Beyond Standard Model: From Theory to Experiment (BSM-2021)

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Evaporation of rotating BTZ Black hole under Hawking radiation

Contents

2. Interior Volume of black hole

- Introduction to BH interiors.

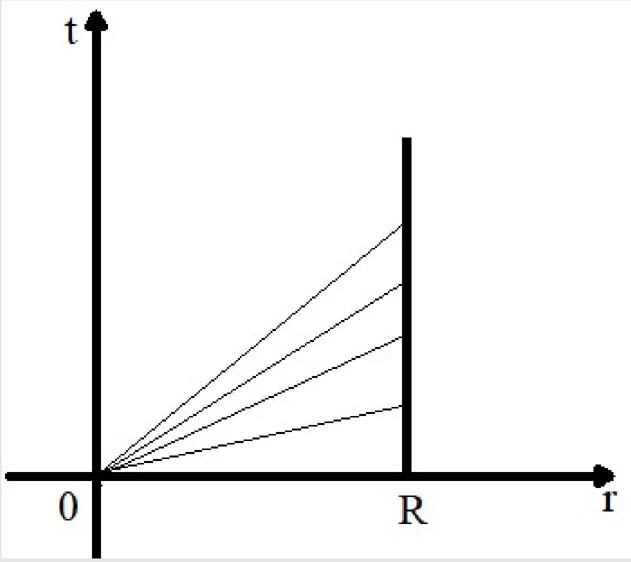
a. In Flat/ Minkowski Space-time

- In Minkowski spacetime the interior volume of spherically symmetric two-sphere “S” needs to choose a space like bound surface.
- For as sphere of radius R Area $= \pi R^2$, $V = \frac{4}{3}\pi R^3$ 1
- If there are many space-like bounded surfaces, then

a. $\sum$ lie on the same simultaneity surface as S.

b. $\sum$ is the largest spherically symmetric bounded surface of S.

- $\delta V = 0$ and $V_R > V_{R'}$
- For a sphere in For a Sphere in curved space-time there are radial space like curves of infinite length and radial time like curves equal to proper time.



b. In curved spacetime

- In GR, the definition of **simultaneity surface** is not valid in curved spacetime (1).
- Schwarzschild geometry in Eddington Finkelstein coordinates

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Omega^2 \qquad 2$$

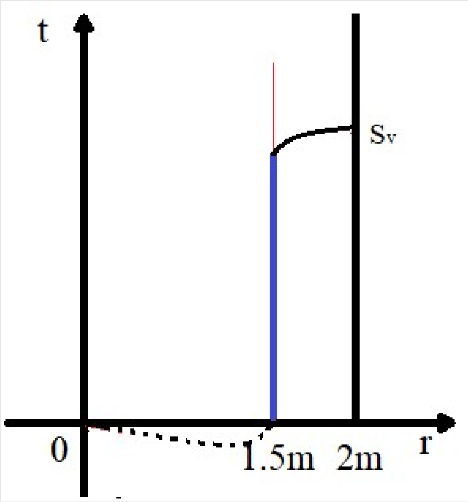
Or

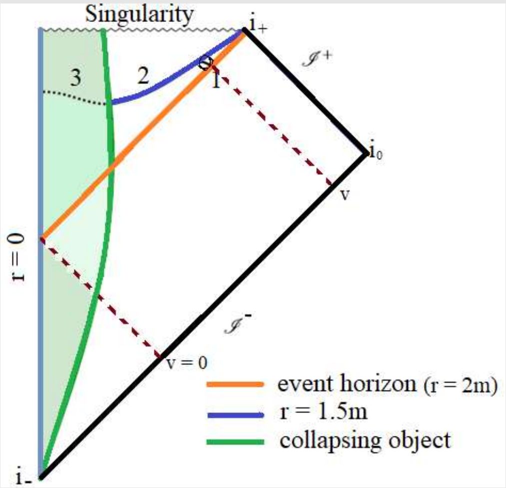
$$ds^2 = (-f(r)\dot{v}^2 + 2\dot{v}\dot{r})d\lambda^2 + r^2 d\Omega^2 \qquad 3$$

Where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and the dot represents differentiation with respect to λ.

