Higher-order $\beta$-functions in the Standard Model and beyond

Florian Herren

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What are $\beta$-functions?

Why don’t we know the SM $\beta$-functions at 4 loops?

SM gauge coupling $\beta$-functions at 4 loops

Beyond the SM

Backup
β-functions determine the energy dependence of coupling constants:

$$\mu^2 \frac{d}{d\mu^2} \frac{\alpha_i(\mu)}{\pi} = \beta_i (\{\alpha_j\} ; \epsilon) .$$

β_i depends on all couplings {α_j} of the theory.

QCD: {α_j} = \left\{ \frac{g_s^2}{4\pi} \right\}
What are $\beta$-functions?

$\beta$-functions determine the energy dependence of coupling constants:

$$
\mu^2 \frac{d}{d\mu^2} \frac{\alpha_i(\mu)}{\pi} = \beta_i \left( \{\alpha_j\}; \epsilon \right).
$$

$\beta_i$ depends on all couplings $\{\alpha_j\}$ of the theory.

SM: $\{\alpha_j\} = \left\{ \frac{\alpha_{\text{QED}}}{\cos^2 \theta_W}, \frac{\alpha_{\text{QED}}}{\sin^2 \theta_W}, \frac{g_s^2}{4\pi}, \frac{y_f^2}{4\pi}, \frac{\lambda}{4\pi} \right\}$
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⇒ precision of $\alpha_s$ determinations made five-loop calculation necessary
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Vacuum stability

Stability of the electroweak vacuum depends on sign of quartic coupling $\lambda$

$\Rightarrow$ three-loop calculation necessary to gain confidence in results

[Zoller '13]
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Beyond the SM

- 5-loop QCD $\beta$-function
  - [Baikov, Chetyrkin, Kühn '16], [Herzog, Ruijl, Ueda, Vermaseren, Vogt '17],
  - [Luthe, Maier, Marquard, Schroder '17]
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• 3-loop SM gauge coupling β-functions
  [Mihaila, Salomon, Steinhauser '12], [Bednyakov, Pikelner, Velizhanin '12]

• 3-loop SM Yukawa coupling β-functions
  [Chetyrkin, Zoller '12], [Bednyakov, Pikelner, Velizhanin '13,'14]

• 3-loop SM self-coupling β-functions
  [Chetyrkin, Zoller '12,'13], [Bednyakov, Pikelner, Velizhanin '13]
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State of the art for gauge theories

- 3-loop 2HDM gauge and Yukawa coupling $\beta$-functions
  [FH, Mihaila, Steinhauser ’19]

- 3-loop gauge coupling $\beta$-function for arbitrary gauge group
  [Poole, Thomsen ’19]
Why don’t we know the SM $\beta$-functions at 4 loops?
Relating $\beta$-functions to counterterms for couplings

Bare and renormalized couplings are related by:

$$\alpha_i^0 = \mu^2 \varepsilon Z_{\alpha_i} \alpha_i$$

Taking the derivative w.r.t. $\mu$ and solving for $\beta_i$:

$$\beta_i = - \left[ \frac{\varepsilon \alpha_i}{\pi} + \frac{\alpha_i}{Z_{\alpha_i}} \sum_{j \neq i} \frac{\partial Z_{\alpha_i}}{\partial \alpha_j} \beta_j \right] \left( 1 + \frac{\alpha_i}{Z_{\alpha_i}} \frac{\partial Z_{\alpha_i}}{\partial \alpha_i} \right)^{-1}$$

To obtain $\beta_i$ at $L$ loops, we need to know $Z_{\alpha_i}$ at $L$ loops and all $\beta_j$ at $L - 1$ loops (at most)
Relating counterterms to Green’s functions

Couplings renormalization constants computed via

\[ Z_{\alpha_i} = \frac{Z_V^2}{\prod \phi Z_{\phi}} \]

