The full angle-dependence of the four-loop cusp anomalous dimension in QED

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The angle-dependent cusp anomalous dimension

• Heavy quark scattering  \( q^\mu = m_Q (v_1 - v_2)^\mu \)

• HQET: Renormalization group

• IR divergences of massive scattering

• Ingredient in resummation

[Korchemsky, Radyushkin, '86]
Wilson lines

- **Momentum space: Eikonal propagators**
  \[
  \int \frac{d^Dk}{k^2(k \cdot v_1)(k \cdot v_2)}
  \]

- **Configuration space**
  \[
  \frac{i}{k \cdot v_1} = \int_0^\infty e^{is_1 k \cdot v_1} ds_1
  \]

- \( \Gamma_{\text{cusp}}(\phi) \) in UV and IR divergences of cusped Wilson line
  \[
  W = \frac{1}{N} \text{P} \exp \left( ig \int_C dx^\mu A_\mu(x) \right)
  \]

(Korchemsky, Radyushkin, '92)
Wilson lines

- Momentum space: Eikonal propagators
  \[ \int \frac{d^D k}{k^2 (k \cdot v_1)(k \cdot v_2)} \]

- Configuration space
  \[ \frac{i}{k \cdot v_1 + \delta} = \int_0^\infty e^{is_1 (k \cdot v_1 + \delta)} ds_1 \]

- \( \Gamma_{\text{cusp}}(\phi) \) in UV and IR divergences of cusped Wilson line
  \[ W = \frac{1}{N} \text{P exp} \left( ig \int_C dx^\mu A_\mu(x) \right) \]

[Korchemsky, Radyushkin, '92]
Four-loop $\Gamma_{\text{cusp}}(\phi)$

- Many colour structures already computed
  \[(T_F n_f)^3 C_R, (T_F n_f)^2 C_R C_F, (T_F n_f)^2 C_R C_A\]
  \[(T_F n_f) C_R C_F^2, (T_F n_f) C_R C_F C_A\]

- Casimir scaling $\Gamma_{\text{cusp}}(\phi) = C_R(\ldots) + \mathcal{O}(\alpha_s^4)$

- First non-planar corrections
  \[\left(\begin{array}{ccc}
  \text{Wilson line} \\
  \end{array}\right)^2 = \ \ + \]

- Fermion box diagrams $n_f d_R d_F / N_R, (T_F n_f) C_R C_F^2$
Universality conjecture

• Use light-like cusp as effective coupling:

\[ a \sim \Gamma_{\text{cusp}}(\phi \to i\infty) \sim K \quad \Gamma_{\text{cusp}}(\phi) = \sum_{k \geq 1} a^k \Omega^{(k)}(\phi) \]

the coefficients are matter independent

\[ \Omega_{\text{QCD}}(\phi, a) = \Omega_{\text{YM}}(\phi, a) = \Omega_{\mathcal{N}=4}(\phi, a) \]

• Prediction for matter dependent terms

\[ \Gamma_{\text{cusp}}(\phi) = \frac{\alpha_s}{\pi} \Omega^{(1)} + \left( \frac{\alpha_s}{\pi} \right)^2 \left( \Omega^{(2)} + K^{(2)} \Omega^{(1)} \right) + \mathcal{O}(\alpha_s^3) \]

• Violated for some colour structures

\[ n_f d_R d_F / N_R, (T_F n_f) C_R C_A^2 \]

[Grozin, Henn, Korchemsky, Marquard, '16]

[Brüser, Grozin, Henn, Stahlhofen, '19]
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(QED)

(Grözin, Henn, Korchemsky, Marquard, '16)

(Grözin, Henn, Stahlhufen, '19)
Workflow

\[ D = 4 - 2\epsilon \]

(qgraph) \rightarrow (FORM)

HQET \rightarrow Feynman Diagrams \rightarrow Numerator Algebra \rightarrow Feynman Integrals

FIRE6p

Integral Families \rightarrow Master Integrals

IBP

\[ d\vec{f} = dA(\epsilon) \vec{f} \]

\[ d\vec{g} = \epsilon d\tilde{A} \vec{g} \]

[Kotikov, '91 / Remiddi '97 / Gehrmann, Remiddi, '00 / Argeri, Mastrolia, '07] (Henn '13)
Integral families

• All integrals belong to one of six families:
  - ~ 500 master integrals per family
  - ~ 150 sectors per family
  - Largest sector has 17 masters
IBP reduction

• FIRE6p: “numeric” reduction over finite fields
  - Max Planck Supercomputer Cobra
  - ~ 60-100 compute nodes with 40 cores for 3-4 weeks
• Custom reconstruction algorithm for
  - Variables $D, \phi$: 40 x 20 evaluations
  - Rational numbers: 5 evaluations
• Only DEs and Feynman diagrams reconstructed
  - Iterative procedure:
    - Reconstruct DEs
    - Improve basis

(Peraro, '16 / Monteuffel, Schabinger, '15 / Katsireas, Mourrain, Pan, Cuyt, Lee, '11)
Canonical basis

• Block structure:  
  - Diagonal blocks 
  - Off-diagonal blocks 

• Standard algorithms: Epsilon, Fuchsia 

• INITIAL: 
  - E.g. 17 masters sector: 2 min to solve 
  - One integral from canonical basis needed 
  - Can often be found by analysing singularities of integrals 
  - Test $\sim \mathcal{O}(1000)$ integrals 

\[ d\vec{g} = \epsilon d\tilde{A} \, \vec{g} \]
Beyond polylogarithms?

