BOTTOM–QUARK HADROPRODUCTION IN NNLO QCD

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CONTENTS

➤ Introduction;

➤ $q_T$ subtraction formalism for heavy-quark production:
   • Top-pair production within $q_T$ subtraction formalism;
   • Extension to bottom-pair production;

➤ Phenomenological results:
   • Inclusive cross section;
   • Differential distributions;

➤ Summary and outlook.
Bottom-quark production is extensively studied at hadron colliders:

- **CERN Sp̅pS**;
- **Tevatron (CDF, D0)**;
- **CERN LHC (ATLAS, CMS, LHCb)**.

**Heavy-quark production classic test of perturbative QCD!**

- **Low bottom mass (~4-5 GeV)**
- **Slow convergence**
- **Large theoretical uncertainties**
- **NNLO corrections!**
THEORETICAL STATUS: SHORT OVERVIEW

➤ **NLO QCD:**

[Nason, Dawson, Ellis; ‘88], [W. Beenakker, H. Kuijf, W. L. van Neerven, and J. Smith; ’89], [P. Nason, S. Dawson, and R. K. Ellis; ‘90], […]

➤ **b-hadron production:**

[B. A. Kniehl, G. Kramer, I. Schienbein, and H. Spiesberger; 1109.2472], [G. Kramer and H. Spiesberger; 1809.04297];

➤ **Resummation:**

[M. Cacciari and M. Greco; 9311260];

➤ **FONNL:**

[M. Cacciari, M. Greco, and P. Nason; 9803400], [M. Cacciari, S. Frixione, N. Houdeau, M. L. Mangano, P. Nason, and G. Ridolfi; 1205.6344];

➤ **NNLO QCD - total cross section:**

Results can be obtained with HATHOR [M. Aliev, H. Lacker, U. Langenfeld, S. Moch, P. Uwer, and M. Wiedermann; 1007.1327], which implements the computation of [Czakon et al.; 1204.5201, 1207.0236, 1210.6832, 1303.6254];

➤ **Differential NNLO QCD**

[S. Catani, SD, M. Grazzini, S. Kallweit, J. Mazzitelli; 2010.11906]
**BOTTOM QUARK PRODUCTION AT NNLO - INGREDIENTS**

- **DOUBLE REAL**
- **REAL - VIRTUAL**
- **TWO LOOP VIRTUAL**

Fast and stable evaluation with **OPENLOOPS 2**
[Cascioli et al. (2012), Buccioni et al. (2018), Buccioni et al. (2019)]

Numerically available
[Czakon (2008); Barnreuther et al. (2013)]

IR divergences cancel once all contributions are combined (KLN theorem) but they do not allow a straightforward implementation of **numerical techniques**.

We need a method to handle and cancel **IR singularities**!
SUBTRACTION METHODS

➤ NLO methods:
  • Catani-Seymour dipole subtraction [S. Catani, M. Seymour (1996)]
  • FKS subtraction [S. Frixione, Z. Kunszt, A. Signer (1996)]
  • …

➤ NNLO methods:
  • CoLoRfulNNLO [G. Somogyi, Z. Trocsanyi, V. Del Duca (2005)]
  • Antenna subtraction [T. Gehrmann, A. Gehrmann-De Ridder, N. Glover (2005)]
  • \( q_T \) subtraction formalism [S. Catani, M. Grazzini (2007)]
  • STRIPPER formalism [M. Czakon (2010); Boughezal et al (2011)]
  • \( N \)-jettiness subtraction [Boughezal, et al. (2015); Gaunt et al. (2015)]
  • Projection to Born [M. Cacciari, F. A. Dreyer, A. Karlberg, G. P. Salam (2015)]
  • Nested soft-collinear [F. Caola, K. Melnikov, R, Röntsch, (2017)]
  • Geometric subtraction [F. Herzog (2018)]
  • Local analytic sector [L. Magnea et al. (2018)]
  • …
The $q_T$ subtraction formalism is a method to handle and cancel IR divergences, originally developed for **colourless** final states.

