Lorentzian Uses of Non-Lorentzian Theories

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2012.06869, 1912.02638 $\bar{w}$ A. Lipstein, R. Mouland and P. Richmond
2011.06968, 2005.14331 $\bar{w}$ with T. Orchard
1904.07547 $\bar{w}$ A. Lipstein and P. Richmond
1904.05071 $\bar{w}$ R. Mouland
1808.02948 $\bar{w}$ M. Owen

Outline

- What?

  Non-Lorentzian, Lagrangian gauge field theories with 4+ 8 (conformal)-supersymmeries (can enhance to 8+16), an $SU(1,3)$ Conformal symmetry and topological charge

$$n = \frac{1}{8\pi^2}\int \mathrm{tr}(F\wedge F)$$

- So What?

  This matches a 6D SCFT on conformally compactified Minkowski space ($\Omega = -\star\Omega$, $\Omega^2 = -R^{-2}$)

$$ds^2_{Mink} = \frac{-2dx^+\left(dx^- - \frac{1}{2}\Omega_{ij}x^i dx^j\right) + dx^i dx^i}{\cos^2(x^+/R)}$$
$$\cong -2dx^+(dx^- - \tfrac{1}{2}\Omega_{ij}x^i dx^j) + dx^i dx^i$$

$x^+ \in (-\pi R, \pi R)$ with $n =$ Fourier mode number.

What?

6D CFT's are not expected to have Lagrangian descriptions.

Nevertheless there are a variety of 6D multiplets [Bershoeff, van Proeyen, Sezgin],[Ferrara, Sokachev],[Buican,Hayling,Papageorgakis] [Cordova, Dumitrescu, Intrilligator]

We reduce the abelian M5-brane on x^+, find an action, make it non-abelian and generalise. We consider:

- Tensor Multiplets $(\phi, H^+_{\mu\nu\lambda}, \lambda_\alpha)$ $(\alpha = 1, 2)$

  $$(\phi, H^+_{\mu\nu\lambda}, \lambda_\alpha) \xrightarrow{\text{Null Reduction}} (\phi, A_-, A_i, G^+_{ij}, \lambda_\alpha)$$

  Adjoint valued with invariant inner-product $(\ ,\ )$
- Hyper-multiplets (X^α, χ), in any representation with invariant inner product $\langle\ ,\ \rangle$

$$
\begin{aligned}
S = &\frac{1}{g_{YM}^2}\int dx^- d^4x \Bigg[\frac{1}{2}\big(F_{i-}, F_{i-}\big) + \frac{1}{2}\big(\mathcal{F}_{ij}, G^+_{ij}\big) - \frac{1}{2}\big(\nabla_i\phi, \nabla_i\phi\big) \\
&- \big\langle \nabla_i X_\alpha{}^m, \nabla_i X^\alpha{}_m\big\rangle - \frac{1}{2}\big(\bar{\lambda}^\alpha, \gamma_+ D_- \lambda_\alpha\big) + \frac{1}{2}\big(\bar{\lambda}^\alpha, \gamma_i \nabla_i \lambda_\alpha\big) \\
&- \frac{1}{2}\big\langle \bar{\chi}^m, \gamma_+ D_- \chi_m\big\rangle + \frac{1}{2}\big\langle \bar{\chi}^m, \gamma_i \nabla_i \chi_m\big\rangle - \frac{i}{2}\big(\bar{\lambda}^\alpha \gamma_+, [\phi, \lambda_\alpha]\big) \\
&+ \frac{i}{2}\big\langle \bar{\chi}^m \gamma_+, \phi(\chi_m)\big\rangle + i\big\langle \bar{\chi}^m \gamma_+, \lambda_\alpha(X^\alpha{}_m)\big\rangle - i\big\langle \bar{\lambda^\alpha}(X_\alpha{}^m), \gamma_+\chi_m\big\rangle\Bigg]
\end{aligned}
$$

where $m = 1, 2, ...$ labels the hyper-multiplets and

$$
\begin{aligned}
\mathcal{F}_{ij} &= F_{ij} - \frac{1}{2}\Omega_{[i|k}x^k F_{j]-} \\
\nabla_i &= D_i - \frac{1}{2}\Omega_{ij}x^j D_-
\end{aligned}
$$