Slavnov-Taylor identities relate renormalization constants

\[ Z_{\alpha_3} = \frac{\left( \text{diagram} \right)^2}{\left( \text{diagram} \right)^2 \times \text{diagram}} = \frac{\left( \text{diagram} \right)^2}{\left( \text{diagram} \right)^3} = \ldots \]

⇒ need to compute UV-poles of 2- and 3-point Green’s functions
Poles do not depend on masses and momenta in $\overline{\text{MS}}$ scheme
→ neglect all particle masses
Relating counterterms to Green’s functions

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$\mathcal{O}(10^5)$ diagrams per relevant Green’s function at 4 loops
3 gauge parameters
However, many diagrams share the same structure:

Combine diagrams with same colour factors and topology into super-diagrams $\rightarrow \mathcal{O}(10^3)$
SM is a chiral theory, thus $\gamma_5$ appears

In four dimensions:

$$\{\gamma_5, \gamma_\mu\} = 0$$

$$\gamma_5^2 = 1$$

$$\text{tr} \left( \gamma_5 \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\sigma \right) = -4i \epsilon_{\mu\nu\rho\sigma}$$

$$\epsilon_{\mu\nu\rho\sigma} \epsilon_{\mu'\nu'\rho'\sigma'} = g^{[\mu} g^{\nu} g^{\rho} g^{\sigma]}$$

But what about $D$ dimensions? (see also talks by Long Chen and Taushif Ahmed)
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None of these diagrams contribute
Only the second diagram contributes with an $\frac{1}{\epsilon}$ pole

$$\rightarrow \text{tr} \left( \gamma_5 \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\sigma \right) = -4i \epsilon_{\mu\nu\rho\sigma} + \mathcal{O}(\epsilon)$$
$\gamma_5$ and the four-loop gauge coupling beta functions

This fails however at four loops:

In [Bednyakov, Pikelner ’15], [Zoller ’15] a non-cyclic trace was used $\rightarrow$ Result depends on reading point
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In the framework of the local renormalization group Osborn’s equation [Osborn ’89, ’91] can be derived:

\[ \partial_i \tilde{A} = T_{ij} B^J \]

This equation gives rise to the so-called Weyl consistency conditions, relations between coefficients of tensor structures of the \( \beta \) functions.
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Weyl consistency conditions

Most general Lagrangian:

\[
\mathcal{L} = -\frac{1}{4} \sum_u F_{u,\mu\nu}^A F_{u,\mu\nu}^A + \frac{1}{2} (D_\mu \phi)_a (D^\mu \phi)_a + i \psi_i^\dagger \bar{\sigma}^\mu (D_\mu \psi)_i \\
- \frac{1}{2} (Y_{aij} \psi_i \psi_j + \text{h.c.}) \phi_a - \frac{1}{24} \lambda_{abcd} \phi_a \phi_b \phi_c \phi_d.
\]

Covariant derivatives are defined by

\[
D_\mu \phi_a = \partial_\mu \phi_a - i \sum_u g_u V_{u,\mu}^A (T_{\phi, u}^A)_{ab} \phi_b
\]

\[
D_\mu \psi_i = \partial_\mu \psi_i - i \sum_u g_u V_{u,\mu}^A (T_{\psi, u}^A)_{ij} \psi_j
\]
Graph-tensor identification rules [Poole, Thomsen ’19]:

\[ A \sim \sim B = G_{AB}^2 \]  
\[ i \sim j = \delta_{ij} \]  
\[ a \sim b = \delta_{ab} \]

\[ i \rightarrow j \quad = y_{aij} \]  
\[ a \rightarrow d \quad = \lambda_{abcd} \]  
\[ i \rightarrow j \quad = (T^A)_{ij} \]  
\[ a \rightarrow b \quad = (T^A_\phi)_{ab} \]  
\[ A \sim \sim \sim \sim B \quad = G_{AD}^{-2} f^{DBC} \]
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Weyl consistency conditions