The equation for the subtraction formalism is:

$$d\sigma_{(N)NLO}^F = d\sigma_{(N)NLO}^F \bigg|_{q_T=0} + d\sigma_{(N)NLO}^F \bigg|_{q_T \neq 0}$$

$q_T =$ transverse momentum of the system $F$

HARD COLLINEAR COEFFICIENT
Contains information on virtual corrections to the process.

Singularities for $q_t \neq 0$ can be computed with NLO subtraction techniques.

Extra singularities of NNLO type associated to the $q_t \rightarrow 0$ limit need additional subtraction. IR behaviour known from $q_t$ resummation formalism allow us to construct a counterterm.

**q_T SUBTRACTION FORMALISM FOR HEAVY QUARK PRODUCTION**

With the inclusion of extra contributions, \( q_T \) subtraction formalism can be extended to **massive coloured** final states.

\[
d\sigma^{Q\bar{Q}}_{\text{NNLO}} = \mathcal{C}^{Q\bar{Q}}_{\text{NNLO}} \otimes d\sigma^{Q\bar{Q}}_{\text{LO}} + \left[ d\sigma^{Q\bar{Q}+\text{jets}}_{\text{NLO}} - d\sigma^{\text{CT}}_{\text{NNLO}} \right]
\]

Contains the integrations of the additional final-state soft singularities.

It was recently computed by some of us.

[S. Catani, SD, M.Grazzini, J.Mazzitelli, in preparation. See also R. Angeles-Martinez, M. Czakon, S. Sapeta (2018)]

IR behaviour known from studies in \( q_T \) resummation


[Hai Tao Li, Chong Sheng Li, Ding Yu Shao, Li Lin Yang, Hua Xing Zu (2013)];
THE MATRIX PROJECT

MUnciCH

MUlti-ChaNnel Integrator at Swiss (CH) precision

OPEnLOOPs

(Collier, CutTools…)

TWO-LOOP AMP liTUDRES

(VVamp, GiNaC, tdhpl…)

qT subtraction

qT resummation

Munich Automates qT subtraction and Resummation to Integrate X-sections
Public code that allows the evaluation of fully differential cross sections at NNLO QCD for a wide class of processes with **colourless** final state.

- $pp \rightarrow H$
- $pp \rightarrow W\gamma \rightarrow l^+ \nu \gamma$
- $pp \rightarrow Z/\gamma^* (\rightarrow l^+ l^-)$
- $pp \rightarrow ZZ (\rightarrow 4l)$
- $pp \rightarrow W (\rightarrow l^+ \nu)$
- $pp \rightarrow WW (\rightarrow l^+ l^- \nu \nu')$
- $pp \rightarrow \gamma \gamma$
- $pp \rightarrow ZZ/WW \rightarrow ll\nu\nu$
- $pp \rightarrow Z\gamma \rightarrow l^+ l^- \gamma$
- $pp \rightarrow WZ \rightarrow l^+ l^- \gamma$
- $pp \rightarrow WW \rightarrow \ell^+ \ell^- \gamma$
- $pp \rightarrow WW \rightarrow \ell^+ \ell^- \nu \nu'$
- $pp \rightarrow ZZ/WW \rightarrow \ell^+ \ell^- \nu \nu'$
- $pp \rightarrow ZZ/WW \rightarrow \ell^+ \ell^- \nu \nu'$
- $pp \rightarrow ZZ/WW \rightarrow \ell^+ \ell^- \nu \nu'$
- $pp \rightarrow ZZ/WW \rightarrow \ell^+ \ell^- \nu \nu'$
TOP QUARK PRODUCTION IN MATRIX

[S. Catani, SD, M. Grazzini ,S.Kallweit, J. Mazzitelli, H. Sargsyan (2019)]

Inclusive cross section can be compared with the results of TOP++

<table>
<thead>
<tr>
<th>$\sigma_{\text{NNLO}}$ [pb]</th>
<th>MATRIX</th>
<th>TOP++</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 TeV</td>
<td>238.5(2)+3.9% $-6.3%$</td>
<td>238.6 $+4%$ $-6.3%$</td>
</tr>
<tr>
<td>13 TeV</td>
<td>794.0(8)+3.5% $-5.7%$</td>
<td>794.0 $+3.5%$ $-5.7%$</td>
</tr>
<tr>
<td>100 TeV</td>
<td>35215(74)+2.8% $-4.7%$</td>
<td>35216+2.9% $-4.8%$</td>
</tr>
</tbody>
</table>

