^D k_2}{\left[ -k_1^2 + m^2 \right]^{a_1} \left[ -\left( k_1 + p_1 + p_2 \right)^2 + m^2 \right]^{a_2} \left[ -\left( k_2 + p_1 \right)^2 \right]^{a_8} \left[ -\left( k_1 - p_3 \right)^2 \right]^{a_9} \left[ -k_2^2 \right]^{a_3} \left[ -\left( k_2 + p_1 + p_2 \right)^2 \right]^{a_4} \left[ -\left( k_1 + p_1 \right)^2 \right]^{a_5} 1 \left[ -\left( k_1 - k_2 \right)^2 + m^2 \right]^{a_6} \left[ -\left( k_2 - p_3 \right)^2 + m^2 \right]^{a_7} \right].$$
Evaluating planar master integrals for Bhabha scattering

\[ F_{a_1, a_2, \ldots, a_9} = \int \int \int \frac{d^D k_1 \, d^D k_2}{[-k_1^2 + m^2]^{a_1}[-(k_1 + p_1 + p_2)^2 + m^2]^{a_2}[-(k_2 + p_1)^2]^{a_8}[-(k_1 - p_3)^2]^{a_9}[-k_2^2]^{a_3}[-(k_2 + p_1 + p_2)^2]^{a_4}[-(k_1 + p_1)^2]^{a_5}} \times \frac{1}{[-(k_1 - k_2)^2 + m^2]^{a_6}[-(k_2 - p_3)^2 + m^2]^{a_7}}. \]

Solving IBP relations with KIRA or FIRE \( \rightarrow 43 \) master integrals \( g_1, \ldots, g_{43} \).
Solving differential equations

Differential equations

\[ \partial_v g = A_v g , \]

\[ v = s, t, m^2, \partial_v = \frac{\partial}{\partial v} \] and matrices \( A_s, A_t, A_{m^2} \) are rational functions of \( s, t, m^2 \) and \( \epsilon \).
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Turn to an $\epsilon$-basis [J. Henn’13], $g_i \rightarrow f_i$,

$$\partial_v f = \epsilon \tilde{A}_v f$$

with $\tilde{A}_v$ independent of $\epsilon$. 
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We use the strategy of

[T. Gehrmann, A. von Manteuffel, L. Tancredi & E. Weihs’14]
dlog form: \[ df = \epsilon d\tilde{A}f. \]
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Solution

$$f(s, t; \epsilon) = P\exp \left[ \epsilon \int_{\gamma} d\tilde{A} \right] f_0(\epsilon)$$

where $P\exp$ is the path-ordered exponential and $f_0(\epsilon)$ is the initial condition related to the value of $f$ at a specific point. The path $\gamma$ connects the initial point $(s_0, t_0)$ to the generic point $(s, t)$. 
\[
f_1 = \epsilon^2 F_{2,0,0,0,0,2,0,0,0}, \\
f_2 = -\epsilon^2 \frac{1}{2} \sqrt{-s} \sqrt{4m^2} - s F_{0,2,1,0,0,2,0,0,0} \\
\qquad - \epsilon^2 \sqrt{-s} \sqrt{4m^2} - s F_{0,2,0,0,1,0,0,0}, \\
f_3 = -\epsilon^2 s F_{0,2,1,0,0,2,0,0,0}, \\
f_4 = -\frac{1}{2} \epsilon^2 \sqrt{-t} \sqrt{4m^2} - t F_{0,0,0,1,2,2,0,0} \\
\qquad - \epsilon^2 \sqrt{-t} \sqrt{4m^2} - t F_{0,0,0,2,1,2,0,0}, \\
f_5 = -\epsilon^2 t F_{0,0,0,1,2,2,0,0}, \\
f_6 = -\epsilon^2 m^2 F_{0,0,1,0,2,2,0,0,0} \\
f_7 = -\epsilon^3 \sqrt{-s} \sqrt{4m^2} - s F_{0,1,1,0,1,2,0,0,0}, \ldots
\]
The appearance of square roots is the price for having a canonical basis. There are four square roots,
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\begin{align*}
    r_s &= \sqrt{-s\sqrt{4m^2 - s}}, &
    r_t &= \sqrt{-t\sqrt{4m^2 - t}}, \\
    r_u &= \sqrt{-s - t\sqrt{4m^2 - s - t}}, &
    r_{st} &= \sqrt{-s\sqrt{4m^6 - s(m^2 - t)^2}}.
\end{align*}
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The square roots are chosen in such a way that that they are manifestly real at Euclidean values, \( s, t < 0 \).
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\end{align*}
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The standard way to rationalize the first two square roots is to turn to dimensionless variables \(x\) and \(y\)

