Thermodynamic structure of a generic null surface: old wine in a new bottle

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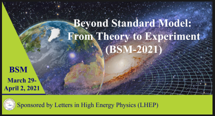
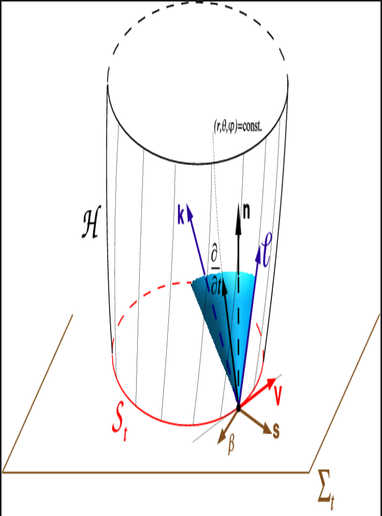


The Setting: The Generic Null Hypersurface



[1]

- $\mathcal{H}$ is any generic null hypersurface in $(\mathcal{M}, g)$. For example, any Black Hole event horizon.
- l^a are the null generators of $\mathcal{H}$, k^a the auxiliary null vector and $q^a{}_b$ the induced metric onto S_t.
- S_t: spacelike transverse submanifold of $\mathcal{H}$.
- The null generators satisfy,

$$\boxed{l^a \nabla_a l^b = \kappa l^b}, \tag{1}$$

where κ is the non-affinity paramter and hence related to the surface gravity of $\mathcal{H}$.

[1] Image adapted from arXiv:gr-qc/0503113

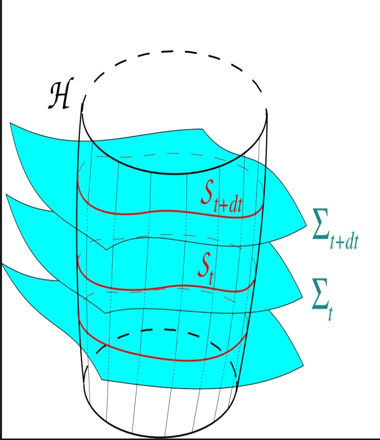
- It has been shown that various projection components of the Ricci tensor R_{ab} onto a generic null hypersurface lead to various physical interpretations.
- Damour showed that the component $R_{ab}l^a q^b{}_c$, for a black hole event horizon in Einstein gravity leads to the Damour Navier-Stokes (DNS) equation. This was later generalized for the case of any general null hypersurface.
- Jacobson (arXiv:gr-qc/9504004) via the component $R_{ab}l^a l^b$ derived the Einstein equations from an equilibrium local constitutive relation as applied to a local causal horizon constructed at any point in the spacetime manifold.
- Padmanabhan and his collaborators (arXiv:1505.05297 [gr-qc]) projected the $R_{ab}l^a k^b$ component of the Einstein equation onto a generic null surface to yield a specific "thermodynamic structure" explicitly in *"Gaussian Normal Coordinates (GNC)*.

$$\boxed{\int_{S_t} d^2x T\delta_\lambda s = \delta_\lambda E + F\delta\lambda \ .} \tag{2}$$

Generalization of $R_{ab}l^a k^b$

- The above thermodynamic interpretation for $R_{ab}l^a k^b$ was brought down explicitly under the GNC (u, r, x^A) adapted to the null hypersurface $\mathcal{H}$. As a result, the thermodynamic quantities are coordinate dependant.
- We show how the above result can be generalized to give the thermodynamic interpretation in a completely COVARIANT fashion without the need for an explicit coordinate system adapted to the null surface.
- We will use a $3 + 1$ foliation of our generic null hypersurface $\mathcal{H}$.

Construction of the Geometrical Arena :



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- Our $\mathcal{H}$ is parametrized by a level function, say $\Phi(x^a) = 1$ i.e $\mathcal{H} = \mathcal{H}_{\Phi=1}$.
- $(\mathcal{M}, g)$ in the neighbourhood of $\mathcal{H}$ is foliated by the family of null surfaces $\Phi = c \in \mathcal{R}$.
- Allows a familly of null surfaces $\mathcal{H}_\Phi$ in the vicinity of $\mathcal{H}$.
- Foliate the family $\mathcal{H}_\Phi$ by a stack of $t(x^a) =$ const spacelike surfaces Σ_t in the spirit of $3 + 1$ induced foliation.
- $S_t = \mathcal{H} \cap \Sigma_t$