- For space-like hypersurface $-f(r)\dot{v}^2 + 2\dot{v}\dot{r} > 0$

which led us $ds^2 > 0$





1. M. Christodoulou and C. Rovelli, Phys. Rev. D 91, no. 6, 064046 (2015), doi:10.1103/PhysRevD.91.064046

- The proper volume of the hypersurface is

$$V = 4\pi\int d\lambda\sqrt{r^4 g_{\mathrm{ab}}dx^a dx^b} = 4\pi \int_0^{\lambda_f} d\lambda \sqrt{r^4(-f(r)\,\dot{v}^2 + 2\dot{v}\dot{r})} \qquad 4$$

- Where the auxiliary metric on the hypersurface is

$$ds^2_{m_{aux}} = r^4\big(-f(\dot{r})v^2 + 2\dot{v}\dot{r}\big) \qquad 5$$

- Solving the eq. (4) we gets

$$\frac{V}{4\pi} = \lambda_f = \int_0^{2m} \frac{r^4 dr}{\sqrt{A^2 + r^4 f(r)}} \qquad 6$$

Where A is a constant

- By Maximizing λ_f for $v \gg m$, the geodesic will spent maximum time along the radius from starting to ending points and $\dot{r} = 0$. So, the auxiliary metric is

$$ds_{m_{aux}} = \sqrt{-r^4 f(r)}dv \qquad 7$$

●The interior volume of spherical symmetric black hole is

$$V_{max} = 4\pi\sqrt{-r_v^4 f(r_v)}v = 4\pi A_c v \ , \qquad A_c = \frac{3}{4}\mathrm{m}^2 \tag{8}$$

●Maximizing $\frac{ds}{dv}$ as $\frac{d}{dr}\sqrt{-r^4 f(r)} = 0 \quad \Rightarrow \quad r_v = \frac{3}{2}m$ so, the interior volume

$$V_{CR} = 3\sqrt{3}\pi m^2 v \tag{9}$$

●For charged black hole (2), $\mathrm{r}_v = \frac{3m\pm\sqrt{9m^2+8q^2}}{4}$, the interior volume is

$$V_{CR} = \frac{\pi\nu}{\sqrt{2}}\sqrt{\left(27M^4 - 36M^2q^2 + 8q^4 + 9M^3\sqrt{9M^2-8q^2} - 8Mq^2\sqrt{9M^2-8q^2}\right)} \tag{10}$$

●In d-dimensional Schwarzschild black hole (3, 4) $r_v = r_h\left(\frac{d}{d-2}\right)^{\frac{1}{d-2}}$ and the volume is

$$V_{CR} = A_{d-1}\left(\frac{d}{2(d-2)}\right)^{\frac{d-1}{d-2}}\left(\frac{d-2}{d}\right)^{\frac{1}{2}} r_h^{\frac{1}{2}} v \tag{11}$$

(2). S-Z Han, J-Z Yang, X. Y Wang and W. B Liu. DOI: 10.1007/s10773-018-3856-6
(3). N. Bhaumik and Bihas Ranjan Majhi work DOI: 10.1142/S0217751X18500112
(4). J-Z Yang, Wen-BiaoLiu ` DOI: 10.1016/j.physletb.2018.05.050

3. Volume in the interior of axially symmetric rotating BTZ BH

- The metric of (2+1)-dimensional rotating BTZ black hole is defined as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(N^{\phi}(r)dt + d\phi\right)^2 \qquad 12$$

Where the lapse and shift functions are

$$f(r) = -m + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad N^{\phi} = \frac{J}{2r^2}, (|J| \leq ml),$$

$\Lambda = -\frac{1}{l^2}$ is the cosmological constant, and ϕ is the period ranging $0 \leq \phi \leq 2\pi$. black hole space-time, J, m, and l are the azimuthal angular momentum, AdS mass and AdS radius corresponding to angular velocity $\Omega(r)$ respectively.

- The mass m and Bekenstein Hawking entropy S_{BH} of BTZ rotating black hole at horizon are

$$m = \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \qquad S_{BH} = 2\pi r_+ \qquad 13$$

- At horizon, the lapse function vanishes so the inner and outer radii are

$$r_{\pm} = \sqrt{\frac{l^2 m}{2}(1 \pm X)}, \qquad X = \sqrt{1 - \left(\frac{J}{lm}\right)} \qquad 14$$

The coordinates singularities for rotating BTZ black hole can be defined at $\frac{J}{lm} = 1$

- Hawking temperature at horizon can be define as

$$T = \frac{f\prime(r_+)}{4\pi} = \frac{mX}{\pi\sqrt{2l^2 m(1+X)}} \qquad 15$$

where the dash (') represents the derivative with respect to r.