\[
\frac{-s}{m^2} = \frac{(1 - x)^2}{x} \quad \frac{-t}{m^2} = \frac{(1 - y)^2}{y}.
\]
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The square root $r_{st}$ does not appear when solving differential equations up to weight 3 for all elements but $f_{37}$ and at weight 4 for all elements but $f_i, i = 35, 36, 37, 38, 39, 41, 43$.
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MPLs

$$G(a_1, \ldots, a_n; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

$$G(0, \ldots, 0; x) = \frac{1}{n!} \ln^n x$$
Then the equations with respect to $y$ can be solved (after checking that the variable $x$ disappears in them) in terms of MPLs of $y$ with the letters $\{0, -1, 1\}$, i.e. harmonic polylogarithms [E. Remiddi & J. Vermaseren’99].
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Using expansion by regions [M. Beneke & VS’98] implemented in the code asy.m [A. Pak & A.V. Smirnov’10] (which is now included in the code FIESTA [A.V. Smirnov’15]; also in pySecDec [E. Villa’21, talk tomorrow])
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Evaluating planar master integrals for Bhabha scattering

\[
f_1 \sim 1 + \frac{\pi^2 \epsilon^2}{6} - \frac{2 \zeta(3) \epsilon^3}{3} + \frac{7 \pi^4 \epsilon^4}{360},
\]
\[
f_6 \sim -\frac{1}{4} - \frac{5 \pi^2 \epsilon^2}{24} - \frac{11 \zeta(3) \epsilon^3}{6} - \frac{101 \pi^4 \epsilon^4}{480},
\]
\[
f_9 \sim -\frac{\pi^2 \epsilon^2}{12} + \frac{1}{4} \epsilon^3 \left(2 \pi^2 \log(2) - 7 \zeta(3)\right)
+ \frac{1}{180} \epsilon^4 \left(13 \pi^4 - 90 \log^4(2) - 180 \pi^2 \log^2(2) - 2160 \text{Li}_4\left(\frac{1}{2}\right)\right),
\]
\[
f_{18} \sim \frac{1}{2} \epsilon^3 \left(2 \pi^2 \log(2) - 3 \zeta(3)\right)
+ \frac{1}{20} \epsilon^4 \left(7 \pi^4 - 20 \log^4(2) - 40 \pi^2 \log^2(2) - 480 \text{Li}_4\left(\frac{1}{2}\right)\right),
\]
\[
f_{19} \sim (-s)^{-\epsilon} \left(-1 + \frac{8 \zeta(3) \epsilon^3}{3} + \frac{\pi^4 \epsilon^4}{30}\right),
\]
\[ f_{22} \sim (-s)^{-\epsilon} \left( -\frac{1}{2} + \frac{4\zeta(3)\epsilon^3}{3} + \frac{\pi^4\epsilon^4}{60} \right) \]

\[ + (-s)^{-2\epsilon} \left( \frac{1}{4} - \frac{\pi^2\epsilon^2}{24} - \frac{14\zeta(3)\epsilon^3}{3} - \frac{67}{480}\pi^4\epsilon^4 \right), \]

\[ f_{23} \sim (-s)^{-2\epsilon} \pi^2 \left( \epsilon^2 + 2\epsilon^3 \log(2) + 2\epsilon^4 \left( \pi^2 + \log^2(2) \right) \right), \]

\[ f_{25} \sim (-s)^{-\epsilon} \pi^2 \left( -\epsilon^2 - 2\epsilon^3 \log(2) - \frac{1}{2}\epsilon^4 \left( \pi^2 + 4\log^2(2) \right) \right) \]

and \( f_i \sim 0 \), i.e. \( f_i = o(s, t) \) for all the other elements.
For example,