Examples:

$$= \text{Tr} \left( T^A T^B \right)$$

$$= y_b \tilde{y}_a y_c \lambda_{bcda}$$
Weyl consistency conditions

\( \tilde{A} \) can be decomposed into coefficients \( a_i \) and tensor structures, e.g. at three loops [Poole, Thomsen '19]:

\[
\tilde{A} \supset a^{(3l)}_{10} + a^{(3l)}_{11}
\]

Derivative acts on couplings, corresponding to gauge lines, fermion-scalar vertices and quartic scalar vertices:

\[
\partial_I = \quad + \\
\partial_I = \quad + \\
\partial_I = \quad + \\
\]

[Diagram showing the decomposition and derivative action]
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Weyl consistency conditions

Overall, we get:

$$\partial_1 \tilde A \supset a^{(3l)}_{10} \left( \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array} \right) + a^{(3l)}_{11} \left( \begin{array}{c}
\text{Diagram 3} \\
\text{Diagram 4}
\end{array} \right)$$

In a similar way $T_{IJ}$ and $B^J$ can be decomposed:

$$T_{IJ} \supset t_{1}^{(1l)} + t_{4}^{(2l)}$$

$$B^J \supset g_{6}^{(2l)} + g_{7}^{(2l)} + n_{1}^{(1l)} + n_{2}^{(1l)}$$
Identifying tensor structures, we obtain 4 equations:

\[ a^{(3l)}_{10} = t^{(1l)}_1 g^{(2l)}_6, \quad a^{(3l)}_{11} = t^{(1l)}_1 g^{(2l)}_7 \]

\[ 2a^{(3l)}_{10} = t^{(2l)}_4 n^{(1l)}_1, \quad 2a^{(3l)}_{11} = t^{(2l)}_4 n^{(1l)}_2 \]

Which can be solved for

\[ g^{(2l)}_7 n^{(1l)}_2 = g^{(2l)}_6 n^{(1l)}_1 \]
Taking a closer look at the $\gamma_5$ contributions
Taking a closer look at the $\gamma_5$ contributions

Let’s connect the external lines in each of them...

All problematic contributions can be expressed through unproblematic ones [Poole, Thomsen '19]
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Calculational setup

QGRAF

[Nogueira 1993]

q2e & exp

[Harlander, Seidensticker, Steinhauser 1997]

Build superdiagrams

FORCER

Ruijl, Ueda, Vermaseren '17
**Gauge coupling $\beta$-functions in the SM at four loops**

Combining the various ingredients we obtained the 4-loop gauge coupling $\beta$-functions in the SM \cite{DaviesFH19}:

