Decoupage at 5 loops

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Work based on 2104.08272
with V. Goncalves, E. Panzer, R. Pereira, A. V. Smirnov and V. A Smirnov
We computed massless 5-loop p-integrals in $4 - 2\epsilon$ dimensions.
We computed massless 5-loop $p$-integrals in $4 - 2\epsilon$ dimensions.

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We computed massless 5-loop p-integrals in $4 - 2\epsilon$ dimensions. They depend on a single external momenta $p$, which we choose as $p^2 = 1$, as $(p^2)^{-\omega}$ where $\omega = \sum a_i - \ell d/2$.

Then a general p-integral $P$ can be written as

$$P(d) = (p^2)^{-\omega(P)} \sum_{n \in \mathbb{Z}} c_n(P) \epsilon^j$$

where all the $c_n$ for our case are rational numbers, $\zeta(i)$ or $\zeta(i, j)$ values.
Motivation

Loop integrals can be used to compute counterterms in different theories. They appear as boundary conditions in differential equations. Can be used for computing different physical quantities (structure constants, anomalous dimension, etc).

Cross-check that our results for $\hat{\zeta}_{p_i q}$ and $\hat{\zeta}_{p_i j q}$ transformation hold for all p-integrals (not only position space) and compute the missing momentum space non-planar contribution.
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- L-loop p-integrals can be used to compute l+1-loop counterterms.
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- Cross-check that our results for $\hat{\zeta}(i)$ and $\hat{\zeta}(i, j)$ transformation hold for all p-integrals (not only position space) and compute the missing momentum space non-planar contribution.
How?

We want to use the Glue-and-Cut method to compute the p-integrals.
Let us consider a vacuum integrals with $\omega = 0$ and no other sub-divergences.

\[
\frac{d^D p}{p^2 + m^2} \frac{P(\epsilon)}{(p^2)^{1+\epsilon}} = \frac{c_0}{(l+1)\epsilon} + O(\epsilon)
\]

The value of the p-integral is then:

\[
= \frac{c_0}{p^2 + O(\epsilon))}
\]

By cutting different edges we can construct convergent p-integrals that have the same value at $\epsilon^0$ order.
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- Generate vacuum 6-loop graphs. Check for “convergence” of vacuum integrals, allowing also for numerators.
We need vacuum integrals with at least 12 propagators. This is the minimal number in 4 dimensions to have $\omega = 0$. Higher number of propagators can be generated by adding numerators. For our computation we used convergent vacuum integrals with up to 2 numerators and 14 propagators.
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- Generate vacuum 6-loop graphs. Check for “convergence” of vacuum integrals, allowing also for numerators.
- Construct all possible cuts. Map integrals to a specific p-integral family.
We now need to cut all the possible vacuum integrals.
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In order to produce p-integrals with a higher number of propagators we also have to blow up higher valence vertices.

We then need to map all the p-integrals into one of the 64 possible 3-valent maximal topologies.
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- Generate vacuum 6-loop graphs. Check for “convergence” of vacuum integrals, allowing also for numerators.
- Construct all possible cuts. Map integrals to a specific p-integral family.
- Apply constraints and equivalence of the different cuts of the same vacuum integral.
How can we learn something from the equivalence of finite p-Integrals?
Constraints

How can we learn something from the equivalence of finite p-Integrals? With integral by parts reductions!

- IBPs are integration of a total derivative:

\[
0 = \int \frac{d^D \ell_1}{i\pi^{D/2}} \cdots \frac{d^D \ell_L}{i\pi^{D/2}} \sum_{j=1}^{L} \frac{\partial}{\partial \ell_j^\mu} \frac{\nu_j^{\mu}}{D_1^{\nu_1} \cdots D_m^{\nu_m}},
\]

- They can be used to express a general integral as a combination of Master Integrals:

\[
I = \sum_{i=1}^{N} d_i I_i,
\]
The constraints are then:

- Degree of divergence of a l-loop 2-point integral does not exceed l. Important as IBP coefficients can have poles in $4 - 2\epsilon$.
- Cancellation of poles as p-integrals obtain through cutting are finite.
Reduction of two-loop propagator integral

\[
\begin{array}{c}
\text{\begin{tikzpicture}
\draw[thick] (0,0) -- (1,0);
\draw[thick] (0.5,0.5) -- (0.5,-0.5);
\end{tikzpicture}}
= \frac{2(3d - 10)(8 - 3d)}{(d - 4)^2}
\end{array}
- \frac{2(d - 3)}{d - 4}
\begin{tikzpicture}
\draw[thick] (0,0) -- (1,0);
\draw[thick] (0.5,0.5) -- (0.5,-0.5);
\end{tikzpicture}
\]
Reduction of two-loop propagator integral

\[
\frac{2(3d - 10)(8 - 3d)}{(d - 4)^2} - \frac{2(d - 3)}{d - 4}
\]

The spurious pole give a constrain on two epsilon orders. Convergence of the starting integral and the insertion of the value of the trivial integral \( M_{\text{Dbubble}} = \frac{1}{\epsilon^2} \) gives:

\[
M_{\text{Sun}} = -\frac{1}{4\epsilon} - \frac{5}{8} - \frac{27}{16}\epsilon
\]
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\[ \frac{2(3d - 10)(8 - 3d)}{(d - 4)^2} - \frac{2(d - 3)}{d - 4} \]

