Two-loop helicity amplitudes for $gg \rightarrow ZZ$ with full top mass dependence

RADCOR-LoopFest 2021


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ZZ production at the LHC

- Significant contribution to off-shell Higgs production through interference [Kauer, Passarino (2012)]
- Constrain Higgs width [Caola, Melnikov (2013)]
- Measuring anomalous $t\bar{t}Z$ coupling; importance of longitudinal modes [Azatov, Grojean, Paul, Salvioni (2016)], [Cao, Yan, Yuan, Zhang (2020)]
- Important channel for BSM searches

- $gg \to ZZ$ formally NNLO at LHC
- High gluon luminosity $\Rightarrow$ large contribution
- Provides $\sim 60\%$ of the total NNLO correction [Cascioli, German, Grazzini, Kallweit, Maierhöfer, von Manteuffel, Pozzorini, Rathlev, Tancredi, Weihs (2014)]
- Increase of 5% to the full NNLO result from $gg \to ZZ$ at NLO [Grazzini, Kallweit, Wiesemann, Yook (2018)]
Status of the calculation

\( gg \rightarrow ZZ \):

- Known exactly at 1-loop [Glover, van der Bij (1988)]
- Massless internal fermions at 2-loops [von Manteuffel, Tancredi (2015)], [Caola, Henn, Melnikov, Smirnov, Smirnov (2015)]
- Large top-mass approximation at 2-loops [Dowling, Melnikov (2015)], [Caola, Dowling, Melnikov, Röntsch, Tancredi (2016)] with Padé approximants [Campbell, Ellis, Czakon, Kirchner (2016)]
- Expansion around \( t\bar{t} \) threshold with Padé approximants [Gröber, Maier, Raum (2019)]
- Small top-mass expansion with Padé approximants [Davies, Mishima, Steinhauser, Wellman (2020)] (See Go Mishima’s talk)

Other similar gluon-induced calculations involving massive internal loops :

- HH production at 2-loops with full top-mass dependence [Borowka, Greiner, Heinrich, Jones, Kerner, Schlenk, Zirke (2016)] and in small top-mass expansion [Davies, Mishima, Steinhauser, Wellmann (2018)] (See Joshua Davies’ talk)
- ZH amplitudes at 2-loops with full top-mass dependence [Chen, Heinrich, Jones, Kerner, Klappert, Schlenk (2020)] and in small and large top-mass regions [Davies, Mishima, Steinhauser (2020)] (See Matthias Kerner’s talk)
- WW amplitudes at 2-loop with 3rd generation quarks [Brønnum-Hansen, Wang (2020)] (See Chen-Yu Wang’s talk, also for ZZ)
Multiloop calculations

Recipe for a multi-loop amplitude:

1. Generation of unreduced amplitude
2. IBP reduction
   - Major bottleneck for processes with many scales and/or legs
   - Significant progress with syzygy based approaches and finite-field methods
3. Insertion of IBP identities into the amplitude
   - Significant blow-up for intermediate results and final reduced amplitude
   - Numerical instabilities in final coefficients
   - Use of multivariate partial fractioning to tame the computational complexity and improve numerical performance
4. Evaluation of master integrals
   - Internal masses => Functions beyond multiple polylogarithms
   - Use of numerical methods instead, improved with the use of finite integrals
Syzygies

• Integration-By-Parts reduction to reduce all the integrals to a basis set
• Generate linear relations between integrals [Chetyrkin & Tkachov (1981)]
• Systematically construct and reduce a linear system to a basis set of master integrals -> Laporta’s algorithm [Laporta (2000)]. Public codes available AIR, FIRE6, Kira, LiteRed, Reduze 2, etc.