N.B. ∇_i has torsion $[\nabla_i, \nabla_j] = -\Omega_{ij}D_-$

These Lagrangians admit 4 + 8 (conformal-)supersymmetries

They have a topological conserved $U(1)$ current $J_T \sim \star\mathrm{tr}(F\wedge F)$ with associated charge

$$n = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{tr}(F\wedge F)$$

There is an $SU(1,3)$ conformal symmetry which includes a Lifshitz scaling $x^- \to \zeta^2 x^-$, $x^i \to \zeta x^i$:

$$\begin{aligned} \phi &\longrightarrow \zeta^{-2}\phi & \qquad X^\alpha_m &\longrightarrow \zeta^{-2}X^\alpha_m \\ A_- &\longrightarrow \zeta^{-2}A_- & \qquad A_i &\longrightarrow \zeta A_i \\ G^+_{ij} &\longrightarrow \zeta^{-4}G^+_{ij} \end{aligned}$$

as well as translations $x^- \to x^- + a^-$ and

$$x^i \to x^i + a^i \qquad x^- \to x^- + \tfrac{1}{2}\Omega_{ij}a^i x^j$$

Special cases:

1: If we take two hypermultiplets in the adjoint and impose

$$X_\alpha{}^m = (X^\alpha{}_m)^* = \varepsilon_{\alpha\beta}\varepsilon^{mn}X^\beta{}_n \qquad \chi^m = (\chi_m)^* = \varepsilon^{mn}\chi_n .$$

then we double the supersymmetries to 8 + 16, corresponding to the $(2,0)$ theory

2: If we set $\Omega_{ij} = 0$ then the G^+_{ij} imposes $F = -\star F$.

- Path integral localises to quantum mechanics on the moduli space of instantons[Mouland]
- The action admits a Galilean boost $\delta x^- = 0$, $\delta x^i = v^i x^-$

$$\begin{aligned}\delta A_- &= -v_i A_i \\ \delta G^+_{ij} &= v_i F_{-j} - v_j F_{-i} + \varepsilon_{ijkl} v_k F_{-l}\end{aligned}$$

The $SU(1,3)$ conformal symmetry leads to a complex structure

$$z_{ab} = x_a^- - x_b^- + \frac{1}{2}\Omega_{ij}x_a^i x_b^j + \frac{i}{4R}|x_a^i - x_b^i|^2$$

and constrains the correlation functions:

$$\langle \mathcal{O}_{p_1}^{(1)} \dots \mathcal{O}_{p_N}^{(N)} \rangle = \delta_{0,p_1+\dots+p_N} \prod_{a<b}^{N} (z_{ab}\bar{z}_{ab})^{-\alpha_{ab}/2} \left(\frac{z_{ab}}{\bar{z}_{ab}}\right)^{p_{ab}R/N} \times H\left(\frac{|z_{ab}||z_{cd}|}{|z_{ac}||z_{bd}|}, \frac{z_{ab}z_{bc}z_{ca}}{\bar{z}_{ab}\bar{z}_{bc}\bar{z}_{ca}}\right)$$

$$\alpha_{ab} = \frac{1}{N-2}(\Delta_a + \Delta_b) - \frac{1}{(N-1)(N-2)}(\Delta_1 + ... + \Delta_N)$$

where H is undetermined. In particular at 2-points H is a constant and 3-points $H = H(Arg(z_{12}z_{23}z_{31}))$.

- Δ_a is the T eigenvalue
- p_a is the P_+ eigenvalue - a central extension of $SU(1,3)$

So What?

Let us look at a 6D CFT on

$$\begin{aligned} ds^2_{Mink} &= -2d\hat{x}^+ d\hat{x}^- + \delta_{ij} d\hat{x}^i d\hat{x}^j \\ &= \frac{-2dx^+ \left(dx^- - \frac{1}{2}\Omega_{ij} x^i dx^j\right) + dx^i dx^i}{\cos^2(x^+/R)} \\ &\cong -2dx^+(dx^- - \tfrac{1}{2}\Omega_{ij} x^i dx^j) + dx^i dx^j \end{aligned}$$

This CFT has an $SO(2,6)$ conformal group with generators

$$\hat{P}_\mu \qquad \hat{M}_{\mu\nu} \qquad \hat{K}_\mu \qquad \hat{D}$$

Restricting to $\partial_+ = 0$ breaks $SO(2,6)$ to the centrally extended $SU(1,3)$ algebra that we had before (*e.g.* $T = \hat{D} - \hat{M}_{+-}$).