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\[ M_{Sun} = -\frac{1}{4\epsilon} - \frac{5}{8} - \frac{27}{16} \epsilon \]

Same holds at higher loops
The constraints are then:

- Degree of divergence of a 1-loop 2-point integral does not exceed 1. Important as IBP coefficients can have poles in $4 - 2\epsilon$.
- Cancellation of poles as p-integrals obtained through cutting are finite.
- Different cuts of the same vacuum integral are the same.
The constraints are then:

- Degree of divergence of a 1-loop 2-point integral does not exceed 1. Important as IBP coefficients can have poles in $4 - 2\epsilon$.
- Cancellation of poles as p-integrals obtain through cutting are finite.
- Different cuts of the same vacuum integral are the same.

With this constraint we are able to relate all the coefficients of 5-loop master integrals, up to transcendental weight 9, to recursive 1-loop integrals and a product integral.
Recursive 1-loop

By recursively applying the bubble integration we can compute several integrals that appear at all loops.

\[
\int \frac{d^d \ell}{\pi^{d/2}} \frac{1}{\ell^{2a} (p - \ell)^{2b}} = (p^2)^{d/2-a-b} G(a, b),
\]

with \( G \) defined as

\[
G(a, b) = \frac{\Gamma(a + b - d/2) \Gamma(d/2 - a) \Gamma(d/2 - b)}{\Gamma(a) \Gamma(b) \Gamma(d - a - b)}.
\]
By recursively applying the bubble integration we can compute several integrals that appear at all loops.

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with $G$ defined as

$$G(a, b) = \frac{\Gamma(a + b - d/2)\Gamma(d/2 - a)\Gamma(d/2 - b)}{\Gamma(a)\Gamma(b)\Gamma(d - a - b)} . \tag{1}$$

For example we can compute the sunset diagram at two loops:

$$= G(1, 1) G(2 - d/2, 1)$$

The same recursion can be done to compute the general watermelon diagram at each loop.
The constraints are then:

- Degree of divergence of a l-loop 2-point integral does not exceed l. Important as IBP coefficients can have poles in $4 - 2\epsilon$.
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- Different cuts of the same vacuum integral are the same.

With this constraint we are able to relate all the coefficients of 5-loop master integrals, up to transcendental weight 9, to recursive 1-loop integrals and a product integral.
The constraints are then:

- Degree of divergence of a $l$-loop 2-point integral does not exceed $l$. Important as IBP coefficients can have poles in $4 - 2\epsilon$.
- Cancellation of poles as p-integrals obtain through cutting are finite.
- Different cuts of the same vacuum integral are the same.

With this constraint we are able to relate all the coefficients of 5-loop master integrals, up to transcendental weight 9, to recursive 1-loop integrals and a product integral.
There seem to exist a ($\epsilon$ dependent) redefinition of the $\zeta$ values such that the $\pi$ dependent terms cancel for all the p-integrals.

$$\hat{\zeta}(3) = \zeta(3) + \frac{3\epsilon}{2} \zeta(4) - \frac{5\epsilon^3}{2} \zeta(6) + \frac{21\epsilon^5}{2} \zeta(8),$$

$$\hat{\zeta}(5) = \zeta(5) + \frac{5\epsilon}{2} \zeta(6) - \frac{35\epsilon^3}{4} \zeta(8),$$

$$\hat{\zeta}(7) = \zeta(7) + \frac{7\epsilon}{2} \zeta(8),$$

$$\hat{\zeta}(3, 5) = \zeta(3, 5) - \frac{29}{12} \zeta(8) - \frac{15\epsilon}{2} \zeta(4)\zeta(5),$$

$$\hat{\zeta}(9) = \zeta(9).$$
There seem to exist a (ε dependent) redefinition of the ζ values such that the π dependent terms cancel for all the p-integrals.

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\hat{\zeta}(7) = \zeta(7) + \frac{7\varepsilon}{2} \zeta(8), \\
\hat{\zeta}(3, 5) = \zeta(3, 5) - \frac{29}{12} \zeta(8) - \frac{15\varepsilon}{2} \zeta(4) \zeta(5), \\
\hat{\zeta}(9) = \zeta(9).
\]

This type of relations, through the no-π theorem are related to some cancellations/relations of π terms in correlations functions, anomalous dimensions and β-functions.
Using the Glue-and-Cut method we were able to compute all the 5-loop 2 point master integral. Our result are up to transcendental weight $\zeta(9)$, which is the $\epsilon^0$ order of the finite $p$-integrals.

We have checked that our $\hat{\zeta}$ representation works also for momentum space $p$-integrals in the non-planar case.
Conclusions & Outlook

- Using the Glue-and-Cut method we were able to compute all the 5-loop 2 point master integral. Our result are up to transcendental weight $\zeta(9)$, which is the $\epsilon^0$ order of the finite $p$-integrals.
- We have checked that our $\hat{\zeta}$ representation works also for momentum space $p$-integrals in the non-planar case.

- A major bottleneck are the IBP reductions. If one would like to push this to higher loops we need better tools.
- Would be interesting to apply the same method to different dimensions, for example $d = 3$. 
Thank You!