• In Baikov representation [Baikov (1996)]:

\[
0 = \int \left( \prod_{i=1}^{L} dz_i \right) \sum_{i=1}^{N} \frac{\partial}{\partial z_i} \left( f_i(z_1, \ldots, z_N) P^{(d-L-E-1)/2} \prod_{i=1}^{N} \frac{1}{z_i^{\nu_i}} \right)
\]

\[
0 = \int \left( \prod_{i=1}^{L} dz_i \right) \sum_{i=1}^{N} \left( \frac{\partial f}{\partial z_i} + \frac{d - L - E - 1}{2P} f_i \frac{\partial P}{\partial z_i} - \frac{\nu_i f_i}{z_i} \right) P^{(d-L-E-1)/2}
\]

• Require:
  • No dimension-shifting terms
  • No integrals with doubled propagators
Syzygies

Disadvantages:
• Such integrals don’t appear in amplitudes
• Significantly larger linear system to reduce for the appearance of auxiliary integrals

Avoiding doubled propagators:
• Generating vectors using Groebner basis [Gluza, Kajda, Kosover (2010)]
• Linear algebra based approach [Schabinger (2011)]
• Differential geometry [Zhang (2014)]

\[ f_i \frac{\partial P}{\partial z_i} \sim P \]

Dimension shifting term

\[ f_i \sim z_i \]

Doubled propagator term

• Explicit solutions known [Boehm, Georgoudis, Larsen, Schulze, Zhang (2017)]
  [Abreu, Cordero, Ita, Page, Zeng (2017)]
• Polynomials of degree 1 in Baikov parameters
• Straightforward to write

• Trivial to write explicit solutions
Syzygies

• Simultaneous solution for the two constraints highly non-trivial
• Compute module intersection of the two syzygy modules
  • Syzygies for top-level topologies inaccessible

• Developed a new linear algebra approach based on finite fields [Agarwal, Jones, von Manteuffel (2020)]
  • Map the problem of module intersection to row reduction of a matrix; Finred - finite field based solver for the linear algebra
  • Solutions produced up to a requested degree in $z_i$
  • Much faster for our purpose than the Groebner basis approach; can run in a highly distributed manner
  • Able to generate the required syzygies for this calculation

• Use Finred - finite field based solver, to compute the required IBP reductions
• Also use this approach for the 2-loop amplitudes for diphoton+jet production [Agarwal, Buccioni, von Manteuffel, Tancredi (2021)], [Agarwal, Buccioni, von Manteuffel, Tancredi (2021)] (see Federico Buccioni’s talk)
Syzygies

- Total size of syzygies $\sim 2GB$
- Largest syzygy $\sim 230MB$
- Up to $s = 4$ integrals
- 2 scales $s, t$ ($m_t, m_Z$ set to numbers)
- Extremely complicated due to internal masses

- Total size of syzygies $\sim 1GB$
- Largest syzygy $\sim 40MB$
- Up to $s = 5$ integrals
- 4 scales $s_{23}, s_{34}, s_{45}, s_{51}$ ($s_{12} = 1$)
Finite integrals

- Feynman integrals often have UV and IR divergences
- Sector decomposition standard method to resolve IR poles \[\text{[Binoth, Heinrich (2000)] [Bogner, Weinzierl (2007)]}\]

Public codes: Fiesta4, pySecDec, etc.

Why use finite integrals instead?
- Much better behaved numerically
- Require fewer orders in epsilon expansion in general
- Poles drop out into the coefficients => Easier to take \(d \rightarrow 4\) limit

Constructing finite integrals:
- Dimension shifted integrals \[\text{[Bern, Dixon, Kosower (1992)]}\]
- Existence of a finite basis \[\text{[Panzer (2014)] [von Manteuffel, Panzer, Schabinger (2014)]}\]
- Reduze 2 to find such integrals, usually involving doubled propagators (dots) and dimension shifts
Finite integrals

Divergent integral in $d = 4 - 2\epsilon$

Divergent integral in $d = 4 - 2\epsilon$ with a numerator

Finite integral in $d = 6 - 2\epsilon$

Finite integral in $d = 6 - 2\epsilon$ with a dot
Finite integrals

However:

- Integrals with dots and dimension-shifts often hard to reduce e.g. need reductions for integrals with 4 dots for the required finite integrals
- Higher dots implies higher powers of $F$ polynomial in the denominator $=>$ worse contour deformation which leads to numerical instabilities

Alternate approach - combining divergent integrals into finite linear combinations. Advantages:

- Integrals often already appearing in the amplitude $=>$ avoid computing extra reductions
- More “natural” $d = 4$ representation
- Finite at the integrand level i.e. integrand free of non-integrable divergences

- In general a highly non-trivial task to find these numerators
- Algorithmically construct finite linear combinations in $d = 4$ from a list of seed integrals [Agarwal, Jones, von Manteuffel (2020)]
- Arbitrary integrals with numerators, dots, dimension shifts, subsector integrals etc allowed as seed integrals
Finite integrals

