Importing pertubative QCD methods into gravitational wave physics

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arXiv:2101.07254 (PRL), Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, MZ
OUTLINE

1. Background - Precision GW Physics

2. The Amplitudes Approach

3. Collider methods - method of regions, differential equations, reverse unitarity...

4. Results & comparisons with numerical relativity
Background
DISCOVERIES OF OUR TIMES

Two of the fundamental discoveries of our times: Higgs boson (2012), gravitational waves (2015). Spectacular confirmation of SM / GR.

ATLAS Collaboration, arXiv:1207.7214
Two of the fundamental discoveries of our times: Higgs boson (2012), gravitational waves (2015). Spectacular confirmation of SM / GR.

- **Intense precision theory efforts** on two types experiments - chance for cross-fertilization?

LIGO & VIRGO collaborations, arXiv:1602.03837
INSPIRAL WAVEFORMS

Weak-field perturbative expansions: Post-Newtonian, Post-Minkoskian, Self force, Effective one body...

LIGO-Virgo GW signals

GW150914
LVT151012
GW151226
GW170104
GW170814
GW170817

0 1 2

time observable (seconds)
FUTURE GW DETECTORS

- aLigo/Virgo
- KAGRA
- IndIGO
- TianQin
- DECIGO
- LISA
- Einstein

Plot: sensitivity of future GW detectors.
NEW RESULTS FOR CONSERVATIVE DYNAMICS

$G(1 + v^2 + v^4 + v^6 + v^8 + \ldots)$

$G^2 (1 + v^2 + v^4 + v^6 + v^8 + \ldots)$

$G^3 (1 + v^2 + v^4 + v^6 + v^8 + \ldots)$

$G^4 (1 + v^2 + v^4 + v^6 + v^8 + \ldots)$

$G^5 (1 + v^2 + v^4 + v^6 + v^8 + \ldots)$

$G^6 (1 + v^2 + v^4 + v^6 + v^8 + \ldots)$

5PN: distinguish BH / NS

6PN: LISA, ET...

Breathrough after 30+ year hiatus.

[adapted from Mikhail Solon's slide]
The Amplitudes Approach
Massive particles (scalar field) coupled to gravity.

Lagrangian: \( S = S_{\text{Einstein-Hilbert}} + S_{\text{point-particle}} + S_{\text{finite-size}} \)

Point-particle: scalar field in dynamic metric.

Finite-size (tidal) effect: **highly suppressed** for compact objects \( \sim \mathcal{O}(G^5) \), even more so for black holes \( \sim \mathcal{O}(G^6) \).
GRAVITY AMPLITUDES FROM YANG-MILLS

- Gravity = (Yang-Mills)$^2$. 3-point amplitude example:

\[ A_3(1^-2^-3^+) = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}, \quad M_3(1^-2^-3^+) = \frac{\langle 12 \rangle^6}{\langle 23 \rangle^2 \langle 31 \rangle^2} \]

- 4-points and above: generally a sum of YM × YM expressions.
  - Early example: Kawai-Lewellen-Tye relations (string theory)
  - Local Feynman diagram-like version: Double copy from color-kinematic duality [Bern, Carrasco, Johanson, 08]
CLASSICAL LIMITS FROM AMPLITUDES

- **Eikonal exponentiation**: scattering angle from saddle-point / stationary phase approximation.

  [Glauber, '59; Levy, Sucher, '69; Soldate, '87; 't Hooft, '87; Amati, Ciafoloni, Veneziano, '87, '88, '90, '07; Muzinich, Soldate, '88; Kobat, Ortiz, '92; Bern, Ita, Parra-Martinez, Ruf, '20; Parra-Martinez, Ruf, MZ, '20; Di Vecchia, Heissenberg, Russo, Veneziano, '21...]

- **Two-body potential**, defined in an EFT inspired by NRQED / NRQCD, has smooth classical limit.

  [Cheung, Rothstein, Solon, '18]

- Suitable **physical observables** from S-matrix have well-defined classical limits.

  [Kosower, Maybee, O'Connell, '18]

- Hyperbolic scattering can be **analytically continued** to elliptic orbits.

  [Kalin, Porto, '20]
MATCHING TO NON-RELATIVISTIC EFT

[Lagrangian: two scalars, no antiparticles, no particle creation]

\[ \mathcal{L} = \sum_{i=1,2} \int_k \phi_i^\dagger(-k) \left( i \partial_t - \sqrt{k^2 + m_i^2} \right) \phi_i(k) \]

- kinetic term

+ instantaneous potential \( \sim V(k, k') \phi_1^\dagger \phi_1 \phi_2^\dagger \phi_2 \)

1. point particle EFT

2. non-relativistic EFT

[Goldberger, Rothstein, '04]

[Cheung, Rothstein, Solon, 1808.02489]
Importing collider methods
GW physics needs 3 loops and beyond, but exact evaluation is very difficult already at 2 loops: most results are planar, with $m_1 = m_2$.

