Nested soft-collinear subtractions for deep inelastic scattering

Konstantin Asteriadis; BNL | May 20, 2021
RADCOR-LoopFest 2021
IR finite differential cross section @NNLO QCD

- To test the SM, predictions for fully-differential cross sections through higher orders in QCD are needed.
- Contributions to next-to-next-to-leading order partonic cross section

\[ d\hat{\sigma}_{\text{nnlo}} = d\hat{\sigma}_{\text{rr}} + d\hat{\sigma}_{\text{rv}} + d\hat{\sigma}_{\text{vv}} + d\hat{\sigma}_{\text{pdf}} \]

- Double real emission contributions
- Real-virtual contributions
- Double virtual contribution
IR finite differential cross section @NNLO QCD

- To test the SM, predictions for fully-differential cross sections through higher orders in QCD are needed.
- Contributions to next-to-next-to-leading order partonic cross section contain explicit infrared poles in $1/\epsilon$

$$d\hat{\sigma}_{\text{nnlo}} = d\hat{\sigma}_{\text{rr}} + d\hat{\sigma}_{\text{rv}} + d\hat{\sigma}_{\text{vv}} + d\hat{\sigma}_{\text{pdf}}$$

contain infrared singularities that become poles in $1/\epsilon$ only upon phase space integration

- In dimensional regularization ($d = 4 - 2\epsilon$) the explicit poles of 1-loop and 2-loop amplitudes are known independent of the hard matrix element [Catani ’98; Becher, Neubert ’09]

$$\mathcal{M}_{1\text{-loop}}(\{p\}) = \left[ \frac{e^{\gamma_E}}{\Gamma(1-\epsilon)} \Sigma_i \left( \frac{1}{\epsilon} + \frac{g_i}{T_i^2 \epsilon} \right) \Sigma_{j\neq i} \frac{T_i \cdot T_j}{2} \left( \frac{\mu^2}{-s_{ij}} \right)^\epsilon \right] \mathcal{M}_{\text{tree}}(\{p\}) + \mathcal{M}_{1\text{-loop}}^{\text{fin}}(\{p\})$$
Singularities of real emission contributions

- Singularities of QCD amplitudes come in two varieties: soft ($E \to 0$) and collinear ($\vec{p}_i \parallel \vec{p}_j$)

\[ \frac{1}{(p-k)^2} \approx \frac{1}{E_p \times E_k \times (1 - \vec{n}_p \cdot \vec{n}_k)} \to \infty \quad \text{for} \quad E_k \to 0 \quad \text{soft singularity} \]

\[ \text{for} \quad \vec{n}_p \parallel \vec{n}_k \quad \text{collinear singularity} \]

- The corresponding limits of amplitudes are generic and independent of a hard process

- For example, the soft ($E_k \to 0$) limit of a single real emission DIS amplitude is [Altarelli, Parisi, ’77]

\[ \begin{aligned} 2 \\ E_k \to 0 \end{aligned} \begin{array}{c} \approx 2C_F g_s^2 \times \frac{p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} \times \end{array} \begin{array}{c} 2 \\ p_1 \quad p_2 \end{array} \]

- Two new genuine NNLO singularities: double soft and triple collinear

- Factorization formulas for double soft and triple collinear singularities are known [Catani, Grazzini ‘99; …]

- They are structurally similar to the NLO case.
Entangled soft and collinear limits

- Entangled soft/collinear limits

\[ \frac{1}{(p - k_1 - k_2)^2} \sim \frac{1}{2p \cdot k_1 + 2p \cdot k_2 - 2k_1 \cdot k_2} \quad \rightarrow \quad k_1 \parallel p \quad k_2 \rightarrow 0 \]

- For a **given amplitude** it can be checked explicitly that entangled **soft/collinear** singularities do not occur

- This observation is general thanks to a phenomenon known as **colour coherence** (a soft gluon does not resolve details of a collinear splitting) [Caola, Melnikov, Röntsch, ‘17]

**As a result …**

... known soft and collinear limits of amplitudes are sufficient to construct all relevant subtraction terms;

... soft and collinear limits can be treated independently;

... can be used to extend FKS subtraction @NLO to NNLO [Frixione, Kunszt, Signer ’96].

→ **Nested soft-collinear subtraction scheme**

[Caola, Melnikov, Röntsch, ‘17]
How to regulate and extract singularities without integration?

- Soft and collinear singularities turn into $1/\epsilon$ poles upon phase space integration.

$$\int \frac{d^{d-1}k}{2E} |M(\{p\}, k)|^2 \sim \int \frac{dE}{E^{1+\epsilon}} \frac{d\theta}{\theta^{1+2\epsilon}} \times |M(\{p\})|^2 \sim \frac{1}{\epsilon^2}$$

- We would like to extract singularities without integration over resolved phase space. Currently two approaches used: slicing and subtraction.