Integrand = \( a_1 \frac{1}{D_1 \ldots D_N} + a_2 \frac{D_{N+1}}{D_1 \ldots D_N} + a_3 \frac{D_j}{D_1 \ldots D_j \ldots D_N} + \ldots \)

Corner integral  Numerator integral  Subsector integral

• Combine over a common denominator using the general formula for Feynman parametric representation [Agarwal, Jones, von Manteuffel (2020)]

\[
I(\nu_1, \ldots, \nu_N) = (-1)^{r+\Delta t} \Gamma(\nu - L d/2) \int \left( \prod_{j \in \mathcal{N}_T} dx_j \right) \left( \prod_{j \in \mathcal{N}_i} \frac{x_j^{\nu_j-1}}{\Gamma(\nu_j)} \right) \delta \left( 1 - \sum_{j \in \mathcal{N}_T} x_j \right) \]

\[
\left[ \left( \prod_{j \in \mathcal{N}_T} \frac{\partial^{\nu_j}}{\partial x_j^{\nu_j}} \right) \left( \prod_{j \in \mathcal{N}_{\Delta t}} \frac{\partial^{\nu_j+1}}{\partial x_j^{\nu_j+1}} \right) \frac{\mathcal{U} \nu-(L+1)d/2}{\mathcal{U} \nu-Ld/2} \right] \bigg|_{x_j=0 \ \forall \ j \in \mathcal{N}_\Lambda}
\]

• Constrain \( a_i \) requiring absence of non-integrable divergences in the integrand
## Finite integrals

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<td>Linear Combination</td>
<td>$\sim 1 \times 10^{-4}$</td>
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\[
I = (m_Z^2 - s - t)(sI_1-I_6) + s(I_2+I_3-I_4-I_5) - (m_Z^2 - t)I_7
\]
## Finite integrals

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\[ I = (m_{2}^2 - s - t)(sI_1 - I_6) + s(I_2 + I_3 - I_4 - I_5) - (m_{2}^2 - t)I_7 \]
Multivariate partial fractioning

- All unreduced integrals expressed in terms of the optimised finite basis
- Need to insert these identities into the amplitude to obtain the “reduced” amplitude
- Resulting coefficients are coefficients in kinematics and d

Challenges:
- This is computationally very difficult; IBPs size of over 200 GB
- Intermediate steps require TB of disk space and computationally very expensive
- Numerical performance issues due to presence of spurious poles
Multivariate partial fractioning

- Certain choices lead to spurious poles with denominators depending on both kinematics and \( d \); want to avoid such poles, e.g.

\[
\frac{1}{1250 - 500d - 9000t + 3600dt + 16200t^2 - 6480d t^2 - 4050s + 1575ds + 19440st - 8100dst - 52488s t^2 + 20412d st^2 - 29160s^2 t + 11664d s^2 t}
\]

In \( d \to 4 \) this becomes:

\[
\frac{1}{-125 + 375s + 900t - 2160st + 2916s^2t - 1620t^2 + 4860st^2}
\]

- Spurious poles lead to numerical instabilities
- Choose d-factoring basis to avoid such denominators [Smirnov, Smirnov (2020)], [Usovitsch(2020)]
- Employ multivariate partial fractioning
Multivariate partial fractioning


• Use Singular to perform partial fractioning using a Groebner basis to prevent new denominators from appearing. E.g. naive partial fractioning in Mathematica:

\[
\frac{1}{25 - 270t + 324st - 5 + 18t + 9s} = \frac{-1}{(5 + 18t)(-5 + 36t)(-5 + 18t + 9s)} + \frac{36t}{(5 + 18t)(-5 + 36t)(25 - 270t + 324st)}
\]

New denominators

• Instead use a Groebner basis approach; Find relations between all appearing denominators to reduce them to simpler ones

• Unique decomposition for a chosen ordering of denominator polynomials

• Handle nasty degree 6 denominators:

\[
105625 - 468000t - 797850t^2 + 3863700t^3 + 2001105t^4 - 5904900t^5 + 2125764t^6 - 3676500s + 17309700st - 19260180s^2 + 25850340s^3 + 35901792s^4t + 8503056s^5t^2 + 25891650s^2t^2 + 73614420s^2t^3 - 75149694s^2t^3 - 12754584s^2t^4 + 50490540s^3 + 80752788s^3t - 60466176s^3t^2 + 8503056s^3t^3 + 29452329s^4 - 18187092s^4t + 2125764s^4t^2
\]
Multivariate partial fractioning