- The angular velocity around the axis of rotation in its actual form can be defined as

$$d\phi = \Omega(\mathrm{r})dt = \frac{J}{2r^2}\,\mathrm{dt} = -\mathrm{N}^{\phi}(r)dt$$

- The positive heat capacity due to axially symmetric angular momentum J and constant angular velocity Ω are

$$C_J = T\left(\frac{\partial S}{\partial T}\right)_J = \frac{C_0 X}{2-X}\sqrt{\frac{1+X}{2}}, \qquad C_\Omega = T\left(\frac{\partial S}{\partial T}\right)_\Omega = 4\pi\sqrt{\frac{m(1+X)}{2}}$$

- The metric of Eq. (12) becomes

$$ds^2 = -f dv^2 + 2dvdr + r^2\left(N^{\phi}(r)dt + d\phi\right)^2 \quad 16$$

Or we can write as

$$ds^2 = (-f\dot{v}^2 + 2\dot{v}\dot{r})d\lambda^2 + r^2\left(N^{\phi}(r)dt + d\phi\right)^2 \quad 17$$

Here λ is an arbitrary parameter to define the curve.

- Using CR analogy the maximal volume in the interior of an axially symmetric rotating BTZ is found to be

$$V_{max} = 2\pi\sqrt{-r_v^2 f(r_v)v}$$

Where using $$r_v = \frac{\sqrt{l^2 m\left(\sqrt{3X^2+1}+2\right)}}{\sqrt{6}} \quad 18$$

Which is numerically found to be $r_v = 0.45m$

$$V_{max} = \frac{1}{3}\pi v\sqrt{-9J + 3J^2 + 2l^2(2 + \sqrt{4 - \frac{3J^2}{l^2m^2}})m^2(-2 + 3m\Big)} \quad 19$$

This equation shows that the interior volume of BH increases with Eddington time.

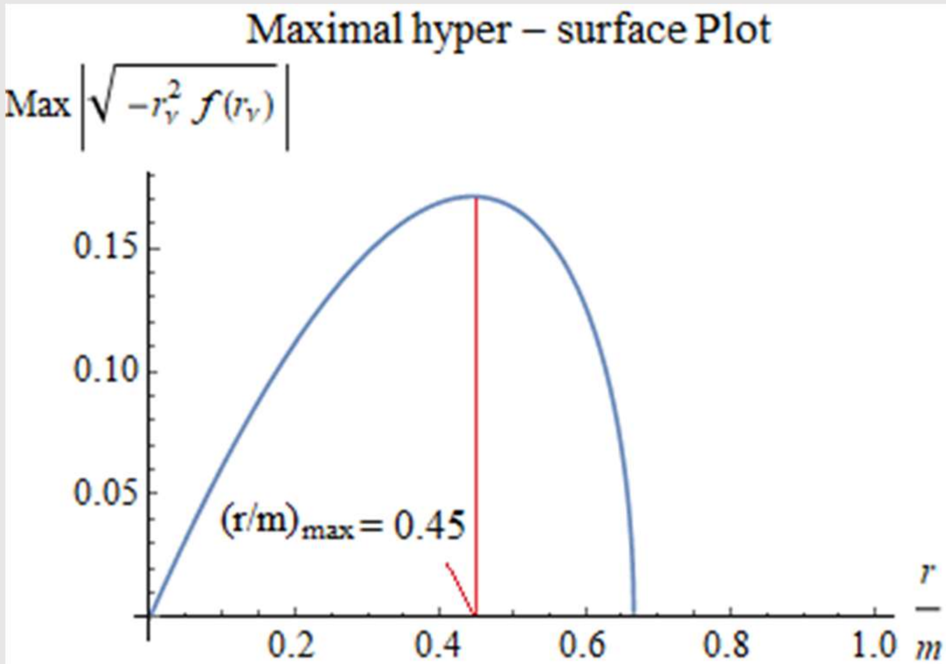


Figure : Plot of $Max\left|\sqrt{-r_v^2 f(r_v)}\right|$vs $\frac{m}{\mathrm{r}}$. The position of maximal hyper-surface in the interior of black hole maximal volume reaches to 0.45m at values of $l^2 = -1$ and J = 0.5