- One sector seems to have no canonical form
- Second-order DE at $\epsilon^0$
- Also no form $d\tilde{g} = (\epsilon + \frac{1}{2}) d\tilde{A} \tilde{g}$
- Integrals not relevant for pole and therefore not needed for $\Gamma_{cusp}^{(4)}$
- Matrices have singularities $x = e^{i\phi}$

$$\{x, 1 + x, 1 - x, 1 + x^2, 1 - x + x^2, \frac{1-\sqrt{-x}}{1+\sqrt{-x}}, \frac{1-\sqrt{-x+x}}{1+\sqrt{-x+x}}\}$$

$$d\tilde{g} = \epsilon \, d\tilde{A} \, \tilde{g}$$
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- Matrices have singularities $x = e^{i\phi}$

\[
\begin{align*}
\{x, 1 + x, 1 - x, 1 + x^2, 1 - x + x^2, \frac{1-\sqrt{-x}}{1+\sqrt{-x}}, \frac{1-\sqrt{-x+x}}{1+\sqrt{-x+x}} \}
\end{align*}
\]

do not appear in $\Gamma^{(4)}_{\text{cusp}}$

\[d\tilde{g} = \epsilon d\tilde{A} \tilde{g}\]
Results

• QED result: one-loop function

\[ A(x) = -\frac{1 + x^2}{1 - x^2} \log x - 1 \]

\[ \Gamma_{\text{cusp}}(x, \alpha) = \gamma(\alpha) A(x) + \left( \frac{\alpha}{\pi} \right)^4 n_f B(x) + O(\alpha^5) \]

constant main new result

• Conjectured result:

\[ B_c(x) = \left( \frac{\pi^2}{6} - \frac{\zeta_3}{3} - \frac{5\zeta_5}{3} \right) A(x) \approx -0.484 A(x) \]

polylogarithms

\[ B(x) = \frac{1 + x^2}{1 - x^2} B_1 + \frac{x}{1 - x^2} B_2 + \frac{1 - x^2}{x} B_3 + B_4 \]
Results

- Very well approximated by rescaled one-loop function

\[ B_c(x) \approx -0.484A(x) \]

\[ \varphi = -i\phi \quad \text{Minkowskian angle} \]

\[ x = e^{i\phi} = e^{-\varphi} \]

\[ B(x) = \frac{1 + x^2}{1 - x^2} B_1 + \frac{x}{1 - x^2} B_2 + \frac{1 - x^2}{x} B_3 + B_4 \]

\[ A(x) = -\frac{1 + x^2}{1 - x^2} \log x - 1 \]
Results

- Deviation in analytic continuation

\[ B_c(x) \approx -0.484 A(x) \]

\[ x = e^{i\phi} = e^{-\varphi} \]

\[ B(x) = \frac{1 + x^2}{1 - x^2} B_1 + \frac{x}{1 - x^2} B_2 + \frac{1 - x^2}{x} B_3 + B_4 \]

\[ A(x) = -\frac{1 + x^2}{1 - x^2} \log x - 1 \]
Checks and limits

• Small angle (low-energy) limit $\phi \to 0, \ x \to 1$
  - agreement with known terms up to $O(\phi^6)$  
    
    $B(x) = \log x \left( \frac{\pi^2}{6} - \frac{\zeta_3}{3} - \frac{5\zeta_5}{3} \right)$

• Light-like (high-energy) limit $\phi \to i\infty, \ x \to 0$

    $B_c(x) = \left( \frac{\pi^2}{6} - \frac{\zeta_3}{3} - \frac{5\zeta_5}{3} \right) A(x)$

• Anti-parallel lines (threshold) limit $\phi \to \pi, \ x \to -1$
  - conformal transformation $\Rightarrow$ quark-antiquark potential

    $B(x) = \frac{\pi}{\phi - \pi} \left( \frac{79\pi^2}{72} - \frac{23\pi^4}{48} + \frac{5\pi^6}{192} + \frac{l_2\pi^2}{2} + \frac{l_2\pi^4}{12} - \frac{l_2^2\pi^4}{4} - \frac{61\pi^2\zeta_3}{24} + \frac{21\pi^2\zeta_3 l_2}{4} \right) + O(\phi - \pi)$

    $l_2 = \log(2)$

[Grazin, Henn, Stahlhofen, '17]
[Lee, Smirnov, Steinhauser, '16]
[Lee, Smirnov, Smirnov, Steinhauser, '19 / Henn, Peraro, Stahlhofen, Wasser, '19]
Summary and outlook

- $\Gamma_{\text{cusp}}^{(4)}$ in QED $n_f d_R d_F/N_R$
- High computational power needed for IBP reduction
- Size of DEs show automation of canonical form
- Result surprisingly close to rescaled one-loop function

- Missing terms: $(T_F n_f) C_R C_A^2$, $d_R d_A/N_R$