A 6D Minkowski space scalar operator $\hat{\mathcal{O}}(\hat{x}^+, \hat{x}^-, \hat{x}^i)$ maps to

$$\mathcal{O}(x^+, x^-, x^i) = \cos^{-\Delta}(x^+/2R)\hat{\mathcal{O}}(\hat{x}^+(x), \hat{x}^-(x), \hat{x}^i(x))$$

We can Fourier expand $x^+ \in (-\pi R, \pi R)$

$$\begin{aligned}\mathcal{O}_n(x^-, x^i) &= \frac{1}{2\pi R}\int_{-\pi R}^{\pi R} dx^+ e^{inx^+/R}\mathcal{O}(x^+, x^-, x^i) \\ &= \frac{(-1)^n}{\pi}\int_{-\infty}^{\infty} du \frac{(u-i)^{n+\Delta/2-1}}{(u+i)^{n-\Delta/2+1}}\hat{\mathcal{O}}(2Ru, \hat{x}^-(x), \hat{x}^i(x))\end{aligned}$$

A key point is that this is invertable

$$\hat{\mathcal{O}}(\hat{x}) = \left(\frac{4R^2}{4R^2 + (\hat{x}^+)^2}\right)^{\Delta/2} \sum_{n\in\mathbb{Z}} \left(\frac{2R - i\hat{x}^+}{2R + i\hat{x}^+}\right)^n \mathcal{O}_n(x^-, x^i)$$

We can compute correlators from the 6D theory *e.g.* at 2-points

$$\left\langle \mathcal{O}^{(1)}_{n_1} \mathcal{O}^{(2)}_{-n_2} \right\rangle = \frac{(-1)^{n_1+n_2}}{\pi^2} \int_{-\infty}^{\infty} d^2u \prod_{a=1}^{2} \frac{(u_a - i)^{n_a+\Delta/2-1}}{(u_a+i)^{n_a-\Delta/2+1}}$$

$$\times \langle \hat{\mathcal{O}}^{(1)}(2Ru_1, \hat{x}_1^-, \hat{x}_1^i) \hat{\mathcal{O}}^{(2)}(2Ru_2, \hat{x}_2^-, \hat{x}_2^i) \rangle$$

$$= \delta_{n_1,-n_2} d(\Delta_1, n_1) \hat{C}_{12} \frac{1}{(z_{12}\bar{z}_{12})^{\Delta/2}} \left(\frac{z_{12}}{\bar{z}_{12}}\right)^{n_1}$$

And these indeed solve the 5D $SU(1,3)$ Ward identities for a specific coefficient:

$$d(\Delta_1, n_1) = (-2R\,i)^{-\Delta_1} \binom{n_1 + \frac{\Delta_1}{2} - 1}{n_1 - \frac{\Delta_1}{2}}$$

We can also reduce 3-point functions to find the function H

Since this operator map is invertible we can also compute the 6D correlator if we know all the 5D ones:

$$\langle \hat{\mathcal{O}}^{(1)}(\hat{x}_1)\hat{\mathcal{O}}^{(2)}(\hat{x}_2)\rangle = \cos^\Delta\left(\frac{x_1^+}{2R}\right)\cos^\Delta\left(\frac{x_2^+}{2R}\right)\langle \mathcal{O}(x_1(\hat{x}_1))\,\mathcal{O}(x_2(\hat{x}_2))\rangle$$

$$= \cos^\Delta\left(\frac{x_1^+}{2R}\right)\cos^\Delta\left(\frac{x_2^+}{2R}\right)\sum_{n=\Delta/2}^{\infty} e^{-inx_{12}^+/R}\left\langle \mathcal{O}_n(x_1^-, x_1^i)\mathcal{O}_{-n}(x_2^-, x_2^i)\right\rangle$$

$$= \hat{C}_{12}(-2R\,i)^{-\Delta}\cos^\Delta\left(\frac{x_1^+}{2R}\right)\cos^\Delta\left(\frac{x_2^+}{2R}\right)(z_{12}\bar{z}_{12})^{-\frac{\Delta}{2}}$$

