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Two-Loop Rational Terms

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based on

JHEP 05 (2020) 077 [2001.11388]
JHEP 10 (2020) 016 [2007.03713]
[2106.XXXX]

RADCOR & LoopFest 2021

Motivation: automated two-loop calculations

- Higher-order calculations are usually performed in $D = 4-2\varepsilon$ dimensions to regularise divergences in Feynman integrals, but D-dim vectors cannot be implemented in a numerical program
- Automated one-loop numerical algorithms usually construct numerators of loop integrands in 4-dim, and the extension to two-loop amplitudes is under development [see Max Zoller's talk]
- **Rational counterterms** reconstruct missing terms originating from $(D-4)$-dim part of loop numerator

  $\Rightarrow$ one loop: rational terms of type R_2 [Ossola, Papadopoulos, Pittau]

  $\Rightarrow$ in this talk:

  - General structure of **two-loop rational counterterms**
  - Extension to **generic renormalisation schemes**
  - **New techniques** for calculations in spontanesouly broken gauge theories

Introduction to one-loop rational terms

Amplitude of an one-loop diagram γ in $D = 4 - 2\varepsilon$ dimensions

$$\bar{\mathcal{A}}_{1,\gamma} \;=\; \mu^{2\varepsilon} \int \mathrm{d}\bar{q}_1 \, \frac{\bar{\mathcal{N}}(\bar{q}_1)}{D_0(\bar{q}_1)\cdots D_{N-1}(\bar{q}_1)} \quad \text{with} \quad D_k(\bar{q}_1) = (\bar{q}_1 + p_k)^2 - m_k^2$$

Rational term emerges by splitting D-dim numerator into 4-dim and ε-dim parts

$$\bar{\mathcal{N}}(\bar{q}_1) \;=\; \mathcal{N}(q_1) + \tilde{\mathcal{N}}(\bar{q}_1) \quad \text{with} \quad \begin{cases} \bar{q} &= q + \tilde{q} \\ \bar{\gamma}^{\bar{\mu}} &= \gamma^{\mu} + \tilde{\gamma}^{\tilde{\mu}} \\ \bar{g}^{\bar{\mu}\bar{\nu}} &= g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}} \end{cases}$$

leads to

$$\bar{\mathcal{A}}_{1,\gamma} \;=\; \underbrace{\mathcal{A}_{1,\gamma}}_{\substack{\text{computed}\\ \text{numerically}}} \;+\; \underbrace{\delta\mathcal{R}_{1,\gamma}}_{\substack{\text{computed}\\ \text{analytically}}}$$

- $\delta\mathcal{R}_{1,\gamma}$ from interplay between ε-dim $\tilde{\mathcal{N}}$ and $\frac{1}{\varepsilon}$ UV poles $\Rightarrow$ require technique to extract UV poles

Tadpole decomposition [Chetyrkin, Misiak, Münz]

The UV divergence can be captured by **massive tadpole decomposition** of denominators

$$\frac{1}{D_k(\bar{q}_1)} = \underbrace{\frac{1}{\bar{q}_1^2 - M^2}}_{\substack{\text{leading UV tadpole} \\ \mathcal{O}(1/\bar{q}_1^2)}} + \underbrace{\frac{\Delta_k(\bar{q}_1, p_k)}{\bar{q}_1^2 - M^2} \frac{1}{D_k(\bar{q}_1)}}_{\substack{\text{subleading UV term} \\ \mathcal{O}(1/\bar{q}_1^3)}}$$

with

$$\Delta_k(\bar{q}_1, p_k) = -p_k^2 - 2\,\bar{q}_1 \cdot p_k + m_k^2 - M^2$$

Apply recursively to obtain **tadpole expansion ($\mathbf{S}_X$)** up to order $(1/\bar{q}_1)^{X+2}$

$$\frac{1}{D_k(\bar{q}_1)} = \underbrace{\sum_{\sigma=0}^{X} \text{UV-div. tadpoles}}_{\mathbf{S}_X(1/D_k)} + \text{UV-finite remainder}$$

Rational terms from UV divergences

- Use tadpole expansions to fully **isolate UV divergent part**

$$\bar{\mathcal{A}}_{1,\gamma} = \underbrace{\int \mathrm{d}\bar{q}_1 \sum_{\sigma=0}^{X} \frac{(\mathcal{N}(q_1) + \tilde{\mathcal{N}}(\tilde{q}_1))\Delta^{(\sigma)}}{\left(\bar{q}_1^2 - M^2\right)^{N+\sigma}}}_{\text{UV-divergent tadpoles } \mathbf{S}_X \bar{\mathcal{A}}_{1,\gamma}} + \text{ UV-finite remainder}$$