Per-mille accuracy in $\sim$1000 CPU days

- **Differential Distributions** $\sim$ S. Catani, SD, M. Grazzini ,S.Kallweit, J. Mazzitelli: JHEP 07 (2019) 100
- **Predictions in the $\overline{MS}$ scheme** $\sim$ S. Catani, SD, M. Grazzini ,S.Kallweit, J. Mazzitelli: JHEP 08 (2020) 08, 027
A TOP–BOTTOM APPROACH

Implementation of bottom-pair production in the MATRIX framework:

Extend the implementation of top-pair production to:

- lower heavy quark mass: $m = 173.3$ GeV $\rightarrow m = 4.92$ GeV
- different number of light flavours: $n_f = 5$ $\rightarrow n_f = 4$

Massive bottom quark, top quark decoupled from the process

Two-loop grid [Czakon (2008); Barnreuther et al. (2013)] so far used only for top-pair production.

Is the phase space sufficiently sampled also for bottom-pair production?

- Effect of the two loop contribution: $\mathcal{O}(1\%)$
- Effect of reducing the number of points of the grid by a factor of 2: $\mathcal{O}(1\%)$

The results of [Czakon (2008); Barnreuther et al. (2013)] can be safely used in our computation!
A TOP–BOTTOM APPROACH

[S. Catani, SD, M. Grazzini, S. Kallweit, J. Mazzitelli: JHEP 03 (2021) 029]

The small mass of the bottom quark might challenge the cancellation of IR singularities

\[ d\sigma_{(N)NLO}^{F} = \mathcal{H}_{(N)NLO}^{F} \otimes d\sigma_{LO}^{F} + \left[ d\sigma_{(N)LO}^{F+jets} - d\sigma_{(N)LO}^{CT} \right] \]

\[ d\sigma_{(N)LO}^{F+jets} \text{ and } d\sigma_{(N)LO}^{CT} \text{ are separately divergent.} \]

In practice, qT subtraction is implemented as a slicing method:

➤ introducing a cutoff \( r_{\text{cut}} = Q/M \);
➤ performing the limit \( r_{\text{cut}} \to 0 \)

Stability of the subtraction procedure can be understood looking at the \( r_{\text{cut}} \) dependence:

- Hathor
- \( \sigma_{\text{NLO}} \)
- \( \sigma_{\text{NLO}}^{r} \)

\[ pp \to bb @ 1.96 \text{ TeV}, \mu_0 = m_b \]

\[ pp \to bb @ 13 \text{ TeV}, \mu_0 = m_b \]
**Central scale:**  
$\mu_0 = m_b = 4.92 \, \text{GeV}$

**7-points scale variations:**  
\[ \frac{1}{2} \mu_0 < \mu_F, \mu_R < 2\mu_0 \]
\[ \frac{1}{2} < \frac{\mu_F}{\mu_R} < 2 \]

Excellent agreement with HATHOR [1007.1327]!

### Matrix Predictions

<table>
<thead>
<tr>
<th>$\sigma$ [(\mu\text{b})]</th>
<th>LO</th>
<th>NLO</th>
<th>NNLO</th>
<th>$K_{NLO}$</th>
<th>$K_{NNLO}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p\bar{p} \text{ @ } 1.96 , \text{TeV}$</td>
<td>34.66$^{+51%}_{-32%}$</td>
<td>60.23$^{+54%}_{-28%}$</td>
<td>75.4(3)$^{+22%}_{-21%}$</td>
<td>1.74</td>
<td>1.25</td>
</tr>
<tr>
<td>$pp \text{ @ } 7 , \text{TeV}$</td>
<td>138.7$^{+51%}_{-46%}$</td>
<td>219.8$^{+61%}_{-39%}$</td>
<td>288(2)$^{+30%}_{-24%}$</td>
<td>1.58</td>
<td>1.31</td>
</tr>
<tr>
<td>$pp \text{ @ } 13 , \text{TeV}$</td>
<td>249.0$^{+59%}_{-51%}$</td>
<td>378.6$^{+65%}_{-45%}$</td>
<td>508(3)$^{+32%}_{-25%}$</td>
<td>1.52</td>
<td>1.34</td>
</tr>
</tbody>
</table>