\[ f_{42} = \ldots + \varepsilon^4 (-\pi^2 G(-1; y)G(0, x) + \frac{1}{2} \pi^2 G(0; y)G(0, x) - \frac{1}{3} \pi^2 G(1; y)G(0, x) - 36 G(-1, -1, 0; y)G(0, x) + 24 G(-1, 0, 0; y)G(0, x) - 12 G(-1, 1, 0; y)G(0, x) + 24 G(0, -1, 0; y)G(0, x) - 10 G(0, 0, 0; y)G(0, x) + 8 G(0, 1, 0; y)G(0, x) - 12 G(1, -1, 0; y)G(0, x) + 8 G(1, 0, 0; y)G(0, x) - 4 G(1, 1, 0; y)G(0, x) + 11 \zeta(3) G(0, x) - \frac{4}{3} \pi^2 G(-1, x)G(0; y) + 2 \pi^2 G(-1; y)G(-1/y; x) - \frac{1}{6} \pi^2 G(0; y)G(-1/y; x) - 2 \pi^2 G(-1; y)G(-y; x) + \frac{3}{2} \pi^2 G(0; y)G(-y; x) - \frac{1}{3} \pi^2 G(-1, 0, x) - 12 G(-1, 0, x)G(-1, 0; y) - 4 \pi^2 G(-1, 0; y) + \pi^2 G(-1, -1/y; x) - \pi^2 G(-1, -y, x) - 2 \pi^2 G(0, -1, y) + 8 G(-1, 0, x)G(0, 0; y) + 2 G(-1, -1/y; x)G(0, 0; y) - 2 G(-1, -y, x)G(0, 0; y) + \frac{7}{2} \pi^2 G(0, 0; y) - 4 G(-1, 0, x)G(1, 0; y) - \frac{4}{3} \pi^2 G(1, 0; y) + \pi^2 G(-1/y, -1; x) + 6 G(-1, 0; y)G(-1/y, 0; x) - 4 G(0, 0; y)G(-1/y, 0; x) + 2 G(1, 0; y)G(-1/y, 0; x) - \frac{1}{6} \pi^2 G(-1/y, 0; x) - G(0, 0; y)G(-1/y, -1/y; x) - \frac{1}{2} \pi^2 G(-1/y, -1/y; x) + G(0, 0; y)G(-1/y, -y; x) + \ldots) \]
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It can be rationalized by the following further change of variables $x \rightarrow w$:

$$x = \frac{2 \left( (1 - w) (y^2 - y + 1)^2 - 2y^2 \right)}{(1 - w^2) (y^2 - y + 1)^2}.$$
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The equations are solved, first, in $w$ and then in $y$. The results are written in terms of $G(\ldots, w)$ and $G(\ldots, y)$. 
The letters in $G(\ldots, w)$ and $G(\ldots, y)$ are cumbersome and the result is rather complicated, the contributions of weight 4 take $\sim 60\text{mb}$. Still we obtain an answer to the question about the class of functions: these are MPLs, with the exception of $f_{14}$. 
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Evaluating the weight 4 results with GiNaC [C. W. Bauer, A. Frink & R. Kreckel’00; J. Vollinga & S. Weinzierl’04] meets certain problems connected with timing and stability, so that such results become impractical.
For these complicated elements, we prefer to apply the recently developed code DiffExp to evaluate Feynman integrals numerically using differential equations [M. Hidding’20; talk at this session].
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With a canonical basis, the code works much better.
Elliptic sector

\[ f_{14} \equiv \epsilon^4 \bar{f} = -\epsilon^4 \sqrt{-s-t} \sqrt{4m^2-s-t} \text{ times} \]
Elliptic sector

\[ f_{14} \equiv \epsilon^4 \bar{f} = -\epsilon^4 \sqrt{-s-t} \sqrt{4m^2 - s - t} \text{ times} \]