- Extract **full UV part** with numerator split into 4-dim and ε-dim

$$\mathbf{S}_X \bar{\mathcal{A}}_{1,\gamma} = \underbrace{-\delta Z_{1,\gamma}}_{\frac{1}{\varepsilon}\,\overline{\mathrm{MS}}\text{ pole}} + \underbrace{\delta \mathcal{R}_{1,\gamma}}_{\substack{\text{finite} \\ \text{rational term}}} + \text{ finite part}$$

- $\delta\mathcal{R}_{1,\gamma}$ and $\delta Z_{1,\gamma}$ from same UV divergence $\Rightarrow$ $\delta\mathcal{R}_{1,\gamma}$ local counterterm like $\delta Z_{1,\gamma}$

Renormalisation of irreducible two-loop diagrams

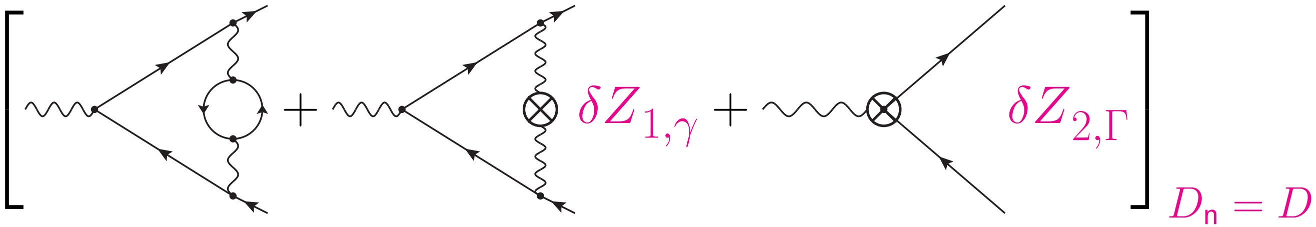
Renormalisation of D-dim amplitude of diagram Γ with $\mathbf{R}$-operation [BPHZ; Caswell and Kennedy]

$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} \;=\; \bar{\mathcal{A}}_{2,\Gamma} + \sum_{\gamma} \underbrace{\delta Z_{1,\gamma}}_{\substack{\text{sub-div.}\\ \text{subtraction}}} \cdot \bar{\mathcal{A}}_{1,\Gamma/\gamma} + \underbrace{\delta Z_{2,\Gamma}}_{\substack{\text{local two-loop}\\ \text{div. subtraction}}}$$

- At 1PI vertex function level ($\Gamma =$ set of diagrams), $\delta Z_{1,\gamma}$ and $\delta Z_{2,\Gamma}$ are local counterterms that can be implemented in the Lagrangian

Example: single QED diagram

$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} =$



where $D_{\mathsf{n}} =$ numerator dimension

Structure of two-loop UV rational terms [Pozzorini, Zhang, Zoller]

Master formula for renormalisation of amplitude in $D_\mathrm{n} = 4$ numerator dimension

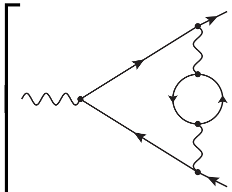
$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} = \Bigg[\mathcal{A}_{2,\Gamma} + \sum_{\gamma} \underbrace{\left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right)}_{\substack{\text{sub-divergence subtraction} \\ +\ \text{rational part reconstruction}}} \cdot \mathcal{A}_{1,\Gamma/\gamma} + \underbrace{\left(\delta Z_{2,\Gamma} + \delta\mathcal{R}_{2,\Gamma}\right)}_{\substack{\text{local two-loop divergence} \\ \text{subtraction} \,+\, \text{reconstruction}}} \Bigg]_{D_\mathrm{n}=4} + \mathcal{O}(\varepsilon)$$

- $\delta\tilde{Z}_{1,\gamma}$ is a new one-loop counterterm $\propto \tilde{q}^2/\varepsilon \equiv \mathcal{O}(1)$ that arises from quadratically divergent one-loop subdiagrams in $D_\mathrm{n} = 4$ (see backup slides)

- $\delta\mathcal{R}_{2,\Gamma}$ originates from local two-loop divergence $\Rightarrow$ **process-independent** local couterterm (proof in [2001.11388])

Example: single QED diagram

$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma}\Big|_{D_\mathrm{n}=D} = \Bigg[$$



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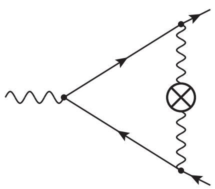
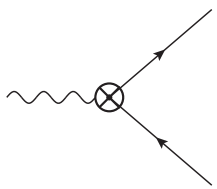


$\left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right) +$



$$\left(\delta Z_{2,\Gamma} + \delta\mathcal{R}_{2,\Gamma}\right)\Bigg]_{D_\mathrm{n}=4} + \mathcal{O}(\varepsilon)$$