- Drastic simplification of coefficients after partial fractioning
  
  Intermediate size: $O(TB)$

  Size after partial fractioning: < 1 MB per coefficient

E.g. Complexity reduces significantly for one of the hardest coefficients in the amplitude

$$\text{coefficient} = \frac{\text{num}(s, t, d)}{\text{den}(s, t, d)}$$

$$\{\text{deg(num, s)} + \text{deg(den, s)}, \text{deg(num, t)} + \text{deg(den, t)}, \text{deg(num, d)} + \text{deg(den, d)}\} = \{107, 117, 38\}$$

After partial fractioning, worst term = \{20, 15, 9\}

Total number of terms after partial fractioning = 10842
Multivariate partial fractioning

- Partial fraction in $d$ to separate the poles
- Set $d = 4$. Allowed since the basis is finite

Factorised form: 
$$\frac{1}{(-1 + d)(-3 + d)^2(-4 + d)(-7 + 2d)} = \left(\frac{1}{3} + \frac{2\epsilon}{9}\right)(1 + 2\epsilon)^2\left(\frac{-1}{2\epsilon}\right)(1 + 4\epsilon)$$

Partial fractioned:
$$\frac{1}{3(-4 + d)} + \frac{5}{4(-3 + d)} + \frac{1}{2(-3 + d)^2} + \frac{1}{60(-1 + d)} + \frac{-16}{5(-7 + 2d)} = \frac{-1}{6\epsilon} + \frac{-13}{9}$$

- Prevents proliferation of terms
- Partial fraction in kinematics to arrive at final form
- Resulting coefficients smaller than 1MB in size. Total size of all coefficients $O(100)$ MB
- Very fast numerical evaluation
Results

Comparison of $\sqrt{s}$ dependence of the unpolarised interference with expansion results at fixed $\cos \theta = -0.1286$. Exact results from [Agarwal, Jones, von Manteuffel (2020)]. Expansion and Padé results from [Davies, Mishima, Steinhauser, Wellmann (2020)]. Error bars for the exact result are plotted but they are too small to be visible.
Comparison of $\cos \theta$ dependence of the unpolarised interference with expansion results at fixed energy $\sqrt{s} = 403$ GeV. Exact results from [Agarwal, Jones, von Manteuffel (2020)]. Expansion and Padé results from [Davies, Mishima, Steinhauser, Wellmann (2020)].

Comparison of $\cos \theta$ dependence of the unpolarised interference with expansion results at fixed energy $\sqrt{s} = 814$ GeV. Exact results from [Agarwal, Jones, von Manteuffel (2020)]. Expansion and Padé results from [Davies, Mishima, Steinhauser, Wellmann (2020)].
IR scheme dependence

• For previous results, “$q_T$” subtraction scheme

• Transformation between Catani’s original scheme and $q_T$ scheme

$$A_{i}^{(2), fin, Catani} = A_{i}^{(2), fin, q_T} + \Delta I_1 A_{i}^{(1), fin}$$

$$\Delta I_1 = - \frac{1}{2} \pi^2 C_A + i\pi\beta_0 \sim 15$$

• For interference terms, 1-loop result multiplied by $\sim 30$ => Leads to a very different qualitative behaviour

• Relative comparisons highly dependent on IR scheme
Comparison of $\sqrt{s}$ dependence of the polarised interference with expansion results at fixed $\cos \theta = -0.1286$. Exact results from [Agarwal, Jones, von Manteuffel (2020)]. Expansion and Padé results from [Davies, Mishima, Steinhauser, Wellmann (2020)].
Conclusions

• Results for two-loop corrections for $gg \rightarrow ZZ$ with full top mass dependence

• Use of syzygies and finite field methods for IBP reduction including presenting our new algorithm for constructing syzygies

• Method of finite integrals with new general approach to construct finite integrals

• Multivariate partial fractioning to drastically simplify amplitude coefficients

• IR scheme dependence of qualitative comparisons between the exact calculation and expansion results