The differential equation equations give
\[
\frac{\partial}{\partial x} \bar{f}(x, y) = \frac{1}{(x - 1)x \sqrt{(x + y)(xy + 1)}(x^2 y + xy^2 - 4xy + x + y)} \\
\times \left[ (x - 1) G(0, x) \left( 2 \left( 3x^2 y + x(y - 1)^2 + y \right) G(0, 0, y) + \pi^2 \left( x^2 - 1 \right) y \right) \\
- (x + 1) \left( 2 G(0, y) \left( x \left( y^2 - 1 \right) G(0, 0, x) + (x - 1)^2 y \left( G \left( \frac{-1}{y}, 0, x \right) - G(-y, 0, x) \right) \right) \right) \right]
\]
\[
- 2(x - 1)^2 y \left( - G \left( \frac{-1}{y}, 0, 0, x \right) - G(-y, 0, 0, x) + 2 G(0, 0, 0, x) - 2 G(1, 0, 0, x) \\
+ G(0, 0, 0, y) - 2 G(1, 0, 0, y) - \zeta(3) \right) + (x - 1)^2 y \left( 2 G(0, 0, y) + \pi^2 \right) G \left( \frac{-1}{y}, x \right) \\
+ (x - 1)^2 y \left( 2 G(0, 0, y) + \pi^2 \right) G(-y, x) \right].
\]
\[
\frac{\partial}{\partial x} \bar{f}(x, y) = \frac{1}{(x - 1)x \sqrt{(x + y)(xy + 1)} (x^2 y + xy^2 - 4xy + x + y)} 
\times \left[ (x - 1) G(0, x) \left( 2 \left( 3x^2 y + x(y - 1)^2 + y \right) G(0, 0, y) + \pi^2 \left( x^2 - 1 \right) y \right) \right. 
\left. - (x + 1) \left( 2G(0, y) \left( x \left( y^2 - 1 \right) + (x - 1)^2 y \left( G \left( -\frac{1}{y}, 0, x \right) - G(-y, 0, x) \right) \right) \right) 
\right.
\left. - 2(x - 1)^2 y \left( -G \left( -\frac{1}{y}, 0, 0, x \right) - G(-y, 0, 0, x) + 2G(0, 0, 0, x) - 2G(1, 0, 0, x) \right) 
\right.
\left. + G(0, 0, 0, y) - 2G(1, 0, 0, y) - \zeta(3) \right) + (x - 1)^2 y \left( 2G(0, 0, y) + \pi^2 \right) G \left( -\frac{1}{y}, x \right) 
\right.
\left. + (x - 1)^2 y \left( 2 \left( G(0, 0, y) + \pi^2 \right) G(-y, x) \right) \right]
\].