\[
\begin{align*}
\beta_1 &= \alpha_1^2 \left( \frac{82}{(4\pi)^2} \right) - \frac{\alpha_1^2}{5} + \frac{\alpha_1^2}{(4\pi)^2} \left( \frac{398\alpha_1}{25} \right) + \frac{\alpha_1^2}{5} + \frac{176\alpha_3}{5} - \frac{\alpha_1^2}{5} \left( \frac{34\alpha_4}{6000} \right) - \frac{123\alpha_1\alpha_2}{40} - \frac{54\alpha_1\alpha_3}{75} \\
&+ \frac{789\alpha_2^2}{16} - \frac{12\alpha_2\alpha_3}{5} + \frac{1188\alpha_4^2}{5} - \frac{2827\alpha_1\alpha_4}{5} - \frac{471\alpha_2\alpha_4}{8} - \frac{116\alpha_3\alpha_4}{5} + \frac{189\alpha_4^2}{4} + \frac{54\alpha_1\alpha_7}{25} + \frac{18\alpha_2\alpha_7}{5} - \frac{36\alpha_7^2}{5} \\
&+ \frac{\alpha_1^2}{(4\pi)^5} \left[ - \alpha_1^3 \left( \frac{143035700}{1080000} \right) + \frac{1638851\zeta_3}{5625} \right] - \alpha_1^2\alpha_2 \left( \frac{3819731}{24000} - \frac{16529\zeta_3}{125} \right) - \alpha_1^2\alpha_3 \left( \frac{3629273}{6750} - \frac{720304\zeta_3}{1125} \right) \\
&+ \alpha_1^2\alpha_4 \left( \frac{14971}{90} - \frac{7472\zeta_3}{15} \right) - \alpha_2^2\alpha_3 \left( \frac{1748}{3} - \frac{2944\zeta_3}{5} \right) + \alpha_2^3 \left( \frac{616}{15} - \frac{18560\zeta_3}{9} \right) + \alpha_1^2\alpha_4 \left( \frac{8978897}{72000} + \frac{2598\zeta_3}{125} \right) \\
&- \alpha_1^2\alpha_4 \left( \frac{11442}{800} + \frac{1122\zeta_3}{25} \right) - \alpha_1^2\alpha_4 \left( \frac{2012}{75} - \frac{408\zeta_3}{25} \right) - \alpha_2^2\alpha_4 \left( \frac{439841}{960} - \frac{616\zeta_3}{5} \right) + \alpha_2^2\alpha_4 \left( \frac{1468}{5} - \frac{1896\zeta_3}{5} \right) \\
&- \alpha_3^2 \alpha_4 \left( \frac{11442}{45} - \frac{3184\zeta_3}{5} \right) + \alpha_1^2\alpha_4 \left( \frac{29059}{160} - \frac{357\zeta_3}{25} \right) + \alpha_2^2\alpha_4 \left( \frac{71463}{160} + \frac{639\zeta_3}{5} \right) + \alpha_3^2 \alpha_4 \left( \frac{1429}{5} - \frac{240\zeta_3}{5} \right) \\
&- \alpha_3^3 \left( \frac{13653}{40} + \frac{102\zeta_3}{5} \right) + \frac{3627\alpha_1^2\alpha_7}{500} + \frac{1917\alpha_1\alpha_2\alpha_7}{50} + \frac{889\alpha_2^2\alpha_7}{20} + \frac{1926\alpha_1\alpha_4\alpha_7}{25} + \frac{162\alpha_2\alpha_4\alpha_7}{5} - \frac{474\alpha_4^2\alpha_7}{5} \\
&- \frac{1269\alpha_1^2\alpha_2^2}{25} - \frac{981\alpha_2\alpha_4^2}{5} + \frac{1188\alpha_4^2\alpha_7}{5} + \frac{624\alpha_7^2}{5}
\end{align*}
\]

For the three gauge couplings, the 4-loop corrections amount to 8%, 5% and 127% of the 3-loop corrections.
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Electroweak corrections cancel pure $\alpha_3$ terms at 3-loops.

$$\Delta = \frac{|\alpha_3^{(4l)} - \alpha_3^{(3l)}|}{|\alpha_3^{(3l)} - \alpha_3^{(2l)}|}$$
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Beyond the SM
Beta functions at order 4-3-2

- Statements concerning $\gamma_5$ hold for any gauge theory
- BSM-landscape is vast $\rightarrow$ dedicated computation for each model unfeasible
- Ansatz by [Poole, Thomsen ’19] covers general theory at order 4-3-2
Beta functions at order 4-3-2

- Directly computing each coefficient using real fields possible, but cumbersome
- Results available in the literature fix 487/510 coefficients
- Adding SM + $\nu_R$ and the type-I 2HDM gives 4 more constraints
- Remaining coefficients need a model with a scalar charged under multiple non-abelian gauge groups
There is one problem with Yukawa matrices starting from 3 loops:

\[ Z_f = 1 - K_\epsilon \left( \sqrt{Z_f} \Sigma(Q^2) \sqrt{Z_f} \right) \]

