Amplitude Evolution beyond Leading Order

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QCD cross sections

\[ d\sigma \sim L \times d\sigma_H(Q) \times PS(Q \rightarrow \mu) \times MPI \times Had(\mu \rightarrow \Lambda) \times ... \]
Setting the scene

QCD description of collider reactions:
Complexity challenges precision.

Hard partonic scattering:
NLO QCD routinely

Jet evolution — parton branching:
NLL sometimes, mostly unclear

Multi-parton interactions
Hadronization

\[ d\sigma \sim L \times d\sigma_H(Q) \times \text{PS}(Q \to \mu) \times \text{MPI} \times \text{Had}(\mu \to \Lambda) \times \ldots \]

[Forshaw, Holguin, Plätzer – JHEP 09 (2020) 014]
[Salam et al. — JHEP 09 (2018) 033]
QCD Coherence

Resummation of observables which globally measure deviations from n-jet limit. Basis of angular ordered parton showers — highest level of analytic control.

\[ s = + q^2 - \delta^2 \]

[Hoang, Plätzer, Samitz — JHEP 1810 (2018) 200]
Non-global Observables

Measure deviation from jet topology only in patch of phase space. Coherent branching breaks down, full complexity of QCD amplitudes strikes back. If non-global bit is isolated can use dipole cascades to resum.

[Dasgupta, Salam, Banfi, Marchesini, Smye, Becher et al. …]
Cross Sections and Amplitudes
Cross Sections and Amplitudes

\[ \sigma[u] = \sum_n \int \text{Tr} [A_n] u(q_1, \ldots, q_n) d\phi(q_1, \ldots, q_n) \]

sum over emissions

‘density operator’ ~ amplitude amplitude

observable and phase space
Cross Sections and Amplitudes

$$A_n(q) = \int_q^Q \frac{dk}{k} \, D_n(k) \, P e^{-\int_q^k \frac{dk'}{k'} \, \Gamma(k')} \, A_{n-1}(k) \, \overline{P} e^{-\int_q^k \frac{dk'}{k'} \, \Gamma^\dagger(k')} \, D_n^\dagger(k)$$

Markovian algorithm at the amplitude level:
Iterate gluon exchanges and emission.

Different histories in amplitude and conjugate amplitude needed to include interference.

[Angeles, De Angelis, Forshaw, Plätzer, Seymour – JHEP 05 (2018) 044]
Tracking colour

Decompose amplitudes in flow of colour charge.

\[
\text{Tr} \left[ A_n \right] = \sum_{\sigma, \tau} A_{\tau \sigma} \langle \sigma | \tau \rangle
\]

\[
\begin{align*}
N^3 & = (123) (123) (123) \\
N^2 & = (123) (213) (312) \\
N & = (123) (213) (312)
\end{align*}
\]

[Plätzer – EPJ C 74 (2014) 2907]
[Angeles, De Angelis, Forshaw, Plätzer, Seymour – JHEP 05 (2018) 044]
Tracking colour

Gluon emission

$$D_n(k)$$

Gluon exchange

$$P e^{-\int_0^k \frac{dk'}{k'} \Gamma(k')} \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$$

$$[\tau | \Gamma|\sigma\rangle = (\alpha_s N)[\tau | \Gamma^{(1)}|\sigma\rangle + (\alpha_s N)^2[\tau | \Gamma^{(2)}|\sigma\rangle + \ldots$$

Explicit suppression in $1/N$

Systematically expand around large-$N$ limit summing towers of terms enhanced by $\alpha_s N$

Non-global Observables and Large-N

Primary application: Non-global observables

\[ E \frac{\partial G_n(E)}{\partial E} = -\Gamma G_n(E) - G_n(E) \Gamma^\dagger + D_\mu^\dagger G_{n-1}(E) D_{n\mu} \ u(E, \hat{k}_n) \]

Utilise colour flow basis, and expand around large-N:

\[ \text{Leading}^{(l)}_{\tau\sigma} [A] = \sum_{k=0}^{l} A_{\tau\sigma} \left| \frac{1}{N^k} \delta \# \text{transpositions}(\tau,\sigma), l-k \right. \]

Re-derive BMS equation: Prototype of constructing a dipole shower

\[ \text{Leading}^{(0)}_{\tau\sigma} [V_n A_n V_n^\dagger] = \delta_{\tau\sigma} \left| V_{\sigma}^{(n)} \right|^2 \text{Leading}^{(0)}_{\tau\sigma} [A_n] \]

\[ \text{Leading}^{(0)}_{\tau\sigma} [D_n A_{n-1} D_n^\dagger] = \delta_{\tau\sigma} \sum_{i,j \text{ c.c. in } \sigma \setminus n} \lambda_i \bar{\lambda}_j R_{ij}^{(n)} \text{Leading}^{(0)}_{\tau \setminus n, \sigma \setminus n} [A_{n-1}] \]

\[ V_{\sigma}^{(n)} = \exp \left( -N \sum_{i,j \text{ c.c. in } \sigma} \lambda_i \bar{\lambda}_j W_{ij}^{(n)} \right) \]

colour connected dipoles
Colour (Flow) Evolution Beyond Leading Order

Include simultaneously unresolved emissions and higher loop structures

\[ E \frac{\partial}{\partial E} A_n(E) = \Gamma_n(E) A_n(E) + A_n(E) \Gamma_n^\dagger(E) - \sum_k R_n^{(k)}(E) A_{n-k}(E) R_n^{(k),\dagger}(E) \]

combination of purely virtual and unresolved real corrections, point-by-point in phase space

resolved real emissions and virtual/unresolved corrections to emissions

Similar in origin to a fixed-order calculation with the subtraction method:

Subtract (unresolved) real emissions — cast virtual corrections into phase-space type integrals instead of integrating subtraction terms.
Leading Order Evolution

\[ \Gamma^{(1)} = \frac{1}{2} \sum_{i,j} \Omega_{ij}^{(1)} \left( \frac{1}{N} T_i \cdot T_j \right) \]

\[ [\tau | \Gamma^{(1)} | \sigma] = \left( \Gamma^{(1)}_{\sigma} + \frac{1}{N^2} \rho^{(1)} \right) \delta_{\sigma \tau} + \frac{1}{N} \Sigma^{(1)}_{\sigma \tau} \]

Expand around colour diagonal limit

Cast e.g. into form of energy ordered observables by doing a contour integral

\[ \Omega_{ij}^{(1)} = i \mu^{2\epsilon} \int \frac{d^d k}{i \pi^{d/2}} \frac{p_i \cdot p_j}{(k^2 + i0)(p_i \cdot k + i0)(p_j \cdot k - i0)} = \int_0^\infty \frac{dE}{E} \left( \frac{\mu^2}{E^2} \right)^\epsilon \omega^{(ij)} \]

\[ \omega^{(ij)} = \frac{(2\pi)^{2\epsilon}}{\pi} \left[ \int \frac{d\Omega^{(d-2)}}{4\pi} \frac{n_i \cdot n_j}{n_i \cdot n \cdot n \cdot n_j} - i\pi \int \frac{d\Omega^{(d-3)}}{2\pi} \right] \]
Beyond Leading Colour

CVolver library implements numerical evolution in colour space. 

|De Angelis, Forshaw, Plätzer — PRL 126 (2021) 11|
|Plätzer – EPJ C 74 (2014) 2907|

Resummation of non-global logarithms at full colour:

Avoid complexity which grows with colour space dimensionality:

- Monte Carlo over colour flows,
- events at intermediate steps carry complex weights.
### Evolution at the Next Order

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Diagram</th>
<th>Colour-factor</th>
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<tbody>
<tr>
<td>(\Omega^{(2)}_{ij})</td>
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<td>((T_i \cdot T_j)(T_i \cdot T_j))</td>
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</tr>
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<td><img src="image" alt="Diagram" /></td>
<td>(i f^{abc} T_i^a T_j^b T_l^c)</td>
</tr>
<tr>
<td>(\Omega^{(2)}_{ij,\text{self-en.}})</td>
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Algorithmic treatment of virtual corrections needed

Feynman tree theorem:

\[
\frac{1}{k^2 + i0(T \cdot k)^2} = \frac{1}{(k^0)^2 - \bar{k}^2 + 2i0(k^0)^2}
\]

\[(T^\mu) = (\sqrt{2}, 0)\]

\[
\frac{1}{[q^2 - i0(T \cdot q)|T \cdot q|]} = \frac{1}{[q^2 + i0(T \cdot q)^2]} + 2\pi i\delta(q^2)\theta(T \cdot q)
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\]

Extend to Eikonal and higher-power propagators:

\[
\frac{1}{2p_i \cdot k - i0(T \cdot p_i)^2} = \frac{1}{2p_i \cdot k + i0(T \cdot p_i)^2} + 2\pi i \delta(2p_i \cdot k)
\]

\[
\frac{1}{[q^2 - i0(T \cdot q)|T \cdot q|^2} - \frac{1}{[q^2 + i0(T \cdot q)^2]} = -2\pi i \theta(T \cdot q) \delta'(q^2)
\]

\[
\omega^{ij} = \frac{(2\pi)^2}{\pi} \left[ \int \frac{d\Omega^{(d-2)}}{4\pi} \frac{n_i \cdot n_j}{n_i \cdot n \cdot n \cdot n_j} - i\pi \int \frac{d\Omega^{(d-3)}}{2\pi} \right]
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\]
Cutting rules
Cutting rules

\[ \mu^{Ae} \int \frac{d^dk}{i\pi^{d/2}} \frac{d^dq}{i\pi^{d/2}} \frac{1}{[p_i \cdot k + i0][-p_j \cdot (k + q) + i0][k^2 + i0][q^2 + i0][1 + 0][k + q]^2 + i0] \]

Eikonal coupling only to hard lines!
see also [Angeles, Forshaw, Seymour – JHEP 12 (2015) 091]
Colour structures imply colour-diagonal **three parton correlations**: Dipoles are not enough. New approaches need to take this into account.
Amplitude level evolution sets level of complexity to understand and design parton shower and resummation algorithms.

Crucial to address effects of Coulomb/Glauber phases, factorisation violation, super-leading logarithms, … — otherwise out of reach.

Investigate soft gluon effects at two loops (and related):

- Understand colour structures for many external legs and systematically expand around large-N limit.
- Design resummation of non-global observables beyond leading log and leading-N, investigate phases.
- Key ingredient for decisive statements about most flexible parton showers beyond leading order.
Thank you!