$$\times\sum_{n=\Delta/2}^{\infty} e^{-inx_{12}^+/R}\begin{pmatrix} n+\frac{\Delta}{2}-1 \\ n-\frac{\Delta}{2}\end{pmatrix}\left(\frac{z_{12}}{\bar{z}_{12}}\right)^n$$

$$= \hat{C}_{12}\left[\frac{2R\,i\left(\bar{z}_{12}e^{ix_{12}^+/2R} - z_{12}e^{-ix_{12}^+/2R}\right)}{\cos\left(\frac{x_1^+}{2R}\right)\cos\left(\frac{x_2^+}{2R}\right)}\right]^{-\Delta}$$

$$= \hat{C}_{12}|-2\hat{x}_{12}^+\hat{x}_{12}^- + \hat{x}_{12}^i\hat{x}_{12}^i|^{-\Delta}$$

and similarly for 3-points.

Relation to DLCQ:

- Consider a $\mathbb{Z}_k$ 'orbifold' by restricting to modes $n \in k\mathbb{Z}$.

- This corresponds to changing the coupling in the gauge theory $g_{YM}^2 \propto R/k$

- Alternatively sending $R, k \to \infty$ with $R_+ = R/k$ fixed sends $\Omega \to 0$, reduces to QM on Instanton moduli space, and hence leads to the DLCQ setup: $x^+ \sim x^+ + 2\pi R_+$

- N-point functions can be reduced: no spatial fall-off but the oscillating part remains *e.g.* at 2-points [Aharony, Berkooz, Seiberg]

$$\left\langle \mathcal{O}_n^{(1),\text{DCLQ}} \mathcal{O}_{-n}^{(2),\text{DCLQ}} \right\rangle \;\propto\; \left(x_{12}^-\right)^{-\Delta} \exp\left(\frac{in}{2R_+} \frac{|x_{12}^i|^2}{x_{12}^-} \right)$$

Conclusions

- Novel 5D gauge theories with 4/8+8/16 (conformal) supersymmetries corresponding to (1,0)/(2,0) 6D SCFT's on conformally compactified Minkowski space.
- non-Lorentzian Liftshitz field theories with a (centrally extended) $SU(1,3)$ conformal group and a KK-like tower of states graded by instanton number
- Interesting in their own right but we explored how they can capture the full 6D correlators
- M5-brane analogue of ABJM:

$$M2 : SU(4) + monopoles \to SO(8)$$

$$M5 : SU(1,3) + instantons \to SO(2,6)$$

- Offers a regularised and invertable DLCQ description

Thank You

In detail the non-zero commutators of the $SU(1,3)$ algebra are

$$
\begin{aligned}
[M_{i+}, P_j] &= -\delta_{ij} P_+ - \tfrac{1}{2}\Omega_{ij} T - \tfrac{2}{R}\delta_{ij} B + \Omega_{ik}\eta^I_{jk} C^I \\
[T, P_-] &= -2P_- \\
[T, K_+] &= 2K_+ \\
[K_+, P_-] &= -2T \\
[P_-, M_{i+}] &= P_i \\
[M_{i+}, M_{j+}] &= -\tfrac{1}{2}\Omega_{ij} K_+ \\
[T, P_i] &= -P_i \\
[T, M_{i+}] &= M_{i+} \\
[P_i, P_j] &= -\Omega_{ij} P_- \\
[C^I, C^J] &= -\varepsilon^{IJK} C^K
\end{aligned}
$$

Here B, C^I, $I = 1,2,3$ are spatial rotations that preserve Ω and P_+ is a central extension.

Future Directions

- Try to perform computations in the 5D theory
- Find conditions for which 5D actions lift to 6D
- Understand the role of instanton operators

  [NL,Papageogakis,Schmidt-Sommerfeld],[Tachikawa]
- Understand the $\mathcal{F} = - \star \mathcal{F}$ constraint
- Apply the conformal compactification to other dimensions with known Lagrangians and maybe make contact with other works [Beem,Lemos, Liendo, Rastelli, van Rees],[Baiguera, Harmark,Wintergerst],[Harmark, Hartong, Menculini, Obers, Yan]