**$K$-factor** - ratio to the previous perturbative order:  
$K_{NLO} = \sigma_{NLO}/\sigma_{LO}$  
$K_{NNLO} = \sigma_{NNLO}/\sigma_{NLO}$

Poor perturbative behaviour:  
- large **scale uncertainties**;  
- large **K-factors**.

The inclusion of NNLO correction significantly improves both!
DIFFERENTIAL DISTRIBUTIONS

➤ We computed single differential distributions at Tevatron (1.96 TeV) and LHC (7 and 13 TeV);

➤ We reduced theoretical uncertainties by considering ratios at different energies;

➤ We compared the pseudorapidity distribution with recent measurements from LHCb [LHCb-PAPER-2016-031].

Renormalisation and factorisation scales, \( \mu_R \) and \( \mu_F \), are chosen of the order of the characteristic hard scale.

➤ We considered the ratio between predictions at LHC with \( \sqrt{s} = 7 \) and 13 TeV;

➤ Theoretical uncertainties are reduced by assuming correlation between scale variations at different energies (reduced sensitivity).

<table>
<thead>
<tr>
<th>Hard scale</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Total cross section</td>
<td>( m_b )</td>
</tr>
<tr>
<td>Rapidity distribution</td>
<td>( m_b )</td>
</tr>
<tr>
<td>Invariant mass distribution</td>
<td>( m_{\bar{b}b} )</td>
</tr>
<tr>
<td>Transverse momentum distribution</td>
<td>( m_T )</td>
</tr>
</tbody>
</table>

[S. Catani, SD, M. Grazzini, S. Kallweit, J. Mazzitelli: JHEP 03 (2021) 029]
Differential Distributions: Tevatron

[S. Catani, SD, M. Grazzini, S. Kallweit, J. Mazzitelli: JHEP 03 (2021) 029]

$p\bar{p} \rightarrow b\bar{b}$ @ 1.96 TeV, $\mu_0 = m_T$

#### Total Cross Section with rapidity cut

\[ \sigma_{LO}^{b\bar{b}}(|y_{b\bar{b}}| < 0.6) = 7.840(3)_{-34}^{+51} \mu b \]

\[ \sigma_{NLO}^{b\bar{b}}(|y_{b\bar{b}}| < 0.6) = 14.282(6)_{-28}^{+53} \mu b \]

\[ \sigma_{NNLO}^{b\bar{b}}(|y_{b\bar{b}}| < 0.6) = 17.87(12)_{-21}^{+22} \mu b \]

\[ \sigma_{Hb}(|y_{Hb}| < 0.6) = 17.6 \pm 0.4_{-2.3}^{+2.5} \mu b \]

- LO and NLO compatible only at low $p_T$, NLO and NNLO compatible in all the phase space;
- The inclusion of a rapidity cut does not change the shape of the distribution;
- Total Cross section with rapidity cut can be compared with CDF measurements for b-hadron production [Phys. Rev. D 71 (2005) 032001].
➤ LO and NLO compatible only at low $p_T$, NLO and NNLO compatible in all the phase space;

➤ Similar behaviour at different energies;

➤ Strong **reduction of uncertainty bands** in the ratio: from $\sim 30\%$ (at low $p_T$), $\sim 10\%$ (in the tail) to $\sim 5\%$. 
Since the impact of QCD correction is **uniform in rapidity**, higher order corrections do not change the shape of the distribution;

- NNLO results completely **overlap** with the NLO predictions and reduce the scale uncertainties;

- The ratio strongly **reduces the scale uncertainties** to ~5%.

