The Hartle–Hawking wavefunction of the universe revisited

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Beyond Standard Model: From theory to experiment (BSM-2021)

Introduction

■ Wavefunctions in Quantum Mechanics ⇒ probabilities

Wavefunctions in Quantum Gravity ⇒ probabilities favoring realistic aspects of the Universe?

■ Hartle–Hawking proposal for spatially closed universes with cosmological constant $\Lambda > 0$. ['83]

■ Homogeneous and isotropic $\quad \mathrm{d}s^2 = -N(t)^2 \mathrm{d}t^2 + a(t)^2 \, \mathrm{d}\Omega_3^2$

Euclidean path integral

$$\Psi(a_0) = \int_{\substack{a_\mathrm{i}=0 \\ a_\mathrm{f}=a_0}} \frac{\mathcal{D}N \, \mathcal{D}a}{\mathrm{Vol(Diff)}} \, e^{-\frac{1}{\hbar} S_\mathrm{E}[N,a]}$$

obeying the **"No-boundary proposal" :** Probability amplitude for creating a Universe of scale factor a_0 from "nothing." [Vilenkin, '82]

Plan

■ **Fix the gauge consistently:** Result independent of the gauge.

■ Field redefinitions of the scale factor are symmetries of the classical action but

$$q = Q(a) \quad \Longrightarrow \quad \mathcal{D}q \neq \mathcal{D}a$$

We obtain **different results for the wavefunctions** at the semi-classical level.

■ However, all prescriptions yield **same quantum predictions,** at least at the semi-classical level.

Gauge fixing of Euclidean time

■ The Euclidean action

$$S_{\mathrm{E}} = 6\pi \int_{x^0_{\mathrm{Ei}}}^{x^0_{\mathrm{Ef}}} \mathrm{d}x^0_{\mathrm{E}}\, \sqrt{g_{00}} \left[a\, g^{00} \left(\frac{\mathrm{d}a}{\mathrm{d}x^0_{\mathrm{E}}} \right)^2 + a - \frac{\Lambda}{3}\, a^3 \right]$$

describes a non-linear σ-model:

- The base is a **line segment $[\boldsymbol{x^0_{\mathrm{Ei}}}, \boldsymbol{x^0_{\mathrm{Ef}}}]$** of metric $g_{00} \equiv N^2$.
- **The target space is parametrized by the scale factor a.**

■ All metrics g_{00} are not equivalent up to a change of coordinate, since the proper length ℓ of a line segment is invariant under a change of coordinate.

Choose a metric $\hat{g}_{00}[\ell]$ in each equivalence class (= choice of gauge) and replace

$$\int \frac{\mathcal{D}N}{\mathrm{Vol(Diff)}} = \int_0^{+\infty} \mathrm{d}\ell \, \cancel{\int_{\mathrm{Diff}} \frac{\mathcal{D}\xi}{\mathrm{Vol(Diff)}}} \; \Delta_{\mathrm{FP}}[\hat{g}_{00}[\ell]]$$

■ Fadeev–Popov determinant

$$1 = \Delta_{\mathrm{FP}}[\hat{g}_{00}[\ell]] \int_0^{+\infty} \mathrm{d}\ell' \int_{\mathrm{Diff}} \mathcal{D}\xi \; \delta\big[\hat{g}_{00}[\ell] - \hat{g}^{\xi}_{00}[\ell']\big]$$

■ Introducing **anticommuting ghosts b^{00}, c_0,**

$$\Delta_{\mathrm{FP}}[\hat{g}_{00}[\ell]] = \int_{\substack{c^0(\hat{x}^0_{\mathrm{Ei}})=0 \\ c^0(\hat{x}^0_{\mathrm{Ef}})=0}} \mathcal{D}c \int \mathcal{D}b \left(b, \frac{\hat{g}[\ell]}{\ell}\right) \exp\left\{4i\pi\,(b, \hat{\nabla}c)\right\}$$

where $(f,h) \equiv \displaystyle\int_{\hat{x}^0_{\mathrm{Ei}}}^{\hat{x}^0_{\mathrm{Ef}}} \mathrm{d}\hat{x}^0_{\mathrm{E}} \sqrt{\hat{g}_{00}[\ell]}\; f^{00}\, h_{00}$

■ By expanding in Fourrier modes on $[\hat{x}^0_{\mathrm{Ei}}, \hat{x}^0_{\mathrm{Ei}}]$ and **using gauge-invariant measures,**

$$\Delta_{\mathrm{FP}}[\hat{g}_{00}[\ell]] = 1$$

NB: For a base with topology of a circle, the result is $1/\ell$.