The function \( \bar{f}(x, y) \) is symmetrical, \( \bar{f}(y, x) = \bar{f}(x, y) \).
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The differential equation is solved on a path which consists of two straight-line segments: the straight line from the point \((1, 1)\) (where the function \(= 0\)) to the point \((1, y), 0 \leq y \leq 1\), and from \((1, y)\) to the general \((x, y)\) in the Euclidean region \(0 < x < 1, 0 < y < 1\).
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Use the variable $\bar{x} = 1 - x$. Here is the result
\[
2\mathcal{E}_4\left(-1_{\frac{1}{y+1}} 1_{\frac{1}{1}} 1_{\frac{1}{1}}; \bar{x}, \bar{a}\right) + 2\mathcal{E}_4\left(-1_{\frac{1}{y+1}} 1_{\frac{1}{1}} 1_{\frac{1}{1}}; \bar{x}, \bar{a}\right) + \left(-3 \log^2(y) - \pi^2\right) \mathcal{E}_4\left(-1_{\frac{1}{1}}; \bar{x}, \bar{a}\right)
\]
\[
+ \left(\log^2(y) + \pi^2\right) \mathcal{E}_4\left(-1_{\frac{1}{y+1}} 1_{\frac{1}{1}} 1_{\frac{1}{1}}; \bar{x}, \bar{a}\right) + \left(\log^2(y) + \pi^2\right) \mathcal{E}_4\left(-1_{\frac{1}{y+1}} 1_{\frac{1}{1}} 1_{\frac{1}{1}}; \bar{x}, \bar{a}\right)
\]
\[
+ 2 \log(y)\mathcal{E}_4\left(-1_{\frac{1}{y+1}} 1_{\frac{1}{1}} 1_{\frac{1}{1}}; \bar{x}, \bar{a}\right) - 2 \log(y)\mathcal{E}_4\left(-1_{\frac{1}{y+1}} 1_{\frac{1}{1}} 1_{\frac{1}{1}}; \bar{x}, \bar{a}\right) + 2\mathcal{E}_4\left(-1_{\frac{1}{1}} 1_{\frac{1}{y+1}} 1_{\frac{1}{1}}; \bar{x}, \bar{a}\right)
\]
\[
+ 2\mathcal{E}_4\left(-1_{\frac{1}{y+1}} 1_{\frac{1}{1}} 1_{\frac{1}{1}}; \bar{x}, \bar{a}\right) + \left(\log^2(y) - \pi^2\right) \mathcal{E}_4\left(-1_{\frac{1}{1}} 1_{\frac{1}{y+1}} 1_{\frac{1}{1}}; \bar{x}, \bar{a}\right) + \left(\log^2(y) + \pi^2\right) \mathcal{E}_4\left(-1_{\frac{1}{1}} 1_{\frac{1}{y+1}} 1_{\frac{1}{1}}; \bar{x}, \bar{a}\right)
\]
\[
- 2 \log(y)\mathcal{E}_4\left(-1_{\frac{1}{y+1}} 1_{\frac{1}{1}} 1_{\frac{1}{1}}; \bar{x}, \bar{a}\right) + 4\mathcal{E}_4\left(-1_{\frac{1}{y+1}} 1_{\frac{1}{1}} 1_{\frac{1}{1}}; \bar{x}, \bar{a}\right) - 4\mathcal{E}_4\left(-1_{\frac{1}{1}} 1_{\frac{1}{y+1}} 1_{\frac{1}{1}}; \bar{x}, \bar{a}\right)
\]
\[
+ 4\mathcal{E}_4\left(-1_{\frac{1}{1}} 1_{\frac{1}{y+1}} 1_{\frac{1}{1}}; \bar{x}, \bar{a}\right) - 4\mathcal{E}_4\left(-1_{\frac{1}{1}} 1_{\frac{1}{y+1}} 1_{\frac{1}{1}}; \bar{x}, \bar{a}\right) + \left(-4\text{Li}_3(-y) - 4\text{Li}_3(y) + 4\text{Li}_2(-y)\log(y)
\]
\[
+ 4\text{Li}_2(y)\log(y) - \frac{2}{3} \log^3(y) + 2 \log(1 - y) \log^2(y) + 2 \log(y + 1) \log^2(y) - \pi^2 \log(y)
\]
\[
+ 2\pi^2 \log(y + 1) - 2\zeta(3)\right) \mathcal{E}_4\left(-1_{\frac{1}{1}}; \bar{x}, \bar{a}\right) + \left(-4\text{Li}_3(-y) - 4\text{Li}_3(y) + 4\text{Li}_2(-y)\log(y)
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\]
\[
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\]
\[
- 2\text{Li}_2(-y) \left(\log^2(y) + \pi^2\right) + 8\text{Li}_3(-y)\log(y) + 8\text{Li}_3(y)\log(y) - 2\zeta(3) \log(y)
\]
\[
- \frac{1}{6} \log^4(y) - \frac{1}{2} \pi^2 \log^2(y) - \frac{3\pi^4}{20}
\]
eMPLs

\[ \mathcal{E}_4 \left( \frac{n_1}{c_1} \cdots \frac{n_k}{c_k} ; x, \vec{a} \right) = \int_0^x dt \, \psi_{n_1} (c_1, t, \vec{a}) \mathcal{E}_4 \left( \frac{n_2}{c_2} \cdots \frac{n_k}{c_k} ; t, \vec{a} \right) \]
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The set of eMPLs in our case is associated with the elliptic curve \( z^2 = P_n(x, y) \), where \( P_n \) is a polynomial of degree \( n = 3 \) or 4. Here

\[ P_4(x, y) = (x - a_1)(x - a_2)(x - a_3)(x - a_4) \]

with

\[ a_1 = y + 1, \quad a_2 = (y - 1) \left( \sqrt{y^2 - 6y + 1 + y - 1} \right) / (2y), \]

\[ a_3 = (y - 1) \left( -\sqrt{y^2 - 6y + 1 + y - 1} \right) / (2y), \quad a_4 = 1 / y + 1 \]
If all the indices $A_i = \left( \frac{n_i}{c_i} \right)$ are equal to $\left( \frac{\pm 1}{0} \right)$, the integral is divergent and a definition with some subtractions is used.
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MPLs are partial cases of eMPLs:

$$E_4\left(\frac{1}{c_1} \cdots \frac{1}{c_k}; x, \vec{a}\right) = G(c_1, \ldots, c_k; x)$$
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Conclusion

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- All the master integrals but one are expressed in terms of MPLs.
- We have derived a compact result for one master integral in terms of eMPLs.