4. Entropy of massless scalar field and its evolution relation with Bekenstein Hawking entropy

- The interior entropy of massless scalar field ϕ in the interior volume of black hole bound by the maximal hypersurface $r = r_v$ can be calculated by using WKB approximation. Where the massless scalar field is $\phi = \exp[iET]\exp[iI(\lambda, \phi)]$(5) So, using Klein Gordon equation we get

$$E^2 - \frac{1}{-f(r)\dot{v}^2 + 2\dot{v}\dot{r}} p_\lambda^2 - \frac{1}{r^2} p_\phi^2 = 0 \qquad 20$$

or

$$p_\lambda = \sqrt{-f(r)\dot{v}^2 + 2\dot{v}\dot{r}} \sqrt{E^2 - \frac{1}{r^2} p_\phi^2} \qquad 21$$

Where, we used

$$P_\lambda = \frac{\partial I}{\partial \lambda},\ P_\phi = \frac{\partial I}{\partial \phi}$$

- Using general form of calculating the total number of quantum state in black hole, we found

$$\text{g(E)} = \frac{1}{(2\pi)^2} \int d\lambda d\phi dP_\lambda dP_\phi = \frac{E^3}{12\ \ ^2} V_{\text{CR}} \qquad 22$$

B. Zhang, Phys. Rev. D 92, no. 8, 081501 (2015) doi:10.1103/PhysRevD.92.081501

- From which the Free energy can be calculated as

$$F(\beta) = \frac{1}{\beta}\int \ln(1 - \exp -\beta E)dg(E) = -\frac{V_{max}\zeta(3)}{4\pi\beta^3} \qquad 23$$

- where $\zeta(3)$ is Riemann Zeta Function. Finally, the entropy of the massless scalar field can be obtained

$$S_\phi = \beta^2\frac{\partial F}{\partial\beta} = \frac{3\zeta(3)V_{max}}{4\pi\ \ ^3} = \frac{\zeta(3)}{2\beta^2}\sqrt{-9J + 3J^2 + 2l^2(2 + \sqrt{4 - \frac{3J^2}{l^2m^2}})m^2(-2 + 3m\Big)}\,v \qquad 24$$

- This equation shows the entropy of interior scalar quantum mode is also proportional the Eddington time and this characteristic could change the statistical quantities in the interior of BTZ black hole.

5. Hawking radiation and Differential Entropy

- If this is true then we could make a relation between the interior and exterior entropy by following Parikh(6) statement for understanding the evaporation of rotating BTZ black hole. For this purpose, let introduce two assumptions

➢ *Black hole radiations as black body radiations*

➢ *Considering the radiation emission process as quasi static*.

- The first assumption could lead us to use Boltzmann law as

$$\frac{dm}{dv} = -\sigma A T^3 \Rightarrow dv = \frac{\beta^3 \gamma}{A} dm \qquad 25$$

Where $A = \pi l \sqrt{2m(X+1)}$ is the area of rotating BTZ black hole and β is the inverse Hawking temperature.

6. M. K. Parikh, Phys. Rev. D 73, 124021 (2006), doi:10.1103/PhysRevD.73.124021

- While the second one guarantees us to calculate the differential form of quantum mode entropy for a small interval of time i.e. $\frac{dm}{dv} \ll 1$.
- Taking the differential form of quantum mode entropy and fixing these assumptions, we get

$$dS_{\phi} = -\frac{\zeta(3)\gamma}{2}\sqrt{-9J + 3J^2 + 2l^2(2 + \sqrt{4 - \frac{3J^2}{l^2m^2}})m^2(-2 + 3m)}\left(\frac{\beta}{A}\right) dm \qquad 26$$