- The stability of the perturbative expansion in the ratio **confirms** the hypothesis of correlation of the scale variation.
Differential Distributions: LHC

We compare with the results presented by the LHCb collaboration in [LHCb-PAPER-2016-031];

- Measured distributions overlap with NNLO predictions at both energies, but the shape is different.

- In the distributions the theoretical uncertainties are larger than the experimental errors, in their ratio the theoretical uncertainties are smaller than the experimental errors.

- In the ratio, data systematically above prediction (systematic and η-independent uncertainty?).
COMPARISON WITH FONLL

[S. Catani, SD, M. Grazzini, S. Kallweit, J. Mazzitelli: JHEP 03 (2021) 029]

➤ At high $p_T$ large logarithms in the expansion: they need to be **resummed** at all orders.

➤ **FONLL** predictions combine NLO calculations with resummed computations, our **NNLO** calculation **ignores** the large logarithms.

➤ Comparison between our NNLO and FONLL: **estimate** of the impact of large logs.

➤ FONLL and NNLO uncertainty bands **overlap**, NNLO has **smaller** uncertainty.

In the phase-space region considered, the NNLO prediction is reliable!
The computation is **fully implemented** within a (at the moment) private release of **MATRIX**;

- Predictions provided upon request;

- Our computation has already been used by experimental collaborations for data-theory comparisons.

Plot from **arXiv:2102.13601** (ALICE collaboration). The NNLO prediction (green) has been computed with **MATRIX**.
SUMMARY & OUTLOOK
We have presented the first differential computation for bottom-quark production at NNLO;

We have shown results for inclusive and differential distributions, comparison with experimental data;

The process is an extension of the computation for top-pair production and has been implemented into the MATRIX framework;

Theoretical uncertainties are still large but significantly reduced by the inclusion of NNLO corrections.
SUMMARY & OUTLOOK

» New public MATRIX release with the inclusion of heavy quark production;
» Improvements of NNLO QCD:
  • b-jet production?
  • resummation effects?
  • fragmentation effects?
BACKUP SLIDES
Single and multi-differential distributions for top-pair production have also been implemented in the MATRIX framework!

Shown here: plots from [JHEP 07 (2019) 100], where we compare our predictions with recent measurements from CMS in the leptons + jet channels [CMS-TOP-17-002].
**TOTAL CROSS SECTION AND UNCERTAINTIES**

![Table with cross section and uncertainties](image)

- **Bottom mass uncertainty**: variation between $m_b = 4.79$ GeV and $m_b = 5.05$ GeV ($m_b = 4.92 \pm 0.13$ GeV);

- **QCD coupling uncertainty**: evaluating the cross section with NNPDF31 NNLO PDFs obtained with $\alpha_S(m_Z) = 0.119$ and $\alpha_S(m_Z) = 0.117$

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Despite the inclusion of NNLO corrections, the missing higher orders in the QCD perturbative expansion still represent the dominant source of theoretical uncertainty!
TOTAL CROSS SECTION AND SCALE CHOICES

<table>
<thead>
<tr>
<th>$\sigma$ [$\mu$b]</th>
<th>$p\bar{p}$ @ 1.96 TeV</th>
<th>$pp$ @ 7 TeV</th>
<th>$pp$ @ 13 TeV</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\mu_0 = m_b$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>LO</td>
<td>34.66 $^{+51%}_{-32%}$</td>
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<td>288(2) $^{+30%}_{-24%}$</td>
<td>508(3) $^{+32%}_{-25%}$</td>
</tr>
<tr>
<td>$\mu_0 = 2m_b$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LO</td>
<td>30.94 $^{+41%}_{-25%}$</td>
<td>145.8 $^{+41%}_{-32%}$</td>
<td>281.9 $^{+41%}_{-37%}$</td>
</tr>
<tr>
<td>NLO</td>
<td>51.16 $^{+33%}_{-23%}$</td>
<td>203.3 $^{+36%}_{-26%}$</td>
<td>362.9 $^{+34%}_{-28%}$</td>
</tr>
<tr>
<td>NNLO</td>
<td>66.7(2) $^{+21%}_{-18%}$</td>
<td>258(1) $^{+20%}_{-18%}$</td>
<td>458(2) $^{+20%}_{-18%}$</td>
</tr>
</tbody>
</table>

The use of the central scale $\mu_0 = m_b$ leads to larger scale uncertainties due to the lower values of $\mu_R$ involved.