[2] Image adapted from arXiv:gr-qc/0503113

Geometrical/Kinematical Quantities

- We consider the Deformation rate tensor of the 2 surface S_t :

$$\boxed{\Theta_{ab} = \frac{1}{2} q^c{}_a q^d{}_b \ \mathcal{L}_{\mathbf{l}} \ q_{cd}} \tag{3}$$

$$\Theta_{ab} = \frac{1}{2} \theta_l \ q_{ab} + \sigma_{ab} \ . \tag{4}$$

- Similarly, we consider the Transversal deformation rate tensor of the 2 surface S_t :

$$\boxed{\Xi_{ab} = \frac{1}{2} q^c{}_a q^d{}_b \ \mathcal{L}_{\mathbf{k}} \ q_{cd}} \tag{5}$$

$$\Xi^{ab} = \frac{1}{2} \theta_{(k)} \ q^{ab} + \sigma_{(k)}{}^{ab} \tag{6}$$

- The Rotation 1-form and the Hajicek 1- form of the null hypersurface are defined as

$$\boxed{\omega_a = l^b \nabla_b k_a \quad \text{and} \quad \Omega_a = q^b{}_a \ \omega_b} \tag{7}$$

- For the *dynamics*, we arrive at the relation,

$$
\begin{aligned}
q^i{}_a q^j{}_b \, \mathcal{L}_{\mathbf{l}} \, \Xi_{ij} &= \frac{1}{2}\left({}^2\mathcal{D}_a \, \Omega_b + {}^2\mathcal{D}_b \, \Omega_a\right) + \Omega_a \Omega_b - \frac{1}{2} \, {}^2R_{ab} \\
&+ \frac{1}{2} q^i{}_a q^j{}_b \, R_{ij} - \left(\kappa + \frac{\theta_{(l)}}{2}\right) \Xi_{ab} \\
&- \frac{\theta_{(k)}}{2} \, \Theta_{ab} + \Theta_{ai} \, \Xi^i{}_b + \Xi_{ai} \, \Theta^i{}_b \, . \qquad (8)
\end{aligned}
$$

- Taking its trace,

$$
\begin{aligned}
-\kappa \, \theta_{(k)} &= \left(- {}^2\mathcal{D}_a \Omega^a - \Omega_a \Omega^a + \theta_{(l)} \, \theta_{(k)} + l^i \nabla_i \theta_{(k)} + \tfrac{1}{2} {}^2R \right) \\
&\quad - \left(R_{ab} l^a k^b + \tfrac{1}{2} R \right) . \qquad (9)
\end{aligned}
$$

This can be regarded as the Null Raychaydhuri Equation (NRE) for k^a of a null hypersurface with vanishing twist or vorticity.

- $k^i = -(dx^i / d\lambda_{(k)})$, where $\lambda_{(k)}$ is the parameter along the k^i field.
- We perform a *"Virtual Displacement"* $\delta\lambda_{(k)}$ along k^a.

The Thermodynamic Identity

- Multiplying both sides of Eq. (9) with the transversal area element $d^2x\sqrt{q}$ of S_t, the overall factor of $\frac{1}{8\pi G}$ and integrating it over S_t, we land up with,

$$\boxed{\int_{S_t} d^2x T\delta_{\lambda(k)}s = \delta_{\lambda(k)}E - \delta\lambda_{(k)}\int_{S_t} d^2x\sqrt{q}\frac{1}{8\pi G}\left[R_{ij}l^i k^j + \frac{1}{2}R\right]} \tag{10}$$

- $T = \frac{\kappa}{2\pi}$ is the temperature of the null surface $\mathcal{H}$ as observed by l^a.
- $s = \frac{\sqrt{q}}{4G}$ is the *Entanglement Entropy Density* assigned to $\mathcal{H}$ by observer along l^a.
- The Energy E is given by,

$$\begin{aligned} E = \int d\lambda_{(k)}\int_{S_t} d^2x\sqrt{q}\frac{1}{8\pi G}\Big[\frac{1}{2}\ {}^2R \\ + l^i\nabla_i\theta_{(k)} + \theta_{(l)}\ \theta_{(k)} - \Omega_a\Omega^a - {}^2\mathcal{D}_A\Omega^A\Big]\ . \end{aligned} \tag{11}$$

- Allowing ourselves the liberty to define a *"Geometric/Gravitational Pressure"* $P = -1/(8\pi G)(R_{ij} l^i k^j + \frac{1}{2} R)$ under $\delta\lambda_{(k)}$, we have,

$$-\delta\lambda_{(k)} \int_{S_t} d^2x \sqrt{q} \frac{1}{8\pi G} \left(R_{ij} l^i k^j + \frac{1}{2} R \right) = \delta\lambda_{(k)} \int_{S_t} d^2x \sqrt{q} P \equiv F \delta\lambda_{(k)}$$

- Under the virtual displacement $\delta\lambda_{(k)}$, we hence have from Eq. (10),

$$\boxed{\int_{S_t} d^2x T \delta_{\lambda(k)} s = \delta_{\lambda(k)} E + F \delta\lambda_{(k)} \ .} \tag{12}$$

The NRE for k^a under the virtual displacement process $\delta\lambda_{(k)}$ yields a structure analogous to the *First Law of Thermodynamics* for **ANY** theory of Gravity.