Derivation of two-loop rational terms

Rational terms can be derived **once and for all** by reverting the master formula, and its calculations can be **simplified into massive tadpole integrals** by tadpole expansion $\mathbf{S}$

$$
\begin{aligned}
\delta\mathcal{R}_{2,\Gamma} &= \mathbf{S}\left[\bar{\mathcal{A}}_{2,\Gamma} + \sum_{\gamma} \delta Z_{1,\gamma} \cdot \bar{\mathcal{A}}_{1,\Gamma/\gamma}\right]_{D_{\mathsf{n}}=D} \\
&\quad - \mathbf{S}\left[\mathcal{A}_{2,\Gamma} + \sum_{\gamma} \left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right) \cdot \mathcal{A}_{1,\Gamma/\gamma_i}\right]_{D_{\mathsf{n}}=4}
\end{aligned}
$$

- Full set of $\delta\mathcal{R}_{2,\Gamma}$ rational terms for QED presented in [2001.11388]

Extension to generic renormalisation schemes [Lang, Pozzorini, Zhang, Zoller]

In a scheme Y with arbitrary finite renormalisations, we write renormalisation constants (RCs) as

$$\mathcal{Z}_\chi^{(Y)} = 1 + \sum_{k=1}^{\infty} \underbrace{\left(t_Y^\varepsilon\right)^k}_{\text{scale factor}} \left(\underbrace{\delta\mathcal{Z}_{k,\chi}^{(\mathrm{MS})}}_{\mathrm{MS}\ \text{pole}} + \underbrace{\delta\mathcal{Z}_{k,\chi}^{(\Delta Y)}}_{\text{finite part}} \right)\Bigg|_{\alpha=\alpha_Y(\mu_R)} \quad \text{for } \chi = \begin{cases} \text{parameter } \theta_i \\ \text{field } \psi_j \end{cases}$$

Renormalised 1PI vertex function in scheme Y

$$\begin{aligned} \mathbf{R}^{(Y)}\,\bar{\mathcal{A}}_{1,\Gamma} &= \mathcal{A}_{1,\Gamma} + \delta Z_{1,\Gamma}^{(Y)} + \delta\mathcal{R}_{1,\Gamma}^{(Y)}\,, \\ \mathbf{R}^{(Y)}\,\bar{\mathcal{A}}_{2,\Gamma} &= \mathcal{A}_{2,\Gamma} + \sum_\gamma \left(\delta Z_{1,\gamma}^{(Y)} + \delta\tilde{Z}_{1,\gamma}^{(Y)} + \delta\mathcal{R}_{1,\gamma}^{(Y)}\right)\cdot \mathcal{A}_{1,\Gamma/\gamma} + \delta Z_{2,\Gamma}^{(Y)} + \delta\mathcal{R}_{2,\Gamma}^{(Y)}\,. \end{aligned}$$

- UV counterterms $\delta Z_{i,\gamma}^{(Y)}$ are fully controlled by RCs at the level of Lagrangian
- One-loop $\delta\mathcal{R}_{1,\gamma}^{(Y)}$ and $\delta\tilde{Z}_{1,\gamma}^{(Y)}$ contain only trivial scheme dependence through scale factor t_Y^ε

$$\delta\mathcal{R}_{1,\gamma}^{(Y)} = t_Y^\varepsilon\,\delta\mathcal{R}_{1,\gamma}^{(\mathrm{MS})}\,, \quad \delta\tilde{Z}_{1,\gamma}^{(Y)} = t_Y^\varepsilon\,\delta\tilde{Z}_{1,\gamma}^{(\mathrm{MS})}\,, \quad \text{e.g. } t_{\overline{\mathrm{MS}}}^\varepsilon = \left(4\pi e^{-\gamma}\frac{\mu_0^2}{\mu_R^2}\right)^\varepsilon$$

Renormalisation scheme dependence of $\delta\mathcal{R}_{2,\Gamma}^{(Y)}$

Universal scheme dependence of $\delta\mathcal{R}_{2,\Gamma}^{(Y)}$ can be written as (set $t_Y^{\varepsilon}=1$)

$$\delta\mathcal{R}_{2,\Gamma}^{(Y)} \;=\; \delta\mathcal{R}_{2,\Gamma}^{(\mathrm{MS})} + D_1^{(\Delta Y)}\,\delta\mathcal{R}_{1,\Gamma}^{(\mathrm{MS})} + \delta\mathcal{K}_{2,\Gamma}^{(\Delta Y)}$$

- It contains **finite multiplicative renormalisation** of one-loop rational term

$$D_1^{(\Delta Y)}\,\delta\mathcal{R}_{1,\Gamma}^{(\mathrm{MS})} \;=\; \left(\sum_i \delta\mathcal{Z}_{1,\theta_i}^{(\Delta Y)}\theta_i\frac{\partial}{\partial\theta_i} + \sum_j \frac{1}{2}\delta\mathcal{Z}_{1,\psi_j}^{(\Delta Y)}\right)\delta\mathcal{R}_{1,\Gamma}^{(\mathrm{MS})}$$