- To illustrate the basic idea of subtraction, consider an integral

$$I = \int_0^1 \frac{dx}{x^{1+\epsilon}} F(x)$$

where $F(0)$ is finite. We then write

$$I = \int_0^1 \frac{dx}{x^{1+\epsilon}} [F(x) - F(0)] + \int_0^1 \frac{dx}{x^{1+\epsilon}} F(0) = \int_0^1 \frac{dx}{x^{1+\epsilon}} [F(x) - F(0)] - \frac{1}{\epsilon} F(0)$$

regulated, finite in the $\epsilon \to 0$ limit

extracted $1/\epsilon$ pole
Regulating the double soft singularity

As an example we consider the double soft singularity (\( E_5 \sim E_6 \to 0 \)). The action of \( \$ \) on the differential cross section is defined as

\[
\$ \left[ \mathcal{N} \int d\text{Lips} (2\pi)^d \delta^{(d)} \left( p_1 + p_2 - \sum_{i=3}^6 p_i \right) |M^\text{tree}(\{p\}, p_5, p_6)|^2 \mathcal{O}(p_3, p_4, p_5, p_6) \right] \]

\[
\equiv g_{s,b}^4 \times \text{Eikonal}(1, 4, 5, 6) \times \mathcal{N} \int d\text{Lips} (2\pi)^d \delta^{(d)} \left( p_1 + p_2 - p_3 - p_4 \right) |M^\text{tree}(\{p\})|^2 \mathcal{O}(p_3, p_4)
\]

independent of the hard matrix element and the observable

We insert unity decomposed as \( I = (I - \$) + \$ \) into the phase space

\[
(I - \$) \times + g_{s,b}^2 \times \text{Eikonal}(1, 4, 5, 6) \times \]

Double-soft singularity regularized but still contains single soft and collinear singularities.

Subtraction term; soft gluons decouple; integrate analytically over phase space of gluons 5 and 6

[Caola, Delto, Frellesvig, Melnikov '18]
... continue iteratively for other singularities

- Gluons ordered in energy → only one single soft singularity (e.g. for $E_6 \to 0$); insert $I = (I - S_6) + S_6$

\[
(I - S) \times 2 = (I - S)(I - S_6) \times 2 + \text{(Subtraction terms)}
\]

- Due to the absence of entangled soft and collinear singularities we can continue with collinear singularities in the same way.

- In case of collinear singularities introduction of partition functions [Frixione, Kunszt, Signer '96] and splitting of the angular phase space into sectors [Czakon '10 '11; Czakon, Heymes '14] to separate overlapping singularities.

\[
\sum_{ij} (I - S)(I - S_6)(I - C_6) \times w_i \times \theta_j \times 2 + \text{(Subtraction terms)}
\]

- Subtraction terms can be integrated analytically over the phase space of one or both gluons and $1/\epsilon$ poles can be extracted explicitly.
The role of deep inelastic scattering

- Most complex singular contributions (both soft and collinear) only depend on the properties of two external partons
- Separation of complex $pp \to N$ processes into simpler building blocks

![Diagram of Drell-Yan process and H → b̄b decay](image)

- Focus on simpler processes → check against analytical results possible
Fully regulated double-real contribution