Window Dressing

- When T and hence κ is independant of (x^A), then,

$$T\delta_{\lambda(k)}S = \delta_{\lambda(k)}E + F\delta\lambda_{(k)} \ . \tag{13}$$

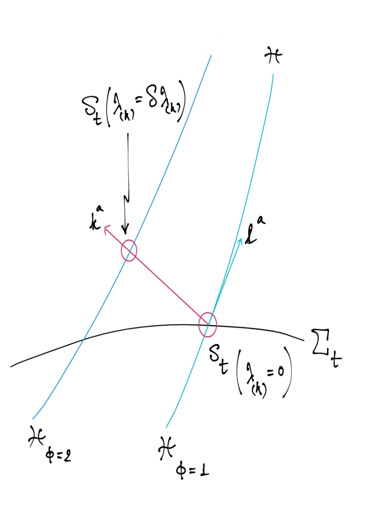
- The Energy expression Eq.(11) can be compactly written as,

$$E = \frac{1}{2}\int d\lambda_{(k)}\left(\frac{\chi}{2G}\right) + \frac{1}{8\pi G}\int d\lambda_{(k)}\int_{S_t} d^2x\sqrt{q}\Big[l^i\nabla_i\theta_{(k)} + \theta_{(l)}\theta_{(k)} -\Omega_a\Omega^a - {}^2\mathcal{D}_A\Omega^A\Big] \ , \tag{14}$$

where, $\chi = \frac{1}{4\pi}\int_{S_t} d^2x\sqrt{q}\ {}^2R$.

What we have is a completely COVARIANT CO0RDINATE INDEPENDANT expression of Energy, thanks to our Null Foliation.

The Physics



- The physics lies in the process of the *Virtual Displacement* of $\mathcal{H}$ along k^a.
- The virtual dispacement is consistent with the Gravitional field equations of the theory.
- As a result of the *Physical Process* involving $\delta\lambda_k$, an amount of Energy $\delta_{\lambda_{(k)}} E$ sweeps **over** $\mathcal{H}$.
- Part of the Energy is used in the Entropy generation of the null surface $\int_{S_t} d^2x T \delta_{\lambda(k)} s$.
- The other part contributes to the virtual work $F\delta\lambda_{(k)}$ done under $\delta\lambda_{(k)}$.

Connection and Differences with previous Results

- We compute the expression of the Energy Eq. (14) for the GNC (u, r, x^A) system for the case of Einstein Gravity,

$$E = \frac{1}{2}\int dr\left(\frac{\chi}{2G}\right) - \frac{1}{8\pi G}\int_{S_t} d^2x\,\partial_u\sqrt{q} - \frac{1}{16\pi G}\int dr\int_{S_t} d^2x\sqrt{q}\left[\frac{1}{2}\beta_A\beta^A + \frac{1}{\sqrt{q}}\partial_A(\sqrt{q}\beta^A)\right] \tag{15}$$

  This exactly matches with the Energy expression as obtained by Padmanabhan (arXiv:1505.05297 [gr-qc]) in the GNC formulation.

- The Gravitational/Geometric Pressure in the case of Einstein Gravity is exatly that from the matter-energy tensor,

$$P = -\frac{1}{(8\pi G)}(R_{ij}l^i k^j + \frac{1}{2}R) = (-T_{ij}l^i k^j)\ . \tag{16}$$

  For the GNC in Einstein gravity,

$$(-T_{ij}l^i k^j) = (-T^{ij}l_i k_j) = -T^a{}_b l_a k^b = T^{ur} = T^r{}_r = T_{ur} = T^u{}_u$$

- For spherically symmetric spacetimes, $T^r{}_r$ is interpreted as the *Radial/Normal Pressure*. Hence $F = \int_{S_t} d^2x \sqrt{q} P = \int_{S_t} d^2x \sqrt{q} T^r{}_r$, is to be interpreted as the average normal force on S_t.
- For our case, irrespective of the theory of gravity, the entropy density is consistently the Entanglement entropy density *as observed by the observer along l^a* . However for Padmanbhan, it is the Bekenstein Hawking entropy density for Einstein Gravity and the Wald entropy density for Lanczos-Lovelock models.
- The Work funtion for Padmanabhan is always $P = -T_{ab} l^a k^b$, either for Einstein or Lanczos-Lovelock gravity.
- We however adopt a geometric/gravitational work function $P = -\frac{1}{(8\pi G)}(R_{ij} l^i k^j + \frac{1}{2} R)$.


Thank you!