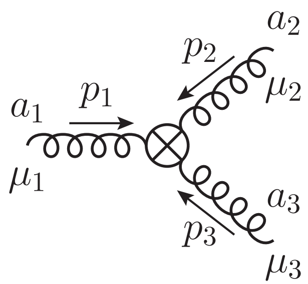
- and additional $\delta\mathcal{K}_{2,\Gamma}^{(\Delta Y)}$ term originates from **non-commutativity** of finite renormalisation ($D_1^{(\Delta Y)}$) and 4-dim projection of loop numerators ($\mathbf{P}_4$)

$$\delta\mathcal{K}_{2,\Gamma}^{(\Delta Y)} \;=\; \left[D_1^{(\Delta Y)}, \mathbf{P}_4\right]\bar{\mathcal{A}}_{1,\Gamma} \;\neq\; 0$$

which can be controlled through a new kind of **process-independent** one-loop counterterms (see [2007.03713] for further details)

Results of two-loop QCD rational counterterms

Example: Triple-gluon function in renormalisation scheme Y



$$= g_s\, f^{a_1 a_2 a_3}\, V^{\mu_1,\mu_2,\mu_3}(p_1,p_2,p_3) \left\{ \sum_{k=1}^{2} \left(\frac{\alpha_s\, t_Y^{\varepsilon}}{4\pi} \right)^k \delta\hat{\mathcal{R}}^{(Y)}_{k,\mathrm{ggg}} \right\},$$

with $V^{\mu_1,\mu_2,\mu_3}(p_1,p_2,p_3) = g^{\mu_1\mu_2}(p_1-p_2)^{\mu_3} + g^{\mu_2\mu_3}(p_2-p_3)^{\mu_1} + g^{\mu_3\mu_1}(p_3-p_1)^{\mu_2}$, and rational terms

$$\delta\hat{\mathcal{R}}^{(Y)}_{1,\mathrm{ggg}} = -\frac{11}{12} C_A - \frac{4}{3} T_F\, n_f\,,$$

$$\begin{aligned}\delta\hat{\mathcal{R}}^{(Y)}_{2,\mathrm{ggg}} = &-\left[\frac{11}{48} C_A^2 + T_F\, n_f \left(\frac{23}{6} C_A - \frac{8}{3} C_F\right)\right] \varepsilon^{-1} + T_F\, n_f \left(\frac{25}{9} C_A - \frac{119}{36} C_F\right) + \frac{145}{288} C_A^2 \\ &\underbrace{-\left(\frac{11}{8} C_A + 2\, T_F\, n_f\right) \delta\hat{\mathcal{Z}}^{(Y)}_{1,\alpha_s} - \left(\frac{13}{4} C_A + 2\, T_F\, n_f\right) \delta\hat{\mathcal{Z}}^{(Y)}_{1,A} + \frac{5}{4} C_A\, \delta\hat{\mathcal{Z}}^{(Y)}_{1,\mathrm{gp}} - \frac{C_A}{24}\, \delta\hat{\mathcal{Z}}^{(Y)}_{1,u} + \frac{4}{3} T_F \sum_{f\in\mathcal{F}} \delta\hat{\mathcal{Z}}^{(Y)}_{1,f}}_{\text{renormalisation scheme dependence} \Rightarrow \text{applicable to any scheme}}\,.\end{aligned}$$

- Compact results of QCD rational counterterms derived once and for all in a generic scheme [2007.03713]

Rational Terms in

Spontaneously Broken Gauge Theories

[Lang, Pozzorini, Zhang, Zoller: 2106.XXXX]

Idea: reduce the complexity of $\delta\mathcal{R}_2$ calculations in the full Standard Model by exploiting the symmetric phase and vev-expansion

Symmetry breaking & vev-expansion

Consider a spontaneously broken theory with Higgs field H, and vev v that generates masses

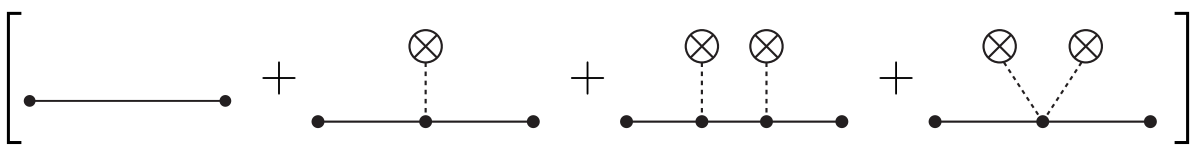
$$\underbrace{\mathcal{L}(H)}_{\text{broken phase}} \quad = \quad \underbrace{\mathcal{L}_{\text{YM}}(\phi)}_{\text{symmetric phase}} \qquad \text{with} \quad \phi = H + v$$