Nevertheless, despite the low value of $m_b = 4.92$ GeV, we do not observe a worrisome perturbative behaviour.
We introduced the **dynamic scale** \( \frac{1}{2} H_T = \frac{m_{T,b} + m_{\bar{T}\bar{b}}}{2} \);

- At LO, \( H_T = m_T \): they indeed show a similar behaviour;
- The choice of a **smaller** scale \( \mu_0 = m_T \) leads to a **better** overlap between NLO and NNLO uncertainty bands.
Differential distributions

- Distributions with rapidity cut $|y| < 0.6$. 
NORMALISED DISTRIBUTIONS: LHC

[S. Catani, SD, M. Grazzini, S. Kallweit, J. Mazzitelli: JHEP 03 (2021) 029]
IR SINGULARITIES - NLO

\[ \Delta \sigma^{NLO} = \int d\sigma^{NLO} = \int_{m+1} \left[ d\sigma^R - d\sigma^{CT} \right] + \left[ \int_{m} d\sigma^V + \int_{1} d\sigma^{CT} \right] \]

**REAL**

- No explicit \( \epsilon \) poles;
- Singular in unresolved limit.

**VIRTUAL**

- Explicit poles up to order \( 1/\epsilon^2 \);
- No additional PS singularity.

Divergent

**CONVERGENT!**
**IR SINGULARITIES - NNLO**

\[
\Delta\sigma_{NNLO} = \int_{m+2} d\sigma^{RR} + \int_{m+1} d\sigma^{RV} + \int_{m} d\sigma^{VV}
\]

More complicated structure due to overlapping singularities!

**DOUBLE REAL**
- No \( \epsilon \) poles;
- Singular in (double) unresolved limit.

**REAL - VIRTUAL**
- Explicit \( 1/\epsilon^2 \) poles;
- Singular in unresolved limit.

**2LOOP VIRTUAL (AND 1LOOP SQUARED)**
- Explicit \( 1/\epsilon^4 \) poles;
- No additional PS singularity.
THE HARD-COLLINEAR COEFFICIENT

\[ d\sigma_{(N)NLO}^F = \mathcal{H}_{(N)NLO}^F \otimes d\sigma_{LO}^F + \left[ d\sigma_{(N)LO}^{F+\text{jets}} - d\sigma_{(N)LO}^{CT} \right] \]

**Colourless final state**

\[ \mathcal{H} \propto \langle \tilde{M} | \tilde{M} \rangle \]

\[ |\tilde{M}\rangle = (1 - I_C) |M\rangle \]

**Subtraction operator**

Removes the poles plus a finite part, taking into account what has been put in the counterterm.

**All loop renormalised virtual amplitude**

**Colourful final state**

\[ \mathcal{H} \propto \langle \tilde{M} \Delta | \tilde{M} \rangle \]

**Additional soft radiative factor**

\[ \mathcal{H} \propto \langle \tilde{M} | \Delta | \tilde{M} \rangle \]

**COMPUTATION**: integration of the additional soft emission from the final state.
**HARD COLLINEAR COEFFICIENT - NLO**

[S. Catani, M. Grazzini, A. Torre: arXiv 1408:4564]

Computation of the soft contribution

Integration of a suitably **subtracted** soft current.