Square root: \[ Z_f = \sqrt{\hat{Z}_f} U^\dagger U \sqrt{\hat{Z}_f} \]
\[ \rightarrow \text{can only determine } \sqrt{Z_f} \text{ up to unitary rotation} \]
Issue with anomalous dimension (similar for $\beta$-function):

$$
\gamma_f = \sqrt{Z_f}^{-1} \mu \frac{d}{d\mu} \sqrt{Z_f}
$$

$$
= U^\dagger \sqrt{Z_f}^{-1} \left( \mu \frac{d}{d\mu} \sqrt{Z_f} \right) U + U^\dagger \mu \frac{d}{d\mu} U
$$

Choice $U = 1$ leads to poles in $\gamma_f$ (and $\beta$) starting from 3 loops
[Bednyakov, Pikelner, Velizhanin '14], [FH, Mihaila, Steinhauser '17]
Do the ambiguities lead to issues?

**RG-finiteness**

The divergent part of *any* set of MS/$\overline{\text{MS}}$ RG functions $(\beta_I, \gamma)$ satisfy

$$\gamma^{(n)} \in g_F \quad \text{and} \quad \beta^{(n)}_I = -\left(\gamma^{(n)} g\right)_I, \quad n \geq 1.$$  

This property of the RG functions is referred to as RG-finiteness.

[ FH, Thomsen '21 ]
It is possible to define an improved $\beta$-function [Fortin, Grinstein, Stergiou '12]:

$$B_I = \beta_I - (\hat{\upsilon} g)_I$$

$B$ is invariant under transformations of the fields with $G_F$ $(\hat{\upsilon} g)_I$ can be computed directly [Fortin, Grinstein, Stergiou '12] and coincides with $(\gamma^{(n)} g)_I$ [FH, Thomsen 21]
In a next step, we plan to derive the relations at orders 5-4-3.

- Allows to determine 3-loop scalar $\beta$-function, many coefficients already known [Steudtner ’21]
- Investigate $\gamma_5$ at this order (3-loop scalar $\beta$-function is safe)
- Does the number of relations grow faster than the number of coefficients?
- Are there non-trivial relations for pure gauge theories at high orders?
- Can one combine Weyl consistency conditions with relations regarding the transcendality structure [Baikov, Chetyrkin ’19]?
- ...

What else can we learn?
• In non-chiral theories 5-loop computations are feasible
• Weyl consistency conditions allow to circumvent an explicit treatment of $\gamma_5$ in certain cases
• Computed SM gauge coupling $\beta$-functions at 4 loops
• Results for a general gauge-Yukawa theory at order 4-3-2 coming soon [Davies, FH, Thomsen TBP]
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Do the ambiguities lead to issues?

Ambiguous terms are unitary $\rightarrow$ do not enter quantities invariant under flavour rotations like:

$$\text{Tr} \left( Y_u^\dagger Y_u \right), \text{Tr} \left( Y_d^\dagger Y_d \right), ...$$

$\rightarrow$ not an issue when running observables
Do the ambiguities lead to issues?

The key observation [FH, Thomsen ’21]:

- Poles are elements of the Lie-Algebra $g_F$ of the flavour group of SM (2HDM) ($G_F = U(3)^5 \times U(1(2))$)
Do the ambiguities lead to issues?

The key observation [FH, Thomsen ’21]:

\[
\left( \frac{\partial}{\partial \ln \mu} + \left( \epsilon \beta_l^{(-1)} + \beta_l \right) \partial I + \int d^d x \mathcal{J}_\beta \gamma^\beta \alpha \frac{\delta}{\delta \mathcal{J}_\alpha} \right) \mathcal{W} \\
= - \sum_{n=1}^{\infty} \frac{1}{\epsilon^n} \left( \beta_l^{(n)} \partial I + \int d^d x \mathcal{J}_\beta \gamma^{(n)} \beta \alpha \frac{\delta}{\delta \mathcal{J}_\alpha} \right) \mathcal{W} \\
= 0 \text{ iff } \beta_l^{(n)} = - (\gamma^{(n)} g), \text{ with } \gamma^{(n)} \in \mathfrak{g}_F
\]