UV divergent part of D-dim amplitudes in broken phase can be **related to massless symmetric phase** (YM) via vev-expansion

$$\underbrace{\mathbf{V}_{[0,X]}\, \bar{\mathcal{A}}_{l,\Gamma}}_{\substack{\text{vev-expansion} \\ \text{in broken phase}}} \quad = \quad \underbrace{\sum_{k=0}^{X} \bar{\mathcal{A}}^{\text{YM}}_{l,\Gamma v^k}}_{\substack{\text{external } \textbf{vev-insertions} \\ \text{in symmetric phase}}} \quad := \quad \underbrace{\sum_{k=0}^{X} \frac{v^k}{k!} \bar{\mathcal{A}}^{\text{YM}}_{l,\Gamma H^k}\Big|_{p_H=0}}_{\substack{\text{external } H\text{-insertions} \\ \text{with 0-momentum}}}$$

Diagrammatic vev-expansion of a loop propagator $\bar{G}_a(\bar{q}, m_a)$ up to $\mathcal{O}(v^2)$

$$\mathbf{V}_{[0,2]}\, \bar{G}_a(\bar{q}, m_a) \quad =$$



$$\Big[\quad\Big]^{\text{YM}}_{D_{\mathsf{n}}=D}$$

turn physical mass dependence into series of **vev-insertions** in symmetric phase

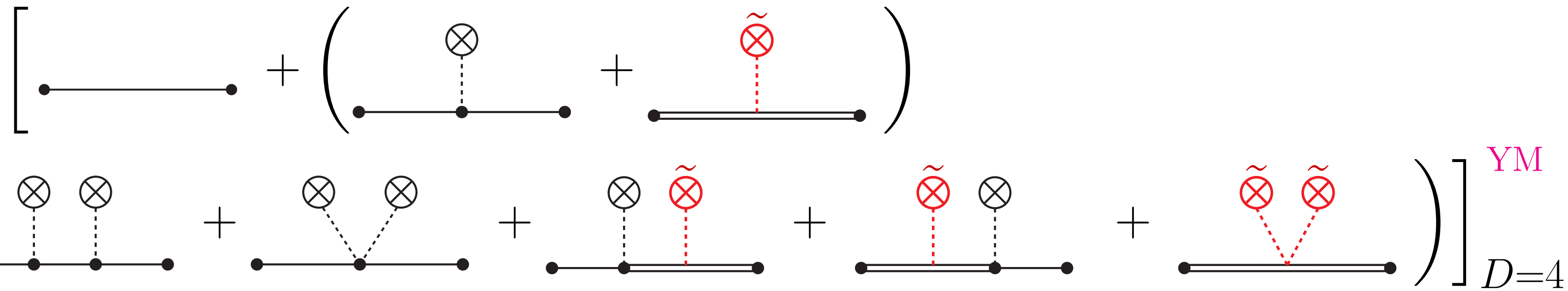
Vev-expansion in $D_\mathsf{n} = 4$ dimensions

In $D_\mathsf{n} = 4$, the naive relation between broken and symmetric phases is violated

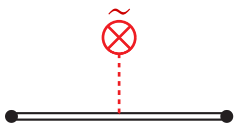
$$\mathbf{V}_{[0,X]}\,\mathcal{A}_{l,\Gamma} \;=\; \sum_{k=0}^{X} \mathcal{A}^{\mathrm{YM}}_{l,\Gamma v^k} \;+\; \underbrace{\Delta\mathbf{V}_{[0,X]}\,\mathcal{A}_{l,\Gamma}}_{\substack{\text{extra } \tilde{q}^2\text{-terms from vev-exp.}\\ \text{of loop propagators in } D_\mathsf{n}=4}}$$

while $\Delta\mathbf{V}_{[0,X]}\,\mathcal{A}_{l,\Gamma}$ part can be described through **auxiliary $\tilde{v}$-counterterm insertions** at the level of loop propagators

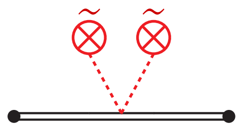
$$\mathbf{V}_{[0,2]}\,G_a \;=$$



In SM, $\tilde{v}$-counterterm insertions are only needed for fermion propagators



$$= \mathrm{i} m_f \frac{\tilde{q}^2}{\bar{q}^4}\,,$$



$$= -\,\mathrm{i} m_f^2 \frac{\not{q}\,\tilde{q}^2}{\bar{q}^6}$$

Rational terms with vev- & $\tilde{v}$-insertions

Vev-expansion for l-loop amplitudes in $D_\mathrm{n} = 4$ reads

$$\mathbf{V}_{[0,X]}\mathcal{A}_{l,\Gamma} \;=\; \underbrace{\sum_{k=0}^{X}\left(\sum_{j=0}^{k}\mathcal{A}^{\mathrm{YM}}_{l,\Gamma v^j \tilde{v}^{k-j}}\right)}_{\substack{\text{vev \& } \tilde{v}\text{-CT insertions of } \mathcal{O}(v^k) \\ \text{in } \textbf{symmetric phase}}}$$

Rational terms at l-loops in broken phase can be related to their symmetric counterparts