**Computation performed in Fourier space.**

- $J_{\text{sub}}$ is the soft current after a proper subtraction of the (known) colourless contribution:

$$J_{i}^\mu(k) = \frac{p_i^\mu}{p_i \cdot k}$$

$$|J_{\text{sub}}(k)|^2 = \sum_{j=3,4} \frac{m_j^2}{(p_j \cdot k)^2} T_j^2 + \frac{2p_3 \cdot p_4}{p_3 \cdot k p_4 \cdot k} T_3 \cdot T_4 + \sum_{i=1,2} \sum_{j=3,4} \left( \frac{2}{p_i \cdot k} \left( \frac{p_i \cdot p_j}{p_j \cdot k} - \frac{p_1 \cdot p_2}{(p_1 + p_2) \cdot k} \right) \right) T_i \cdot T_j$$

Average on the azimuthal degrees of freedom, $\hat{q}_T \rightarrow \hat{b}$.
HARD COLLINEAR COEFFICIENT - NLO

\[ \bar{f}^{(1)}_{c\bar{c} \to Q\bar{Q}} \left( \epsilon, \frac{M^2}{\mu^2} \right) = - \frac{1}{2} \left( \frac{M^2}{\mu^2} \right)^{-\epsilon} \left\{ \frac{1}{e^2} + \frac{i\pi}{e} - \frac{\pi^2}{12} \right\} \left( T_1 + T_2 \right) + \frac{2}{\epsilon} \gamma_c \Gamma_t^{(1)}(y_{34}) + \frac{4}{\epsilon} \Gamma_t^{(1)}(y_{34}) + F_t^{(1)}(y_{34}) \]

Poles reproduce singularities in one-loop amplitudes

\[ F_t^{(1)}(y_{34}) = (T_3^2 + T_4^2) \ln \left( \frac{m_T^2}{m^2} \right) + (T_3 + T_4)^2 \text{Li}_2 \left( -\frac{p_T^2}{m^2} \right) + T_3 \cdot T_4 \frac{1}{v} L_{34} \]

\[ \Gamma_t^{(1)}(y_{34}) = -\frac{1}{4} \left\{ (T_3^2 + T_4^2) (1 - i\pi) + \sum_{\substack{i=1,2 \atop j=3,4}} T_i \cdot T_j \ln \left( \frac{(2p_i \cdot p_j)^2}{M^2 m^2} \right) \right\} + 2 T_3 \cdot T_4 \left[ \frac{1}{2v} \ln \left( \frac{1 + v}{1 - v} \right) - i\pi \left( \frac{1}{v} + 1 \right) \right] \].

\[ L_{34} = \ln \left( \frac{1 + v}{1 - v} \right) \ln \left( \frac{m_T^2}{m^2} \right) - 2 \text{Li}_2 \left( \frac{2v}{1 + v} \right) - \frac{1}{4} \ln^2 \left( \frac{1 + v}{1 - v} \right) + 2 \left[ \text{Li}_2 \left( 1 - \sqrt{\frac{1 - v}{1 + v} e^{y_{34}}} \right) + \text{Li}_2 \left( 1 - \sqrt{\frac{1 - v}{1 + v} e^{-y_{34}}} \right) + \frac{1}{2} y_{34}^2 \right] \]

\[ v = \sqrt{1 - \frac{m^4}{(p_3 \cdot p_4)^2}} \]

\[ y_{34} = y_3 - y_4 \]

We need to do the same at the next order!
**HARD COLLINEAR COEFFICIENT - NNLO**

**NNLO**: the soft current is more complicated. Contributions from:

- **Light quark pair production**:
  \[ J_{\text{sub}}^{\text{NNLO}(q\bar{q})}(k_1, k_2) \]
  \[ \int \frac{d^n k_1}{2\pi^{n-1}} \frac{d^n k_2}{2\pi^{n-1}} \delta_+(k_1) \delta_+(k_2) \left| J_{\text{sub}}^{\text{NNLO}(q\bar{q})}(k_1, k_2) \right|^2 e^{ib \cdot \vec{k}_{T1} + \vec{k}_{T2}} \]

- **Double gluon emission**:
  \[ J_{\text{sub}}^{\text{NNLO}(gg)}(k_1, k_2) \]
  \[ \int \frac{d^n k_1}{2\pi^{n-1}} \frac{d^n k_2}{2\pi^{n-1}} \delta_+(k_1) \delta_+(k_2) \left| J_{\text{sub}}^{\text{NNLO}(gg)}(k_1, k_2) \right|^2 e^{ib \cdot \vec{k}_{T1} + \vec{k}_{T2}} \]