Smirnov, '01; Henn, Smirnov, '13
Two-mass: Heller '21

Heinrich, Smirnov, '04
*(only the $1/\epsilon^2$ pole)*

Leoni, Bianchi, '16; Kreer, Weinzierl, '21
Talk by S. Weinzierl

Heller, von Manteuffel, Schabinger, 19;
Broedel, Duhr, Dulat, Penante, Tancredi, '19

Talk by V. Smirnov, w/ Duhr, Tancredi
**METHOD OF REGIONS**

\[ p_2 = m_2 u_2 + q/2 \quad p_3 = m_2 u_2 - q/2 \]

\[ p_1 = m_1 u_1 - q/2 \quad p_4 = m_1 u_1 + q/2 \]

**Expansion of matter propagators (linearized):**

\[ (1) \quad (k_1 + p_1)^2 - m_1^2 \approx 2m_1 u_1 \cdot k_1 + i0 \]

\[ (2) \quad (k_2 + p_2)^2 - m_2^2 \approx 2m_2 u_2 \cdot k_2 + i0 \]

**Soft region:**

\[ |k_1| \sim |k_2| \sim |q| \ll m_1, m_2, \sqrt{s} \]

**Kinematics:**

\[ u_1^2 = u_2^2 = 1, \quad u_1 \cdot q = u_2 \cdot q = 0, \]

\[ q^2 = t, \quad u_1 \cdot u_2 = \sigma \]

**\( \sigma \): dimensionless velocity parameter - differential equations in \( \sigma \)**

**IBP analytic - two decoupled systems:**

integer and half-integer powers of \( t \)

*(Hard region does not contribute to classical physics)*
Hard gravitons are irrelevant for classical physics, sometimes after nontrivial cancellations.

- Contact potential, no long-range effects
  - Hard
  - Hard

- Scaleless integral
  - Soft
  - Hard

- Suppressed in
  - $\tilde{\eta} \sim |q|/m$, $|q|/\sqrt{s}$
  - Hard
  - Soft

- Suppressed after IR subtraction
  - Soft
  - Hard
FURTHER EXPANSION IN SMALL VELOCITY

Potential region coincides with conservative dynamics up to 2 loops.
- suitable regularization needed: symmetrization prescription

[Cheung, Rothstein, Solon, '18; Cheung, Bern, Roiban, Shen, Solon, MZ, '19; Parra-Martinez, Ruf, MZ, '20]
**DIFFERENTIAL EQUATIONS**

**PN expansion**

\[ q^2 \ll (mv)^2 \ll m^2 \]

**PM expansion**

\[ q^2 \ll m^2, \quad v \sim \mathcal{O}(1) \]

**Consistency conditions:**

Cancellation of spurious singularities

Imported into post-Minkowskian gravity: [Parra-Martinez, Ruf, *MZ*, '20]

Subsequently: [Kalin, Porto, '20; Di Vecchia, Heissenberg, Russo, Veneziano, '21; Herrmann, Parra-Martinez, Ruf, *MZ*, '21; Bjerrum-Bohr, Damgaard, Plante, Vanhove, '21]

\[
\frac{\partial}{\partial v} \left( \frac{1/(2u_1 \cdot l)}{1/(2u_2 \cdot l)} \right) = \epsilon \frac{\partial \log(\sqrt{1 + v^2} - v)}{\partial v} \times \]

\[ u_1^2 = u_2^2 = 1, \quad u_1 \cdot u_2 = y = \sqrt{1 + v^2} \]

Solution for box: constant in potential region. Logarithm in soft region.
\[
\frac{1}{(2u_1 \cdot l)} + \frac{1}{(-2u_2 \cdot l)}
\]

\[
u_1^2 = u_2^2 = 1, \quad u_1 \cdot u_2 = y = \sqrt{1 + v^2}
\]

Rationalization: \( y = \frac{1 + x^2}{2x} \)

Physical region: \( 0 < x < 1 \)

Euclidean region: \( -1 < x < 0 \)

symbol letters: \( x, 1 \pm x, 1 + x^2 \)

Canonical form: [Henn, '13]

Talk by C. Dlapa

Similar to letters in massive cusp anomalous dimension [Bruser, Dlapa, Henn, Yan, '20]

The last letter only appears at 3 loops. For the potential region, smooth static limit implies that \((1 - x)\) is never a first entry.

\[
\frac{\partial}{\partial x} \hat{I} = \epsilon \sum_r \frac{\partial \log W_r}{\partial x} \mathcal{M}^{(r)} \cdot \vec{I}
\]
Results and Comparisons
• Amplitude integrand from 8 generalized unitarity cuts.