Path integral over the scale factor

- Gauge $\hat{g}_{00}[\ell] = \ell^2$

$$\Psi(a_0) = \int_0^{+\infty} \mathrm{d}\ell \int_{\substack{a(0)=0 \\ a(1)=a_0}} \mathcal{D}a \; e^{-\frac{1}{\hbar} S_{\mathrm{E}}[\ell,a]}$$

where the action

$$S_{\mathrm{E}}[\ell, a] = 6\pi \int_0^1 \mathrm{d}\tau \left[\frac{a}{\ell} \left(\frac{\mathrm{d}a}{\mathrm{d}\tau} \right)^2 + \ell \, V(a) \right] , \qquad V(a) = a - \frac{\Lambda}{3} \, a^3$$

is not quadratic $\Longrightarrow$ **semi-classical approximation**

- **steepest-descent method**
  - Find all instanton solutions $(\bar{a}, \bar{\ell})$: Two solutions.
  - Develop at quadratic order and integrate over fluctuations.

$$S_{\mathrm{E}}[\ell, a] = \bar{S}_{\mathrm{E}} + 6\pi^2 \int_0^1 \mathrm{d}\tau \bar{\ell} \Big[\delta a\, \mathcal{Q}\, \delta a + 2\, \delta a\, \frac{V_a(\bar{a})}{\bar{\ell}}\, \delta\ell + \delta\ell\, \frac{V(\bar{a})}{\bar{\ell}^2}\, \delta\ell \Big] + \cdots$$

- Diagonalizing,

$$\Psi(a_0) = \sum_{\epsilon=\pm 1} e^{-\frac{1}{\hbar}\bar{S}_{\mathrm{E}}^{\epsilon}} \int_{\substack{\delta a(0)=0 \\ \delta a(1)=0}} \mathcal{D}\delta a\, \exp\left\{ -\frac{6\pi^2}{\hbar} (\delta a, \mathcal{Q}_{\epsilon}\, \delta a) \right\} \int \mathrm{d}\delta\ell\, \exp\left\{ -\mathcal{K}_{\epsilon}\, \delta\ell^2 \right\} \; (1 + \mathcal{O}(\hbar))$$

- Gaussian (path) integrals $\implies$ $\dfrac{1}{\sqrt{\det \mathcal{Q}_{\epsilon}}} \dfrac{1}{\sqrt{\mathcal{K}_{\epsilon}}}$

$$\Psi(a_0) = \sum_{\epsilon=\pm 1} \frac{1}{\sqrt{\epsilon}} \frac{\exp\left[\epsilon \frac{12\pi^2}{\hbar\Lambda} \left(1 - \frac{\Lambda}{3} a_0^2\right)^{\frac{3}{2}}\right]}{a_0^{\frac{1}{8}} \left(1 - \frac{\Lambda}{3} a_0^2\right)^{\frac{1}{4}}} \left(1 + \mathcal{O}(\hbar)\right)$$

■ Classically, the action is invariant under redefinitions $\boldsymbol{q = Q(a)}$

At the quantum level $\boldsymbol{\mathcal{D}q \neq \mathcal{D}a}$ **due to a Jacobian**

$$\widetilde{\Psi}(q_0) = \int_0^{+\infty} \mathrm{d}\ell \int_{\substack{q(0)=Q(0) \\ q(1)=Q(a_0)}} \mathcal{D}q \; e^{-\frac{1}{\hbar} S_{\mathrm{E}}[\ell^2, q]}$$

$$= \sum_{\epsilon=\pm 1} \frac{1}{\sqrt{\epsilon}} \frac{\exp\left[\epsilon \frac{12\pi^2}{\hbar\Lambda} \left(1 - \frac{\Lambda}{3} a_0^2\right)^{\frac{3}{2}}\right]}{|Q'(a_0)|^{\frac{1}{4}} \, a_0^{\frac{1}{8}} \left(1 - \frac{\Lambda}{3} a_0^2\right)^{\frac{1}{4}}} \left(1 + \mathcal{O}(\hbar)\right)$$

There are **infinitely many different prescriptions for the wavefunctions!**

Wheeler–DeWitt equation

■ For each presciption $\mathcal{D}q$, **all possible states/wavefunctions satisfy an equation** similar to Schrödinger in quantum mechanics

To derive it,

$$0 = \int \frac{\mathcal{D}N\,\mathcal{D}q}{\mathrm{Vol(Diff)}}\,\frac{\delta}{\delta N}\,e^{iS[N,q]} = -i\int \frac{\mathcal{D}N\,\mathcal{D}q}{\mathrm{Vol(Diff)}}\,\frac{H}{N}\,e^{iS[N,q]} \tag{1}$$

where the **classical Hamiltonian is**

$$\frac{H}{N} = -\frac{1}{24\pi}\,\frac{\pi_q^2}{AA'^2} - 6\pi V \quad \text{where} \quad A = Q^{-1}$$

$\Longrightarrow$ **The quantum Hamiltonian vanishes on all states of the Hilbert space.**

■ Classically, we have for arbitrary functions $\rho_1(q), \rho_2(q)$

$$\pi_q^2 = \frac{1}{\rho_1\,\rho_2}\,\pi_q\,\rho_1\,\pi_q\,\rho_2$$

- **canonical quantization**

$$q \longrightarrow q_0\,, \qquad \pi_q \longrightarrow -i\hbar\,\frac{\mathrm{d}}{\mathrm{d}q_0}$$

yields an **ambiguity**

$$\frac{\hbar^2}{24\pi}\,\frac{1}{AA'^2}\,\frac{1}{\rho}\,\frac{\mathrm{d}}{\mathrm{d}q_0}\left(\rho\,\frac{\mathrm{d}\Phi}{\mathrm{d}q_0}\right)+\left(\hbar^2\,\omega-6\pi V\right)\Phi=0$$

where Φ is an arbitrary wavefunction of the Hilbert space.

■ We can find ρ by solving this equation at the semi-classical level using the **WKB method**

$$\Phi(q_0)=\sum_{\epsilon=\pm1}N_\epsilon\,\frac{\exp\left[\epsilon s\,\frac{12\pi^2}{\hbar\,\Lambda}\,\left(1-\frac{\Lambda}{3}a_0^2\right)^{\frac{3}{2}}\right]}{|\rho(q_0)A'(q_0)|^{\frac{1}{2}}\,a_0^{\frac{1}{2}}\,\left(1-\frac{\Lambda}{3}a_0^2\right)^{\frac{1}{4}}}\,(1+\mathcal{O}(\hbar))$$

Comparing with a particular wavefunction, the “no-boundary state”

$$\Longrightarrow \quad \rho(q_0)=a_0^{-\frac{3}{4}}\,|A'(q_0)|^{-\frac{3}{2}}$$

Universality at the semi-classical

■ **Different wavefunction prescriptions $\mathcal{D}q$ and Wheeler–DeWitt equations $\Longrightarrow$ different quantum gravities with same classical limits?**

• To discuss probabilities, we **define inner product in each Hilbert space.** Denoting $\Phi(q_0) \equiv \Phi_A(a_0)$, $\big(a_0 = A(q_0)\big)$

$$\langle \Phi_{A1}, \Phi_{A2} \rangle = \int_0^{+\infty} \mathrm{d}a_0 \, \mu(a_0) \, \Phi_{A1}(a_0)^* \, \Phi_{A2}(a_0)$$

• Imposing **Hermiticity of the Hamiltnonians**

$$\Big\langle \Phi_{A1}, \frac{H}{N} \, \Phi_{A2} \Big\rangle = \Big\langle \frac{H}{N} \, \Phi_{A1}, \Phi_{A2} \Big\rangle$$

$\Longrightarrow$ Differential equation $\Longrightarrow$ $\mu = a_0 \, |A'| \, \rho$

$$\Longrightarrow \quad \sqrt{\mu(a_0)}\,\Phi_A(a_0) = \sum_{\epsilon=\pm 1} N_\epsilon \, \frac{\exp\left[\epsilon\,\frac{12\pi^2}{\hbar\,\Lambda}\left(1-\frac{\Lambda}{3}a_0^2\right)^{\frac{3}{2}}\right]}{\left(1-\frac{\Lambda}{3}a_0^2\right)^{\frac{1}{4}}}\left(1+\mathcal{O}(\hbar)\right)$$

is **independent of ρ and A *i.e.* is independent of the choice of field redefinition, at the semi-classical level**

So is the inner product $\langle \Phi_{A1}, \Phi_{A2}\rangle = \int_0^{+\infty} \mathrm{d}a_0\,\mu\,\Phi_{A1}^*\,\Phi_{A2}$

$\Longrightarrow$ **All probabilities are independent of the choice of measure $\mathcal{D}q$, at least at the semi-classical level**

Conclusion

■ We have considered the **Hartle–Hawking wavefunction for spatially closed universes, with $\Lambda > 0$.**

■ We focussed on a simpler version, for **homogeneous and isotropic universes.**

■ The system can be seen as a non-linear σ-model with a line segment for the base and a target space parametrized by the scale factor.

■ The **gauge fixing of time reparametrization** is done by:

- Integrating over the proper length of the line-segment base.
- The Faddeev–Popov determinant is trivial.
- Using gauge invariant measures.

■ The **reparametrizations of the scale factor** (*i.e.* coordinate in the target space) **yield different measures and path integrals, but the Hilbert spaces are equivalent at least semi-classically.**