- In our case the black hole is axially symmetric and no work is done on the horizon of black hole. This means that as the interior volume is increasing with Eddington time so, there will be a large space in the interior of black hole to store the information lost. Hence, the first law of black hole thermodynamics for a axially symmetric BTZ black hole can be written as

$$dm = \frac{dS_{BH}}{\beta} + \Omega_H dJ \qquad 27$$

So,

$$dS_{\phi} = -\frac{\zeta(3)\gamma}{2}\sqrt{-9J + 3J^2 + 2l^2\left(2 + \sqrt{4 - \frac{3J^2}{l^2m^2}}\right)m^2(-2 + 3m)}\left(\frac{\beta}{A}\right)\left(\frac{dS_{BH}}{\beta} + \Omega_H dJ\right) \qquad 28$$

- This equation gives a direct relation between the two type of entropy with a disturbing term due to angular momentum. As the angular momentum is conserved, hence its distortion will be small at horizon and will be negligibly small at $r = r_v = 0.45m$. So, we can ignore the term $\Omega_H dJ$ during the evaporation phenomena. So, the final relation between the two entropy could be written as

$$dS_{\phi} = -\frac{4\zeta(3)\gamma}{3\pi}F(m,J)dS_{BH} \qquad 29$$

Where $$F(m,J) = \sqrt{-9J + 3J^2 + 2l^2\left(2 + \sqrt{4 - \frac{3J^2}{l^2m^2}}\right)m^2(-2 + 3m)}$$

- As similar to black hole quantum mode entropy is directly related to interior volume, for a (2+1)-dimensional black hole could be maximized by the factor $\sqrt{-r^2 f(r)}$. So we call quantum mode entropy as maximal entropy.

5. Results

- Numerically, the position of this maximal hyper-surface contributing to maximal interior volume is found at $r_v = 0.45m$
- The scalar quantum mode entropy is found related to Eddington time like higher dimensional black holes.
- The plot of $F(m, J)$ vs. mass of the black hole is plotted as shown in the figure.
- The plot shows the evolution relation as a function of black hole mass and it seems similar to the power function of a variable, when the power is fraction between 0 and 1.
- The plot shows that as the BTZ black hole mass m increases from 0, so the slop of curve also increases.

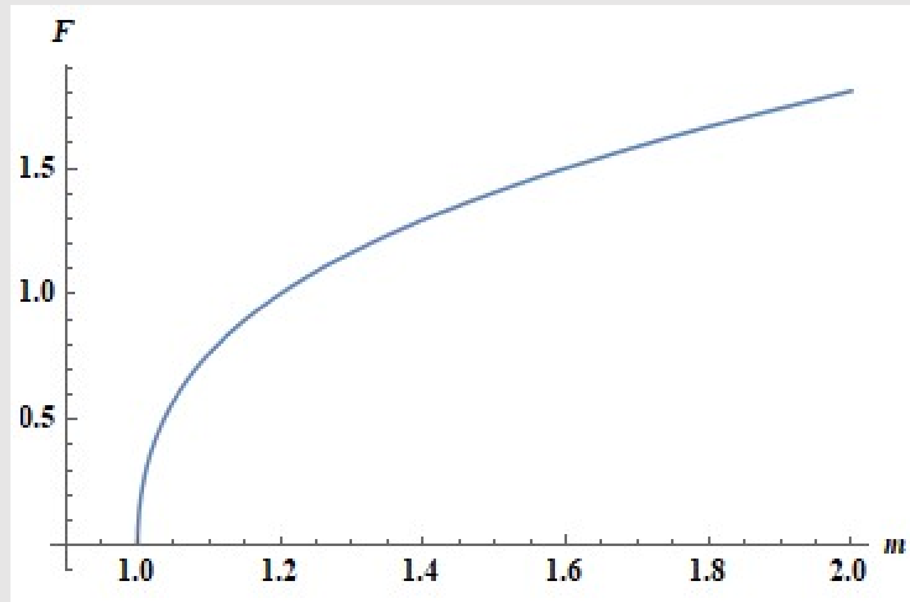


Figure 3: Plot of evolution relation $F(m, J)$ vs. mass m for rotating BTZ black hole.

- At the beginning, the BTZ black hole mass is seem to be constant for some increase in evolution relation or slowly increasing but after acquiring certain mass limit, the evolution relation increases with increase in black hole mass continuously without any divergence.
- This work not only extends our earlier work idea to lower dimension space-time but also confirms the Non-Evaporational character of BTZ black holes.

Thanks for your attention

Any Question