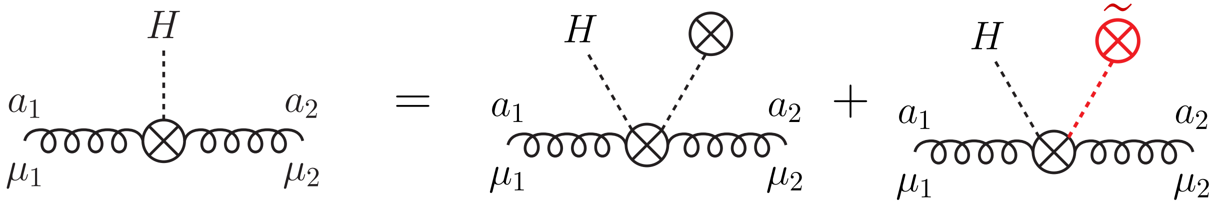
$$\delta\mathcal{R}_{l,\Gamma} \;=\; \sum_{k=0}^{X}\left(\sum_{j=0}^{k}\delta\mathcal{R}^{\mathrm{YM}}_{l,\Gamma v^j \tilde{v}^{k-j}}\right)$$

- Simplified calculations of rational terms in SM based on **massless symmetric** phase.
- For renormalisation scheme dependence including gauge fixing and mixing effects, tadpoles and vev renormalisations, please see upcoming paper [2106.XXXX]

Results for two-loop rational counterterms of $\mathcal{O}(\alpha_\mathrm{s}^2)$ in SM

Example 1: Higgs–gluon three-point function

Broken Phase **Symmetric Phase**



$$= \; \mathrm{i}\, v\, \delta^{a_1 a_2}\, g^{\mu_1 \mu_2} \left\{ \sum_{k=1}^{2} \left(\frac{\alpha_\mathrm{S}\, t^{\varepsilon}}{4\pi} \right)^{k} \delta\hat{\mathcal{R}}_{k,\mathrm{gg}H} \right\}$$

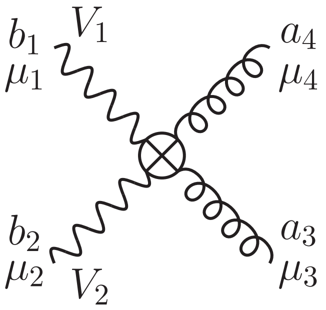
with $m_f = \lambda_f\, v/\sqrt{2}$ and

$$\delta\hat{\mathcal{R}}_{1,\mathrm{gg}H} \;=\; -\,2\,T_\mathrm{F} \sum_{f\in\mathcal{F}} \lambda_f^2$$

$$\delta\hat{\mathcal{R}}_{2,\mathrm{gg}H} \;=\; -\,T_\mathrm{F} \sum_{f\in\mathcal{F}} \left[\left(\frac{1}{2}\, C_\mathrm{A} + 6\, C_\mathrm{F} \right) \varepsilon^{-1} + \frac{19}{12}\, C_\mathrm{A} - 7\, C_\mathrm{F} \right] \lambda_f^2 - 2\,T_\mathrm{F} \sum_{f\in\mathcal{F}} \left[\delta\hat{\mathcal{Z}}_{1,\alpha_\mathrm{s}} + \delta\hat{\mathcal{Z}}_{1,A} + 2\,\delta\hat{\mathcal{Z}}_{1,\lambda_f} \right] \lambda_f^2$$

Results for two-loop rational counterterms of $\mathcal{O}(\alpha_{\mathrm{S}}^2)$ in SM

Example 2: Gluon–vector-boson four-point function



$$= \mathrm{i} \underbrace{\delta_{a_3 a_4} \sum_{N_{\mathrm{gen}}} \frac{1}{2} \mathrm{Tr}\left[\mathrm{I}^{b_1} \mathrm{I}^{b_2}\right]}_{\substack{SU(3)\times SU(2)\times U(1) \\ \text{gauge group structure}}} \left\{ \sum_{k=1}^{2} \left(\frac{\alpha_{\mathrm{s}}\, t_Y^{\varepsilon}}{4\pi} \right)^k \left[\sum_{\beta=\mathrm{I,II}} \left(\mathcal{V}_{\beta}^{\mu_1\mu_2\mu_3\mu_4}\, \delta\hat{\mathcal{R}}^{(\mathrm{A}\beta)}_{k,\mathrm{gg}V_1V_2} \right) \right] \right\},$$

with $\mathcal{V}_{\mathrm{I}}^{\mu_1\mu_2\mu_3\mu_4} = g^{\mu_1\mu_2} g^{\mu_3\mu_4}$, $\mathcal{V}_{\mathrm{II}}^{\mu_1\mu_2\mu_3\mu_4} = g^{\mu_1\mu_3} g^{\mu_2\mu_4} + g^{\mu_1\mu_4} g^{\mu_2\mu_3}$, and

$$\delta\hat{\mathcal{R}}^{(\mathrm{AI})}_{1,\mathrm{gg}V_1V_2} = \frac{2}{3} T_{\mathrm{F}}\,,$$

$$\delta\hat{\mathcal{R}}^{(\mathrm{AI})}_{2,\mathrm{gg}V_1V_2} = -\,T_{\mathrm{F}} \left(\frac{1}{12} C_{\mathrm{A}} + \frac{3}{2} C_{\mathrm{F}} \right) + \frac{2}{3} T_{\mathrm{F}} \left(\delta\hat{\mathcal{Z}}^{(Y)}_{1,\alpha_{\mathrm{s}}} + \delta\hat{\mathcal{Z}}^{(Y)}_{1,G} \right)$$