- **One gluon emission at 1 loop**:
  \[ J_{\text{sub}}^{\text{NNLO}(1L)}(k) \]
  \[ \int \frac{d^n k}{2\pi^{n-1}} \delta_+(k) \left| J_{\text{sub}}^{\text{NNLO}(1L)}(k) \right|^2 e^{ib \cdot \vec{k}_r} \]
AN EXAMPLE: DOUBLE GLUON EMISSION

Square of the soft current:

\[
\left| J^{\text{NNLO}}(g g)(k_1, k_2) \right|^2 = \frac{1}{2} \left\{ J^2(k_1), J^2(k_2) \right\} - C_a \sum_{i,j=1}^{n} T_i \cdot T_j \mathcal{S}_{ij}(k_1, k_2)
\]

\[
\mathcal{S}_{ij}(k_1, k_2) = \mathcal{S}_{ij}^{m=0}(k_1, k_2) + \left( m_i^2 \mathcal{S}_{ij}^{m \neq 0}(k_1, k_2) + m_j^2 \mathcal{S}_{ji}^{m \neq 0}(k_1, k_2) \right)
\]

\[
\mathcal{S}_{ij}^{m=0}(q_1, q_2) = \frac{(1 - \epsilon) p_i \cdot q_1 p_j \cdot q_2 + p_i \cdot q_2 p_j \cdot q_1}{(q_1 \cdot q_2)^2 p_i \cdot (q_1 + q_2) p_j \cdot (q_1 + q_2)}
\]

\[
- \frac{(p_i \cdot p_j)^2}{2 p_i \cdot q_1 p_j \cdot q_2 p_i \cdot q_2 p_j \cdot q_1} \left[ 2 - \frac{p_i \cdot q_1 p_j \cdot q_2 + p_i \cdot q_2 p_j \cdot q_1}{p_i \cdot (q_1 + q_2) p_j \cdot (q_1 + q_2)} \right]
\]

\[
+ \frac{p_i \cdot p_j}{2 q_1 \cdot q_2} \left[ \frac{2}{p_i \cdot q_1 p_j \cdot q_2} + \frac{2}{p_j \cdot q_1 p_i \cdot q_2} - \frac{1}{p_i \cdot (q_1 + q_2) p_j \cdot (q_1 + q_2)} \right]
\]

\[
\times \left( 4 + \frac{(p_i \cdot q_1 p_j \cdot q_2 + p_i \cdot q_2 p_j \cdot q_1)^2}{p_i \cdot q_1 p_j \cdot q_2 p_i \cdot q_2 p_j \cdot q_1} \right)
\]

\[
\mathcal{S}_{ij}^{m \neq 0}(q_1, q_2) = - \frac{1}{4 q_1 \cdot q_2 p_i \cdot q_1 p_i \cdot q_2} + \frac{p_i \cdot p_j p_j \cdot (q_1 + q_2)}{2 p_i \cdot q_1 p_j \cdot q_2 p_i \cdot q_2 p_j \cdot q_1 p_i \cdot (q_1 + q_2)}
\]

\[
- \frac{1}{2 q_1 \cdot q_2 p_i \cdot (q_1 + q_2) p_j \cdot (q_1 + q_2)} \left( \frac{(p_j \cdot q_1)^2}{p_i \cdot q_1 p_j \cdot q_2} + \frac{(p_j \cdot q_2)^2}{p_i \cdot q_2 p_j \cdot q_1} \right)
\]
We computed all the needed integrals:

- **Analytic expression** for \( T_i T_j, T_j T_j \) contributions:

- **Numerical expression** for some pieces of the \( T_3 T_4 \) contribution (regular part of the double gluon emission):
Pole structure can be predicted from 2-loop virtual contribution: it must cancel (part of) its IR singularities.

- **Triple poles**: cancel combining the 3 different contributions;
- **Double poles**: analytic cancellation;
- **Single poles**: analytic cancellation except $T_3 T_4$.

The completion of this calculation allowed the implementation of top pair production and bottom pair production in the **MATRIX** framework!