• In potential region, used \textit{epsilon} [Prauso, '17] to find canonical form, except for one bottom-level sector (elliptic, 3×3 system).
THE 4PM / 3-LOOP CALCULATION

[Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, MZ, '21 (PRL)]

\[ M_4(q) = G^4 M^7 \nu^2 |q| \left( \frac{q^2}{4^{1/3} \tilde{\mu}^2} \right)^{-3\epsilon} \pi^2 \left[ M_4^p + \nu \left( \frac{M_4^t}{\epsilon} + M_4^f \right) \right] + \text{IR divs.} \]

\[ M_4^t = h_4 + h_5 \log(\frac{\sigma+1}{2}) + h_6 \frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}} + h_7 \log(\sigma) - h_2 \frac{2\pi^2}{3} + h_8 \frac{\text{arccosh}^2(\sigma)}{\sigma^2 - 1} + h_9 \left[ \text{Li}_2 \left( \frac{1-\sigma}{2} \right) + \frac{1}{2} \log^2 \left( \frac{\sigma+1}{2} \right) \right] \]

\[ + h_{10} \left[ \text{Li}_2 \left( \frac{1-\sigma}{2} \right) - \frac{\pi^2}{6} \right] + h_{11} \left[ \text{Li}_2 \left( \frac{1-\sigma}{2} \right) - \text{Li}_2 \left( \frac{\sigma-1}{\sigma+1} \right) + \frac{\pi^2}{3} \right] + h_{12} \frac{2\sigma(2\sigma^2-3)}{(\sigma^2-1)^{3/2}} \left[ \text{Li}_2 \left( \sqrt{\frac{\sigma-1}{\sigma+1}} \right) - \text{Li}_2 \left( -\sqrt{\frac{\sigma-1}{\sigma+1}} \right) \right] \]

\[ + \frac{2h_3}{\sqrt{\sigma^2 - 1}} \left[ \text{Li}_2 \left( 1-\sigma-\sqrt{\sigma^2 - 1} \right) - \text{Li}_2 \left( 1-\sigma+\sqrt{\sigma^2 - 1} \right) + 5\text{Li}_2 \left( \sqrt{\frac{\sigma-1}{\sigma+1}} \right) - 5\text{Li}_2 \left( -\sqrt{\frac{\sigma-1}{\sigma+1}} \right) + 2 \log \left( \frac{\sigma+1}{2} \right) \text{arccosh}(\sigma) \right] \]

\[ + h_{12} K^2 \left( \frac{\sigma-1}{\sigma+1} \right) + h_{13} K \left( \frac{\sigma-1}{\sigma+1} \right) E \left( \frac{\sigma-1}{\sigma+1} \right) + h_{14} E^2 \left( \frac{\sigma-1}{\sigma+1} \right) , \]

polylogarithms up to transcendental weight 2

complete elliptic integrals of the 1st & 2nd kind

Rational prefactors:

\[ h_1 = \frac{1151 - 3336\sigma + 3148\sigma^2 - 912\sigma^3 + 339\sigma^4 - 552\sigma^5 + 210\sigma^6}{12(\sigma^2 - 1)} , \]

\[ h_2 = \frac{1}{2} \left( 5 - 76\sigma + 150\sigma^2 - 60\sigma^3 - 35\sigma^4 \right) , \]

::
IR DIVERGENCE IN 2-BODY POTENTIAL

Potential region: graviton dominated by spatial momenta

\[ \mathcal{M}_{4}^{\text{pot}}(q) \propto \left[ \frac{\mathcal{M}_{4}^{t}}{\epsilon} + \text{finite} \right] + \text{Iterations} \]

Radiation region: small energy and spatial momenta \( \sim 1/\lambda \)

\[ \mathcal{M}_{4}^{\text{rad}}(q) \propto \left[ -\frac{\mathcal{M}_{4}^{t}}{\epsilon} + 2 \log(v^2) + \text{finite} \right] \]

Divergence cancels in the sum, leaving \( \log(v) \) term analogous to Lamb shift in QED with \( \log(\alpha) \) term.

Next slides: preliminary NR comparison

To be calculated in PM expansion!
4PM BINDING ENERGY VS. NUMERICAL RELATIVITY

[Khalil, Buonanno, Steinhoff, Vines, preliminary]

GW cycles before merger

\[ E_0 / \mu \]

\[ 40 \quad 20 \quad 10 \quad 5 \quad 4 \quad 3 \quad 2 \]

\[ -0.02 \]
\[ -0.03 \]
\[ -0.04 \]
\[ -0.05 \]
\[ -0.06 \]
\[ -0.07 \]

\[ H^{EOB}_{1PM} \]
\[ H^{EOB}_{2PM} \]
\[ H^{EOB}_{3PM} \]
\[ H_{4PM}^{local EOB, \epsilon=0} \]
\[ H^{EOB}_{4PN} \]
\[ H^{4PN}_{4PN} \]
\[ NR \]

mass ratio \( q = 1 \)

4PM \( \sim G^4 \)

4PN \( \sim G^n (v^2)^{5-n}, \quad 0 \leq n \leq 5 \)

4PM (potential region)

NR
Same plot shown as relative deviation from NR.

Post-Minkowskian prediction starts to become competitive with post-Newtonian predictions!
DISCUSSIONS & OUTLOOK

• Obtained new results for post-Minkowskian binary dynamics, beyond the best classical calculations.

• Start to compete with post-Newtonian theory, and offers new analytic insights.

• Relies on modern methods for scattering amplitudes and advanced integration techniques developed in QCD:
  - Method of regions
  - IBP & Differential equations
  - Polylogarithms & (iterated) elliptic integrals
  - Reverse unitarity  Talk by Julio Parra-Martinez

• Rich physics to be explored - spin, tidal effects, radiation reaction... Preparing for coming decades of GW physics!