Similarly for $\delta\hat{\mathcal{R}}^{(\mathrm{AII})}_{k,\mathrm{gg}V_1V_2}$ terms.

- γ_5 in Korner-Kreimer-Schilcher (KKS) scheme (anticommuting + reading-point prescription)
- Full set of $\mathcal{O}(\alpha_{\mathrm{S}}^2)$ rational terms of $f\bar{f}S, f\bar{f}V, ggS, ggV, gggV, ggS_1S_2, ggV_1V_2$ vertices in SM will appear in [2106.XXXX]

Summary

- **Renormalised** D-dim two-loop amplitude can be constructed by amplitude with 4-dim numerator $+$ **rational counterterms in a generic renormalisation scheme** Y

$$\mathbf{R}^{(Y)}\,\bar{\mathcal{A}}_{2,\Gamma} \;=\; \mathcal{A}_{2,\Gamma} + \sum_{\gamma}\left(\delta Z^{(Y)}_{1,\gamma} + \delta \tilde{Z}^{(Y)}_{1,\gamma} + \delta\mathcal{R}^{(Y)}_{1,\gamma}\right)\cdot \mathcal{A}_{1,\Gamma/\gamma} + \delta Z^{(Y)}_{2,\Gamma} + \delta\mathcal{R}^{(Y)}_{2,\Gamma}$$

$\Rightarrow$ an important step towards two-loop automation

- We have presented a **generic method to compute** $\delta\mathcal{R}_{2,\Gamma}$ from one-scale tadpoles, and shown that $\delta\mathcal{R}_{2,\Gamma}$ are **process-independent local counterterms**

- Full set of rational terms at two loops in QED [2001.11388] and pure QCD [2007.03713]

- **New vev-expansion technique** that relates rational terms in broken and symmetric phases $\Rightarrow$ calculation of rational terms in SM strongly simplified

- Full set of $\mathcal{O}(\alpha_s^2)$ rational terms at two loops in the SM [2106.XXXX]

Outlook

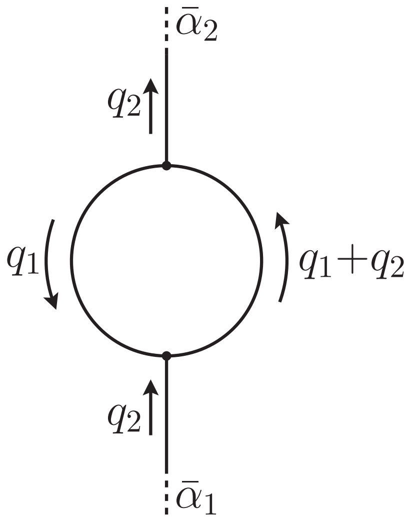
- Study of two-loop rational terms of IR origin (ongoing)

- Rational terms in full SM with EW corrections

Backup

B.1. $\delta\tilde{Z}_{1,\gamma}$ from subdivergences in two-loop diagram

- **Subdivergence** originates from the UV divergent **one-loop subdiagram**
  $\Rightarrow$ needs to be firstly **subtracted** in renormalisation procedure
- **Subdiagram** has D-dim external loop momenta



One-loop subdiagram with D-dim external $\bar{q}_2 = q_2 + \tilde{q}_2$:

$$D_k(\bar{q}_1, \bar{q}_2) = \underbrace{D_k(\bar{q}_1, q_2)}_{\text{4-dim } q_2} + \underbrace{(2\,\bar{q}_1 \cdot \tilde{q}_2 + \tilde{q}_2^2)}_{\varepsilon\text{-dim } \tilde{q}_2}$$

$\Rightarrow$ additional ε-dim terms show up in tadpole expansion

$$\frac{1}{(\bar{q}_1 + q_2 + \tilde{q}_2)^2} = \frac{1}{\bar{q}_1^2 - M^2} + \frac{-(q_2 + \tilde{q}_2)^2 - 2\,\bar{q}_1 \cdot (q_2 + \tilde{q}_2) - M^2}{(\bar{q}_1^2 - M^2)^2} + \dots$$