\[
F_{LM}(1, 4, 5, 6) = \left\langle \frac{\pi^2}{\pi} F_{LM}(1, 4, 5, 6) \right\rangle + \left\langle \left[ I - \frac{\pi^2}{\pi} \right] S_6 F_{LM}(1, 4, 5, 6) \right\rangle \\
+ \sum_{i, j \in \{1, 4\}} \sum_{i \neq j} \left\langle \left[ I - \frac{\pi^2}{\pi} \right] \left[ I - S_6 \right] \left[ C_{5i} w^{5j, 6j} + C_{6i} w^{5j, 6j} \right] \left( \theta_i^{(a)} C_{5i} + \theta_i^{(c)} C_{6i} \right) w^{5i, 6i} \right\rangle \\
\times \left[ dp_5 \right] \left[ dp_6 \right] F_{LM}(1, 4, 5, 6) \\
+ \sum_{i \in \{1, 4\}} \left\langle \left[ I - \frac{\pi^2}{\pi} \right] \left[ I - S_6 \right] \left[ \theta_i^{(b)} C_{56} + \theta_i^{(d)} C_{56} \right] \left[ dp_5 \right] \left[ dp_6 \right] w^{5i, 6i} F_{LM}(1, 4, 5, 6) \right\rangle \\
- \sum_{i, j \in \{1, 4\}} \sum_{i \neq j} \left\langle \left[ I - \frac{\pi^2}{\pi} \right] \left[ I - S_6 \right] C_{5i} C_{6j} \left[ dp_5 \right] \left[ dp_6 \right] w^{5i, 6j} F_{LM}(1, 4, 5, 6) \right\rangle \\
+ \sum_{i \in \{1, 4\}} \left\langle \left[ I - \frac{\pi^2}{\pi} \right] \left[ I - S_6 \right] \left[ \theta_i^{(a)} \omega_i \left[ I - C_{5i} \right] + \theta_i^{(b)} \omega_i \left[ I - C_{56} \right] + \theta_i^{(c)} \omega_i \left[ I - C_{6i} \right] + \theta_i^{(d)} \omega_i \left[ I - C_{56} \right] \right] \left[ dp_5 \right] \left[ dp_6 \right] w^{5i, 6i} F_{LM}(1, 4, 5, 6) \right\rangle \\
+ \sum_{i, j \in \{1, 4\}} \sum_{i \neq j} \left\langle \left[ I - \frac{\pi^2}{\pi} \right] \left[ I - S_6 \right] \left[ 1 - C_{5i} \right] \left[ 1 - C_{5j} \right] \left[ 1 - C_{5i} \right] \left[ 1 - C_{5j} \right] \left[ dp_5 \right] \left[ dp_6 \right] w^{5i, 6j} F_{LM}(1, 4, 5, 6) \right\rangle \\
+ \sum_{i \in \{1, 4\}} \left\langle \left[ I - \frac{\pi^2}{\pi} \right] \left[ I - S_6 \right] \left[ 1 - \omega_i \right] \left( \theta_i^{(a)} \left[ 1 - C_{6i} \right] + \theta_i^{(b)} \left[ 1 - C_{56} \right] \right) \left[ dp_5 \right] \left[ dp_6 \right] w^{5i, 6i} F_{LM}(1, 4, 5, 6) \right\rangle \\
+ \sum_{i \in \{1, 4\}} \left\langle \left[ I - \frac{\pi^2}{\pi} \right] \left[ I - S_6 \right] \left[ \omega_i \right] \left( \theta_i^{(c)} \left[ 1 - C_{5i} \right] + \theta_i^{(d)} \left[ 1 - C_{56} \right] \right) \left[ dp_5 \right] \left[ dp_6 \right] w^{5i, 6i} F_{LM}(1, 4, 5, 6) \right\rangle
\]
Pole structure @NNLO

- Analytic integration of subtraction terms is possible (analytic simplifications after recombining subtractions terms)

\[
\left\langle [1 - S][1 - S_6] C_{54} w^{54,61} + C_{64} w^{51,64} + \left( \theta^{(a)} C_{64} + \theta^{(c)} C_{54} \right) w^{54,64} \right\rangle [dg_5][dg_6] F_{LM}(1, 4, 5, 6)
\]

\[
= \frac{\alpha_s C_F}{\epsilon} \left\langle \sum_{i=1,4} (I - S_5)(I - C_{5i}) w^{5i} \left[ \left( \frac{1}{\epsilon} + Z^{2,2} \right) (2E_4)^{-2\epsilon} - \frac{1}{\epsilon} (2E_5)^{-2\epsilon} \right] \left[ w^{51}_{dc} + w^{54}_{tc} \left( \frac{p_{54}}{4} \right)^{-\epsilon} \right] F_{LM}(1, 4, 5) \right\rangle
\]

\[
+ \frac{\alpha_s^2 C_F^2}{\epsilon^3} \left\langle \left[ \left( \frac{1}{\epsilon} + Z^{2,2} \right) (2E_4)^{-2\epsilon} (2E_{max})^{-2\epsilon} - \frac{1}{2\epsilon} (2E_{max})^{-4\epsilon} \right] \right\rangle
\]

\[
\times \left[ \frac{\langle \Delta_{51} \rangle S_5 - \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} - \frac{2\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} }{F_{LM}(1, 4)} \right]
\]

\[
+ \frac{\alpha_s^2 C_F^2}{\epsilon^2} \left[ \frac{2\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} \right] \left[ \frac{1}{\epsilon} + Z^{2,2} \right] \left[ \frac{1}{\epsilon} + Z^{4,2} \right] \left\langle (2E_4)^{-4\epsilon} F_{LM}(1, 4) \right\rangle
\]

\[
- \frac{\alpha_s^2 C_F^2}{\epsilon^3} \left[ \frac{1}{2\epsilon} + Z^{2,4} \right] \left[ \langle \Delta_{51} \rangle S_5 + \frac{2\epsilon}{2} \frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)} \right] (2E_4)^{-4\epsilon} F_{LM}(1, 4)
\]

\[
- \frac{\alpha_s^2 C_F^2}{\epsilon^2} \left[ \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \right] \int dz \left\langle \left[ \left( \frac{1}{\epsilon} + Z^{2,2} \right) (2E_4)^{-2\epsilon} - \frac{1}{\epsilon} (2E_1)^{-2\epsilon} (1 - z)^{-2\epsilon} \right] \right\rangle (2E_1)^{-2\epsilon} (1 - z)^{-2\epsilon} \bar{p}_{qq}(z) \frac{F_{LM}(z \cdot 1, 4)}{z} \right\rangle.
\]

- The subtraction terms contains the regulated NLO differential cross section → poles cancel against similar terms from real virtual contributions

- Regular and “boosted” LO differential cross section → cancel against double virtual (and collinear renormalization contributions)

- Provide analytic formula for the finite remainder
Finite contributions of integrated subtraction terms

- E.g. abelian boosted quark non-singlet contribution (multiplying the LO matrix element)
- This is one of the more complex contributions, still relatively compact

\[
\begin{align*}
\mathcal{T}_{\text{int}} &= \frac{\pi^2(5 - 2z)}{3(z - 1)} + 5z + 8\mathcal{D}_3(z) + 12\mathcal{D}_3(z) \ln \left( \frac{4E_T^2}{\mu^2} \right) \\
+ \ln \left( \frac{E_1}{E_4} \right) \left( \ln \eta_{14} \left( -2z - 2(z + 1) \ln \left( \frac{4E_T^2}{\mu^2} \right) - 4(z + 1) \ln(1 - z) + 2 \right) \\
+ (2 - 2z) \ln z - \frac{2(z^2 + 1) \ln(4E_T^2/\mu^2) \ln z}{z - 1} - \frac{4(z^2 + 1) \ln(1 - z) \ln z}{z - 1} \right) \\
+ \ln \eta_{14} \left( -3z + 2(z - 1) \ln z + \ln \left( \frac{4E_T^2}{\mu^2} \right) \left( \frac{2(z^2 + 1) \ln z}{z - 1} - 3(z + 1) \right) + \ln(1 - z) \left( \frac{4(z^2 + 1) \ln z}{z - 1} - 6(z + 1) \ln z \right) + \ln \eta_{14} \left( -2z - 4(z + 1) \ln \left( \frac{E_1}{E_{\text{max}}} \right) - 2(z + 1) \ln \left( \frac{4E_T^2}{\mu^2} \right) \right) \\
+ \frac{\pi^2(2z^2 + 2)}{3(1 - z)} + 4(z + 1) \ln(1 + z) + \ln \left( \frac{E_1}{E_4} \right) \left( -z + (-z - 1) \ln \left( \frac{4E_T^2}{\mu^2} \right) \\
- 2(z + 1) \ln(1 - z) \right) - 2(z + 1) \ln(1 - z) \ln^2 \eta_{14} \\
+ \ln \left( \frac{E_1}{E_{\text{max}}} \right) \left( -\frac{1}{2} \ln^2(2)(z + 1) - 2 \ln \eta_{14} \ln(z + 1) + \frac{2}{3} \pi^2(z + 1) \right) \\
+ \ln \eta_{14} \left( 4(z - 1) + 4(z + 1) \ln \left( \frac{4E_T^2}{\mu^2} \right) + 8(z + 1) \ln(1 - z) \right) \right) \\
+ \ln \left( \frac{4E_T^2}{\mu^2} \right) \left( 2z^2 \ln z - 2(z + 1) \ln(1 - z) + \frac{3z^2 + 1) \ln z}{2(z - 1)} \right) \\
+ (2 - 2z) \ln z - \frac{(1 - 7z^2) \ln z}{2(z - 1)} \ln^2(1 - z) - \frac{3(2z^2 + 2z - 7) \ln z}{2(z - 1)} \\
+ \ln(1 - z) \left( \pi^2(27 - 29z^2) \right) \ln z - \frac{7z^2 + 2z - 7 \ln z}{z - 1} + \frac{-7z - (z + 1) \ln^2(2) - 46}{2} \\
+ \frac{5(z^2 + 1) \ln z}{2(z - 1)} + \ln \left( \frac{4E_T^2}{\mu^2} \right) \left( -3z + 2\pi^2(z + 1) + \frac{-4z^2 - 2z + 3) \ln z}{z - 1} \right) \\
+ \ln(1 - z) \left( -z + \frac{2(z^2 + 1) \ln z}{z - 1} - 5 \right) - 6(z + 1) \ln(1 - z) \\
+ \left( \frac{3z^2 + 1) \ln^2 z}{2(z - 1)} - 10 \right) - 4(z + 1) \ln^3(1 - z) + \frac{1}{12} (z + 1) \ln^5(z) \\
+ \text{Li}_2(1 - \eta_{14}) \left( -2z - 4(z + 1) \ln \left( \frac{E_1}{E_{\text{max}}} \right) - 2(z + 1) \ln \left( \frac{4E_T^2}{\mu^2} \right) \right) \\
- 8(z + 1) \ln(1 - z) + 2 \right) + \mathcal{D}_3(z) \left( 8 \ln \left( \frac{E_1}{E_4} \right) \ln \eta_{14} - 16 \ln \left( \frac{E_1}{E_{\text{max}}} \right) \ln \eta_{14} \\
+ 12 \ln \eta_{14} + 6 \ln \left( \frac{4E_T^2}{\mu^2} \right) + 8 \ln(1 - z) + 4 \ln \left( \frac{E_1}{E_4} \right) + 4 \ln^2 \eta_{14} + 4 \ln^2 \left( \frac{4E_T^2}{\mu^2} \right) \right) \\
+ 16 \text{Li}_2(1 - \eta_{14}) - 8\pi^2 + 26 \right) + \left( 4(z + 1) + \frac{4(z^2 + 1) \ln z}{z - 1} \right) \text{Li}_2(-z) \\
+ \text{Li}_2(z) \left( \frac{2(2z^2 - 5)}{z - 1} - 2(z + 1) \ln \left( \frac{4E_T^2}{\mu^2} \right) + \frac{3 - 5z^2 \ln(1 - z)}{z - 1} + \frac{(z^2 + 1) \ln z}{z - 1} \right) \\
+ \frac{9z^2 + 1) \text{Li}_3(1 - z) - 8(z^2 + 1) \text{Li}_3(-z) + \frac{1 - 3z^2) \text{Li}_3(z)}{z - 1} \right) \\
+ \mathcal{D}_3(z) \left( 4 \ln \left( \frac{E_1}{E_4} \right) \ln \eta_{14} + \frac{2z^2}{4E_T^2} \right) + 6 \ln \eta_{14} + \frac{2z^2}{4E_T^2} \right) + 2 \ln \left( \frac{E_1}{E_4} \right) \right) \\
+ \left( 13 - \frac{10\pi^2}{3} \right) \ln \left( \frac{4E_T^2}{\mu^2} \right) + \ln \left( \frac{E_1}{E_{\text{max}}} \right) \left( -8 \ln \eta_{14} \ln \left( \frac{4E_T^2}{\mu^2} \right) \right) + \ln^2(2) \\
+ 4 \ln^2 \eta_{14} - \frac{4\pi^2}{3} + 3 \ln^2 \left( \frac{4E_T^2}{\mu^2} \right) + \left( 8 \ln \left( \frac{E_1}{E_{\text{max}}} \right) + 4 \ln \left( \frac{4E_T^2}{\mu^2} \right) \right) \text{Li}_2(1 - \eta_{14}) \\
+ 16 \xi_3 \right) + \frac{(1 - 11z^2) \xi_3}{z - 1} - 2,
\end{align*}
\]
Numerical results

- Due to the focus on simpler processes as building blocks, results can be extensively tested against known analytic results. [Kazakov et al. ‘90; Zijlstra, van Neerven ‘92; Moch, Vermaseren ‘00]

- In the case of photon-induced deep-inelastic scattering with only up-quarks and gluons in the initial state and $\sqrt{s} = 100\text{ GeV}$, $10\text{ GeV} < Q < 100\text{ GeV}$, $\mu_R = \mu_F = 100\text{ GeV}$, we obtain permille agreement for the NNLO contribution

$$\sigma = \sigma_{\text{LO}} + \Delta\sigma_{\text{NLO}} + \boxed{\Delta\sigma_{\text{NNLO}}}$$

<table>
<thead>
<tr>
<th>channel</th>
<th>numeric result (pb)</th>
<th>analytic result (pb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{q,ns}^{\text{NNLO}}$</td>
<td>$33.1(2) - 2.18(1) \cdot n_f$</td>
<td>$33.1 - 2.17 \cdot n_f$</td>
</tr>
<tr>
<td>$\sigma_{q,s}^{\text{NNLO}}$</td>
<td>$9.19(2)$</td>
<td>$9.18$</td>
</tr>
<tr>
<td>$\sigma_g^{\text{NNLO}}$</td>
<td>$-142.4(4)$</td>
<td>$-142.7$</td>
</tr>
</tbody>
</table>

[KA, Caola, Melnikov, Röntsch ‘19]

- In general, we find that we can get per mill precision on the NNLO total cross section, corresponding to a few percent precision on the NNLO coefficient, running for a few hours on an 8-core machine.
The role of deep inelastic scattering

- Most complex singular contributions (both soft and collinear) only depend on the properties of two external partons
- Separation of complex $pp \rightarrow N$ processes into simpler building blocks

- Permille agreement for all basic processes against analytic results.
- Fast computation time for the NNLO cross section (Drell-Yan $\mathcal{O}(1h)$, $H \rightarrow b\bar{b}$ decay $\mathcal{O}(<1h)$, DIS $\mathcal{O}(60h)$)
- Building blocks under control! What is next?
  
  → Natural next application: factorizable VBF @ NNLO QCD

- Due to focus on simpler processes check against analytical results possible
Vector boson fusion (developing a subtraction scheme)

- Challenging VBF phase space → requires efficient subtraction scheme

- Integrated subtraction terms not Lorentz invariant → complex cancellation between analytically integrated subtractions and regulated real emission contributions.

- Can be checked against known fully differential descriptions @NNLO QCD [Cacciari, Dreyer, Karlberg, Salam, Zanderighi ‘15; Cruz-Martinez, Gehrmann, Glover, Huss ‘18]

Vector boson fusion (physics)

- Important production channel of Higgs boson @LHC (second highest cross section @14TeV)

- Fast numerical implementation → study Higgs decay, anomalous couplings, non-factorizable contributions, ...
Preliminary results

- Setup: 13 TeV, typical VBF cuts
- Computation time for ΔNNLO histograms around ~ 50,000 CPU hours
Conclusion

- HL-LHC requires high precision theoretical predictions for collider processes.

- Despite progress with developing IR subtraction schemes, the “perfect” subtraction scheme is yet to come.

- The presented nested soft-collinear scheme for NNLO descriptions includes many of the desired properties from FKS @NLO.

- Development status: Complete set of analytic building blocks (obtained from studies of colour singlet production, decay and a DIS process) that can be used as building blocks to design subtractions for arbitrary LHC processes.

- Next steps: Application to more complex processes; in the pipeline: Higgs production in vector boson fusion.

**Construction of the nested soft-collinear subtraction scheme is based on ...**

... iterative extraction of soft and collinear singularities;

... partitioning of angular phase space into sectors to obtain well-defined sets of collinear limits;

... (not shown) the possibility to parametrize phase space in a way that makes analytic integration of subtraction terms possible.
Backup
Subtraction @NLO is ...

- **physically transparent**
  "physical" singularities and clear mechanism of cancellation

- **local**
  subtracted matrix elements are finite at any point in the phase-space

- **analytic**
  analytic formulas for integrated subtraction terms

- **modular**
  subtractions for complex processes are built from subtraction terms established in analyses of simpler processes (soft singularities are sensitive to pairs of emittors; collinear singularities factorize on external lines)

- **efficient**
  efficient numerical evaluation (as result of local and analytic)

Situation @NNLO

- Many subtraction schemes at NNLO [Gehrmann-de Ridder, Gehrmann, Glover ‘05; Czakon ‘10, ‘11; Cacciari et al ‘15; Somogyi, Trócsányi, Del Duca ‘05; Caola, Melnikov, Röntsch ‘17; Herzog ‘18; Magnea et al ‘18; …]

- None of the existing subtraction schemes satisfies all of the above criteria ...

  ... but up to now this was not a problem for phenomenology.

- For more complex processes, better subtraction schemes may become a necessity.
The role of deep inelastic scattering

- Most complex singular contributions (both soft and collinear) only depend on the properties of two external partons
- Separation of complex $pp \rightarrow N$ processes into simpler building blocks

Deep inelastic scattering

- Colour charged particles not back-to-back
- As a result: integrated subtraction terms are not numbers but functions in the opening angle $\Theta$
- First frame independent computation (using this scheme)
- Dependence on alpha-like parameter $E_{\text{max}}$ that controls the “amount” of soft subtraction
- Focus on simpler processes $\rightarrow$ check against analytical results possible
Deep inelastic scattering @NNLO QCD

We write the differential cross section as

\[ 2s \cdot d\sigma_{rr} = \int [dg_5][dg_6] \Theta(E_5 - E_6) F_{LM}(1, 4, 5, 6) = \langle F_{LM}(1, 4, 5, 6) \rangle \]

with

\[ F_{LM}(1, 4, 5, 6) = \mathcal{N} \int d\text{Lips} \ (2\pi)^d \delta^{(d)}(p_1 + p_2 - p_3 - p_4 - p_5 - p_6) \]
\[ \times |M^{\text{tree}}(\{p\}, p_5, p_6)|^2 \times \mathcal{O}(p_3, p_4, p_5, p_6) \]
\[ [dg_i] = \frac{d^{d-1}p_i}{(2\pi)^{d-1}2E_i} \Theta(E_{\text{max}} - E_i) \]
Different subtraction schemes and slicing methods

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</table>

Updated and adapted from [Nigel Glover, Amplitudes ‘15]
Collinear singularities

\[ |M^{nnlo}(\{p\}, p_5, p_6)|^2 = 1 \]

\[ = \begin{bmatrix} 5 & 6 \\ 6 & 5 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 6 & 5 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 6 & 5 \\ 5 & 6 \end{bmatrix} + \ldots \]

- In the collinear limits, many different singular configurations exist, but collinear singularities factorize on external legs, therefore either three partons become collinear or two pairs of partons become collinear at once.
- To control which partons these are, the different configurations are separated by introducing partition functions (similarly to NLO)

\[ 1 = w^{51,61} + w^{54,64} + w^{51,64} + w^{54,61} \]

- Singularities in double collinear sectors are separated.
- Different collinear singularities in triple collinear partitions are isolated in the angular phase space.
- We separate them by splitting the phase space into different sectors.
Partition functions

\[ |M_{\text{nnlo}}^{(\{p\}, p_5, p_6)}|^2 = \left| \begin{array}{c}
\begin{array}{cc}
5 & 6 \\
\end{array}
\begin{array}{cc}
\vphantom{\text{\{}} & \\
\vphantom{\text{\{}} & \vphantom{\text{\{}}
\end{array}
\end{array}
\right| + \left| \begin{array}{c}
\begin{array}{cc}
6 & 5 \\
\end{array}
\begin{array}{cc}
\vphantom{\text{\{}} & \\
\vphantom{\text{\{}} & \vphantom{\text{\{}}
\end{array}
\end{array}
\right| + \left| \begin{array}{c}
\begin{array}{cc}
5 & 6 \\
\end{array}
\begin{array}{cc}
\vphantom{\text{\{}} & \\
\vphantom{\text{\{}} & \vphantom{\text{\{}}
\end{array}
\end{array}
\right| + \left| \begin{array}{c}
\begin{array}{cc}
5 & 6 \\
\end{array}
\begin{array}{cc}
\vphantom{\text{\{}} & \\
\vphantom{\text{\{}} & \vphantom{\text{\{}}
\end{array}
\end{array}
\right| + \ldots
\right|
\]

- The different configurations are separated by introducing partition functions in the phase space

\[ 1 = w^{51,61} + w^{54,64} + w^{51,64} + w^{54,61} \]

with

\[ \lim_{5 \parallel l} w^{5i,6j} \sim \delta_{li}, \quad \lim_{6 \parallel l} w^{5i,6j} \sim \delta_{lj} \quad \text{and} \quad \lim_{5 \parallel i} \lim_{6 \parallel j} w^{5i,6j} = 1. \]

- One possible choice

\[ w^{51,61} = \frac{\rho_{54}\rho_{64}}{d_5d_6} \left( 1 + \frac{\rho_{51}}{d_{5641}} + \frac{\rho_{61}}{d_{5614}} \right), \quad w^{51,64} = \frac{\rho_{54}\rho_{61}\rho_{56}}{d_5d_6d_{5614}}, \]

\[ w^{54,64} = \frac{\rho_{51}\rho_{61}}{d_5d_6} \left( 1 + \frac{\rho_{64}}{d_{5641}} + \frac{\rho_{54}}{d_{5614}} \right), \quad w^{54,61} = \frac{\rho_{51}\rho_{64}\rho_{56}}{d_5d_6d_{5641}}, \]

where

\[ d_{i=5,6} \equiv \rho_{1i} + \rho_{4i}, \quad d_{5614} \equiv \rho_{56} + \rho_{51} + \rho_{64}, \quad d_{5641} \equiv \rho_{56} + \rho_{54} + \rho_{61}. \]
Splitting of the angular phase space

\[ |M^{\text{nnlo}}(\{p\}, p_5, p_6)|^2 = \begin{vmatrix} 5 & 6 \end{vmatrix} + \begin{vmatrix} 6 & 5 \end{vmatrix} + \begin{vmatrix} 5 & 6 \end{vmatrix} + \begin{vmatrix} 5 & 6 \end{vmatrix} + \ldots \]

• As example consider partition \( w^{51,61} \): it is singular when (5\|1), (6\|1) and (5\|6)

\[ 1 = \theta \left( \rho_{61} < \frac{\rho_{51}}{2} \right) + \theta \left( \frac{\rho_{51}}{2} < \rho_{61} < \rho_{51} \right) + \theta \left( \rho_{51} < \frac{\rho_{61}}{2} \right) + \theta \left( \frac{\rho_{61}}{2} < \rho_{51} < \rho_{61} \right) \]

• In practice this is done by introducing the unity

• To integrate singularities analytically it is crucial that the phase space is parameterized in such a way that all singularities are made explicit [Czakon]
Subtraction terms before NLO regulation

- Single collinear finial state emission

\[
\left\langle \left[ I - S^5 \right] \left[ I - S_6 \right] \left[ C_{54} w^{54,61} + C_{64} w^{51,64} + \left( \theta^{(a)} C_{64} + \theta^{(c)} C_{54} \right) w^{54,64} \right] [dg_5] [dg_6] F_{LM}(1, 4, 5, 6) \right\rangle \\
= \frac{[\alpha_s] C_F}{\epsilon} \left\langle \left[ \left( \frac{1}{\epsilon} + Z^{2,2} \right) (2E_4)^{-2\epsilon} - (2E_5)^{-2\epsilon} \right] \left( w_{DC}^{51} + w_{TC}^{54} \left( \frac{\rho_{54}}{4} \right)^{-\epsilon} \right) F_{LM}(1, 4, 5) \right\rangle \\
- \frac{[\alpha_s]^2 C_F^2}{\epsilon^3} \left( \frac{1}{2\epsilon} + Z^{2,4} \right) \left\langle \langle \Delta_{51} \rangle S_5 (2E_4)^{-4\epsilon} F_{LM}(1, 4) \right\rangle.
\]

with

\[
Z^{n,m} = -\frac{2}{m\epsilon} \int_0^1 dz \, z^{-n\epsilon} (1 - z)^{-m\epsilon} P_{q\bar{q}}(z) = \frac{3}{2} + \frac{1}{12} \left[ 6 + 21m + 15n - 4n\pi^2 \right] \epsilon + O(\epsilon^2),
\]

\[
\langle \Delta_{51} \rangle_{S_5} = \left( -\frac{1}{\epsilon} \left[ \frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] 2^{-2\epsilon} \right)^{-1} \int d\Omega_5^{(d-1)} \frac{\rho_{14}}{\rho_{15} \rho_{45}} \left[ w_{DC}^{51} + w_{TC}^{54} (\frac{\rho_{54}}{4})^{-\epsilon} \right] = \frac{3}{2} + \epsilon \left( \frac{\ln 2}{2} - 2 \ln \eta_4 \right) + O(\epsilon^2),
\]

\[
w_{DC}^{51} = C_{64} w_{51,64},
\]

\[
w_{TC}^{54} = C_{64} w_{54,64}.
\]

- The subtraction terms contains the NLO differential cross-section with NLO singularities
Single and double soft limit

- Single soft at NLO
  \[
  \left. \frac{2}{E_k \to 0} \right|_{2C_F g_{s,b}^2} \left( \frac{p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} \right)^2 \approx \frac{p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} \times \left. \frac{2C_F g_{s,b}^2}{E_k^2} \right|_{2C_F g_{s,b}^2} \]

- Single soft at NNLO
  \[
  S_6 F_{LM}(1, 4, 5, 6) = g_{s,b}^2 \times \frac{1}{E_6^2} \left[ (2C_F - C_A) \frac{\rho_{14}}{\rho_{16} \rho_{46}} + C_A \left( \frac{\rho_{15}}{\rho_{16} \rho_{56}} + \frac{\rho_{45}}{\rho_{16} \rho_{56}} \right) \right] \times F_{LM}(1, 4, 5)
  \]

- Double soft eikonal
  \[
  E_{\text{Eikonal}}(1, 4, 6, 7) = 4C_F^2 S_{14}(6) S_{14}(7) + C_A C_F \left[ 2S_{12}(6, 7) - S_{11}(6, 7) - S_{22}(6, 7) \right],
  \]
  \[
  S_{ij}(k) = \frac{p_i \cdot p_j}{[p_i \cdot p_k][p_j \cdot p_k]},
  \]
  \[
  S_{ij}(k, l) = S_{ij}^{\text{soft}}(k, l) - \frac{2[p_i \cdot p_j]}{[p_k \cdot p_i][p_j \cdot (p_k + p_i)][p_j \cdot (p_k + p_l)]} \left[ p_i \cdot p_k[p_j \cdot p_l] + [p_i \cdot p_l][p_j \cdot p_k] \left( \frac{1 - \epsilon}{[p_k \cdot p_l]^2} - \frac{1}{2} S_{ij}^{\text{soft}}(k, l) \right) \right] - \frac{[p_i \cdot p_j]^2}{[p_i \cdot p_k][p_j \cdot p_l][p_i \cdot p_l][p_j \cdot p_l]}. \]