$\Rightarrow$ **extra quadratic pole term** $\delta\tilde{Z}^{\alpha}_{1,\gamma}(\tilde{q}_2) \propto \dfrac{\tilde{q}_2^2}{\varepsilon}$ in 4-dim numerator case

$$\bar{\mathbf{K}}\,\mathcal{A}^{\alpha}_{1,\gamma}(\bar{q}_2) = \underbrace{-\,\delta Z^{\alpha}_{1,\gamma}(q_2)}_{\frac{1}{\varepsilon}\ \mathrm{MS}\ \text{pole}} \quad \underbrace{-\,\delta\tilde{Z}^{\alpha}_{1,\gamma}(\tilde{q}_2)}_{\text{extra pole of } \mathcal{O}(1)}$$

B.2. Renormalised one-loop subdiagrams

Subtract poles and rational terms in both D- and 4-dim, we can **identify amplitudes** with

$$\underbrace{\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) - \bar{\mathbf{K}}\,\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2)}_{D\text{-dim full subtraction}} \;=\; \underbrace{\mathcal{A}^{\alpha}_{1,\gamma}(\bar{q}_2) - \bar{\mathbf{K}}\,\mathcal{A}^{\alpha}_{1,\gamma}(\bar{q}_2)}_{4\text{-dim full subtraction}} + \mathcal{O}(\varepsilon,\tilde{q})$$

Recall

$$\bar{\mathbf{K}}\,\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) \;=\; -\delta Z^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) + \delta\mathcal{R}^{\alpha}_{1,\gamma}(q_2) + \mathcal{O}(\varepsilon)$$

$$\bar{\mathbf{K}}\,\mathcal{A}^{\alpha}_{1,\gamma}(\bar{q}_2) \;=\; -\delta Z^{\alpha}_{1,\gamma}(q_2) - \delta\tilde{Z}^{\alpha}_{1,\gamma}(\tilde{q}_2)$$

$\Rightarrow$ **Renormalised one-loop sub-amplitude**

$$\underbrace{\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) + \delta Z^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2)}_{D\text{-dim renormalisation}} \;=\; \underbrace{\mathcal{A}^{\alpha}_{1,\gamma}(q_2) + \delta Z^{\alpha}_{1,\gamma}(q_2) + \delta\tilde{Z}^{\alpha}_{1,\gamma}(\tilde{q}_2)}_{\substack{4\text{-dim renormalisation}\\ \text{compute numerically}}} + \underbrace{\delta\mathcal{R}^{\alpha}_{1,\gamma}(q_2)}_{\text{rational parts}} + \mathcal{O}(\varepsilon,\tilde{q})$$

B.3. One-loop subdiagram example: photon self-energy

Let $D_{\mathsf{n}} \in \{D, 4\}$ be the dimension of numerator, we have

$$D_{\mathsf{n}} = D \quad \Rightarrow \quad \bar{\mathbf{K}} \int \mathrm{d}\bar{q}_1 \frac{-\mathrm{Tr}\left[\bar{\gamma}^{\bar{\alpha}_1} \bar{\not{q}}_1 \bar{\gamma}^{\bar{\alpha}_2} (\bar{\not{q}}_1 + \bar{\not{q}}_2)\right]}{\bar{q}_1^2\,(\bar{q}_1 + \bar{q}_2)^2} \quad = \quad \frac{1}{\varepsilon}\Bigg(\underbrace{-\frac{4}{3}\left(\bar{q}_2^2\, g^{\bar{\alpha}_1\bar{\alpha}_2} - \bar{q}_2^{\bar{\alpha}_1}\bar{q}_2^{\bar{\alpha}_2}\right)}_{-\delta Z_{1,\gamma}(\bar{q}_2)} \quad \underbrace{+\frac{2\,\varepsilon}{3}\,\bar{q}_2^2\, g^{\bar{\alpha}_1\bar{\alpha}_2}}_{\delta\mathcal{R}_{1,\gamma}(q_2) + \mathcal{O}(\varepsilon)} \Bigg)$$

and

$$D_{\mathsf{n}} = 4 \quad \Rightarrow \quad \mathbf{K} \int \mathrm{d}\bar{q}_1 \frac{-\mathrm{Tr}\left[\gamma^{\alpha_1} \not{q}_1 \gamma^{\alpha_2} (\not{q}_1 + \not{q}_2)\right]}{\bar{q}_1^2\,(\bar{q}_1 + q_2 + \tilde{q}_2)^2} \quad = \quad \frac{1}{\varepsilon}\Bigg(\underbrace{-\frac{4}{3}\left(q_2^2\, g^{\alpha_1\alpha_2} - q_2^{\alpha_1} q_2^{\alpha_2}\right)}_{-\delta Z_{1,\gamma}(q_2)} \quad \underbrace{-\frac{2}{3}\,\tilde{q}_2^2\, g^{\alpha_1\alpha_2}}_{-\delta \tilde{Z}_{1,\gamma}(\tilde{q}_2)} \Bigg)$$

$\Rightarrow$ Renormalised photon self-energy insertion: