Integration of subtraction terms at NNLO

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Outline

Subtraction at NNLO

Integrating the counterterms

Methods

Results

Conclusions and outlook

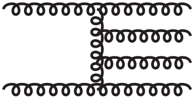
Subtraction at NNLO

The NNLO cross section

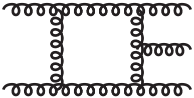
NNLO correction to a generic *m*-jet cross section

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m}$$

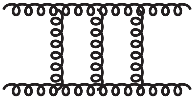
- Doubly-real
  - tree level SMEs with $m+2$ parton kinematics
  - no explicit infrared poles
  - but implicit poles from soft/collinear emission



- Real-virtual
  - one-loop SMEs with $m+1$ parton kinematics
  - explicit infrared poles from loop integration
  - and implicit poles from soft/collinear emission



- Doubly-virtual
  - two-loop SMEs with m parton kinematics
  - explicit infrared poles from loop integration
  - no implicit poles



Need a method to deal with implicit poles!

Approaches

Sector decomposition

(Binoth, Heinrich; Anastasiou, Melnikov, Petriello; Czakon)

- extract poles by expanding integrand in distributions
- numerically integrate resulting pole coefficients
- can it handle complicated final states?

Subtraction

(Catani, Grazzini; Cieri, Ferrera, de Florian; Gehrmann, Gehrmann-De Ridder, Glover; Weinzierl; Del Duca, Trócsányi, GS)

- rearrange the poles between real and virtual contributions by subtracting and adding back suitable approximate cross sections
- cancellation of explicit ϵ poles achieved analytically, remaining PS integrals are finite
- experience from NLO would lead one to expect nice properties
  - process independent solution based on explicit expressions
  - efficient implementation, including automation possible
- definition of subtraction terms is not unique, hence several approaches: $q_\perp$, antenna, local

Local subtraction

Goal: devise a subtraction scheme with

- explicit expressions for general process, including color (sine qua non for automation, color space notation used)
- fully local counterterms, including all spin and color correlations (mathematical rigor, efficiency)
- option to constrain subtractions to near singular regions (efficiency, important check)

Local subtraction

Strategy: IR limits are process independent and known

1. start from defining the subtraction terms based on IR limit formulae
   - they are trivially general, explicit and local
   - done some time ago (2006) for colorless initial states
2. and worry about integrating them later
   - since this is *in principle* a very narrowly defined problem, given 1.
   - but in practice is very cumbersome, due to lack of technology

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However: antenna factorization factorizes essentially $1 \to 3$ and $1 \to 4$ SME's

1. integration well understood from the start
   - technology to integrate antennae over 3 and 4 parton phase spaces known from two-loop calculations by the end of the 90's
   - in practice, it is still a lot of work
2. it should be possible to build subtraction terms from antennae
   - can it be done in an explicitly general way?
   - not completely local, treatment of color is not explicit

Structure

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

Each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

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➠ ${\rm d}\sigma^{\rm RR,A_2}_{m+2}$ regularizes the doubly-unresolved limits of ${\rm d}\sigma^{\rm RR}_{m+2}$

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- $\left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2}\right)^{\rm A_1}$ regularizes the singly-unresolved limit of $\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2}$

General features of counterterms

Built using universal IR limit formulae

- Altarelli-Parisi splitting functions, soft currents (tree level and one-loop, also triple AP functions)
- matching of limits to avoid overlaps: simple and general procedure using physical gauge
- extension away from limits based on momentum mappings that generalize to any number of unresolved partons

Fully local in color $\otimes$ spin space

- no need to consider the color decomposition of real emission ME's
- azimuthal correlations correctly taken into account in gluon splitting

Straightforward to constrain subtractions to near singular regions

- gain in efficiency
- independence of physical results on phase space cut is a strong check

Given completely explicitly for any process with non colored initial state

Integrating the counterterms

Basic setup

Momentum mappings used to define the counterterms

$$\{p\}_{n+p} \xrightarrow{X_R} \{\tilde{p}\}_n$$

- implement exact momentum conservation
- recoil distributed democratically (can be generalized to any p)
- different collinear and soft mappings (R labels precise limit)
- exact factorization of phase space

$$\mathrm{d}\phi_{n+p}(\{p\};Q) = \mathrm{d}\phi_n(\{\tilde{p}\}_n^{(R)};Q)[\mathrm{d}p_{p,n}^{(R)}]$$

Counterterms are products (in color and spin space) of

- factorized ME's independent of variables in $[\mathrm{d}p_{p,n}^{(R)}]$
- singular factors (AP functions, soft currents), to be integrated over $[\mathrm{d}p_{p,n}^{(R)}]$

Strategy for computing the integrals

- explicit parametrization of factorized phase space leads to parametric integral representations
- evaluate the parametric integrals

Types of integrated counterterms

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➡ tree-level and one-loop singly-unresolved integrals

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- tree-level and one-loop singly-unresolved integrals
- tree-level iterated singly-unresolved integrals

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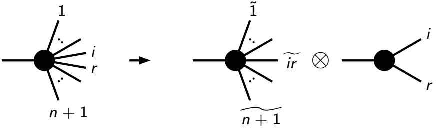
- tree-level and one-loop singly-unresolved integrals
- tree-level iterated singly-unresolved integrals
- tree-level doubly-unresolved integrals

Singly-unresolved integrals

Collinear phase space factorization

$$\mathrm{d}\phi_{n+1}(\{p\};Q) = \mathrm{d}\phi_n(\{\tilde{p}\}^{(ir)};Q)[\mathrm{d}p_{1,n}^{(ir)}(p_r,\tilde{p}_{ir};Q)]$$

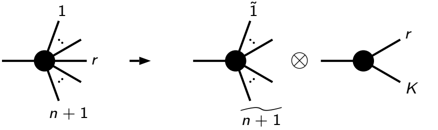
$$[\mathrm{d}p_{1,n}^{(ir)}(p_r,\tilde{p}_{ir};Q)] = \mathrm{d}\alpha(1-\alpha)^{2(n-1)(1-\epsilon)-1}\frac{s_{\widetilde{ir}Q}}{2\pi}\mathrm{d}\phi_2(p_i,p_r;p_{(ir)})$$



Soft phase space factorization

$$\mathrm{d}\phi_{n+1}(\{p\};Q) = \mathrm{d}\phi_n(\{\tilde{p}\}^{(r)};Q)[\mathrm{d}p_{1,n}^{(r)}(p_r;Q)]$$

$$[\mathrm{d}p_{1,n}^{(r)}(p_r;Q)] = \mathrm{d}y(1-y)^{(n-1)(1-\epsilon)-1}\frac{Q^2}{2\pi}\mathrm{d}\phi_2(p_r,K;Q)$$



Collinear integrals

Collinear counterterms defined using Altarelli-Parisi splitting functions

$$\int_0^{\alpha_0} \mathrm{d}\alpha (1-\alpha)^{2d_0-1} \frac{s_{\widetilde{ir}Q}}{2\pi} \int \mathrm{d}\phi_2(p_i, p_r; p_{(ir)}) \frac{1}{s_{ir}^{1+\kappa\epsilon}} P^{(\kappa)}_{f_i f_r}(z_i, z_r; \epsilon)$$

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➠ explicit parametrization of PS gives ($x = 2\tilde{p}_{ir} \cdot Q / Q^2$)

$$\mathrm{d}\phi_2(p_i, p_r; p_{(ir)}) = \frac{s_{ir}^{-\epsilon}}{8\pi} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \mathrm{d}s_{ir}\, \mathrm{d}v\, \delta\Big\{ s_{ir} - Q^2[\alpha(\alpha + (1-\alpha)x)] \Big\}$$
$$\cdot [v(1-v)]^{-\epsilon} \Theta(v) \Theta(1-v)$$

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$$\int_0^{\alpha_0} \mathrm{d}\alpha (1-\alpha)^{2d_0-1} \frac{s_{\widetilde{ir}Q}}{2\pi} \int \mathrm{d}\phi_2(p_i, p_r; p_{(ir)}) \frac{1}{s_{ir}^{1+\kappa\epsilon}} P_{f_i f_r}^{(\kappa)}(z_i, z_r; \epsilon)$$

- Altarelli-Parisi functions can be expressed as linear combinations of

$$\frac{z_r^{k+\delta\epsilon}}{s_{ir}^{1+\kappa\epsilon}} g_I^{\pm}(z_r)\,, \qquad z_r = \frac{p_r \cdot Q}{(p_i + p_r)\cdot Q} = \frac{\alpha + (1-\alpha)xv}{2\alpha + (1-\alpha)x}$$

with $k = -1, 0, 1, 2$ and

δ	Function	$g_I^{\pm}(z)$
0	g_A	1
∓ 1	$g_B^{(\pm)}$	$(1-z)^{\pm\epsilon}$
0	$g_C^{(\pm)}$	$(1-z)^{\pm\epsilon}\,{}_2F_1(\pm\epsilon, \pm\epsilon, 1\pm\epsilon, z)$
± 1	$g_D^{(\pm)}$	${}_2F_1(\pm\epsilon, \pm\epsilon, 1\pm\epsilon, z)$

Soft integrals

Soft counterterms defined using eikonal functions (and its collinear limit)

$$\int_0^{y_0} \mathrm{d}y (1-y)^{d_0'-1} \frac{Q^2}{2\pi} \int \mathrm{d}\phi_2(p_r, K; Q) \left\{ \left(\frac{s_{ik}}{s_{ir} s_{kr}} \right)^{1+\kappa\epsilon}, \left(\frac{1}{s_{ir}} \frac{z_i}{z_r} \right)^{1+\kappa\epsilon} \right\}$$

Soft integrals

Soft counterterms defined using eikonal functions (and its collinear limit)

$$\int_0^{y_0} \mathrm{d}y (1-y)^{d_0'-1} \frac{Q^2}{2\pi} \int \mathrm{d}\phi_2(p_r, K; Q) \left\{ \left(\frac{s_{ik}}{s_{ir} s_{kr}} \right)^{1+\kappa\epsilon}, \left(\frac{1}{s_{ir}} \frac{z_i}{z_r} \right)^{1+\kappa\epsilon} \right\}$$

- choose a frame

$$Q^\mu = \sqrt{s}(1, \ldots), \qquad \tilde{p}_i^\mu = \tilde{E}_i(1, \ldots, 1), \qquad \tilde{p}_k^\mu = \tilde{E}_i(1, \ldots, \sin\chi, \cos\chi)$$

and

$$p_r^\mu = E_r(1, ..\text{`angles'}.., \sin\vartheta \sin\varphi, \cos\vartheta)$$

to find the explicit parametrization of PS

$$\mathrm{d}\phi_2(p_r, K; Q) = \frac{Q^{-2\epsilon}}{16\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \mathrm{d}\varepsilon_r \, \varepsilon_r^{1-2\epsilon} \delta(\varepsilon_r - y)$$
$$\cdot \, \mathrm{d}(\cos\vartheta)(\sin\vartheta)^{-2\epsilon} \mathrm{d}(\cos\varphi)(\sin\varphi)^{-1-2\epsilon}$$

Soft integrals

Soft counterterms defined using eikonal functions (and its collinear limit)

$$\int_0^{y_0} \mathrm{d}y (1-y)^{d_0'-1} \frac{Q^2}{2\pi} \int \mathrm{d}\phi_2(p_r, K; Q) \left\{ \left(\frac{s_{ik}}{s_{ir} s_{kr}}\right)^{1+\kappa\epsilon}, \left(\frac{1}{s_{ir}} \frac{z_i}{z_r}\right)^{1+\kappa\epsilon} \right\}$$

➠ precise definition of the soft momentum mapping implies

$$s_{ik} = (1-\varepsilon_r) s_{\tilde{i}\tilde{k}}\,, \qquad s_{ir} = s_{\tilde{i}r}\,, \qquad s_{kr} = s_{\tilde{k}r}\,, \qquad s_{iQ} = (1-\varepsilon_r) s_{\tilde{i}Q} + s_{\tilde{i}r}$$

hence ($\cos\chi = 1 - 2Y$)

$$\frac{s_{ik}}{s_{ir} s_{kr}} = \frac{4Y}{Q^2} \frac{1-\varepsilon_r}{\varepsilon_r^2} \frac{1}{(1-\cos\vartheta)(1-\cos\chi\cos\vartheta - \sin\chi\sin\vartheta\cos\varphi)}$$

$$\frac{1}{s_{ir}} \frac{z_i}{z_r} = \frac{1}{Q^2} \frac{1}{\varepsilon_r} \left[1 + \frac{2(1-\varepsilon_r)}{\varepsilon_r(1-\cos\vartheta)}\right]$$

Basic one-particle integrals

Computed (semi)analytically

(Aglietti, Del Duca, Duhr, Trócsányi, GS; Bolzoni, Moch, Trócsányi, GS)

- collinear type

$$\mathcal{I} \propto x \int_0^{\alpha_0} \mathrm{d}\alpha\, \alpha^{-1-(1+\kappa)\epsilon} (1-\alpha)^{2d_0-1} [\alpha + (1-\alpha)x]^{-1-(1+\kappa)\epsilon}$$

$$\cdot \int_0^1 \mathrm{d}v [v(1-v)]^{-\epsilon} \left(\frac{\alpha+(1-\alpha)xv}{2\alpha+(1-\alpha)x}\right)^{k+\delta\epsilon} g_l^{(\pm)}\left(\frac{\alpha+(1-\alpha)xv}{2\alpha+(1-\alpha)x}\right)$$

- soft type

$$\mathcal{J} \propto -Y^{1+\kappa\epsilon} \int_0^{y_0} \mathrm{d}y\, y^{-1-2(1+\kappa)\epsilon}(1-y)^{d_0'+\kappa\epsilon} \int_{-1}^{1} \mathrm{d}(\cos\vartheta)\, (\sin\vartheta)^{-2\epsilon}$$

$$\cdot \int_{-1}^{1} \mathrm{d}(\cos\varphi)\, (\sin\varphi)^{-1-2\epsilon} \frac{1}{[(1-\cos\vartheta)(1-\cos\chi\cos\vartheta-\sin\chi\sin\vartheta\cos\varphi)]^{1+\kappa\epsilon}}$$

$$\mathcal{K} \propto \int_0^{y_0} \mathrm{d}y\, y^{-(2+\kappa)\epsilon}(1-y)^{d_0'-1} \int_{-1}^{1} \mathrm{d}(\cos\vartheta)\, (\sin\vartheta)^{-2\epsilon} \left[1+\frac{2(1-y)}{y(1-\cos\vartheta)}\right]^{1+\kappa\epsilon}$$

Iterated singly-unresolved integrals

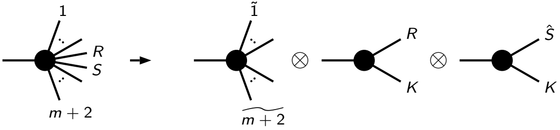
The momentum mapping is of iterated form

$$\{p\}_{m+2} \to \{\hat{p}\}_{m+1} \to \{\tilde{p}\}_m$$

Phase space factorization (schematically)

$$\mathrm{d}\phi_{m+2}(\{p\};Q) = \mathrm{d}\phi_m(\{\tilde{p}\}^{(\hat{S},R)};Q)[\mathrm{d}p^{(R)}_{1,m+1}][\mathrm{d}p^{(\hat{S})}_{1,m}]$$

factorized phase space measures from singly-unresolved case



Examples of iterated one-particle integrals

Collinear-double collinear counterterm from $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_{12}}$ (Bolzoni, Trócsányi, GS)

$$\mathcal{C}_{kt}\mathcal{C}_{ir;kt}^{(0)} = (8\pi\alpha_s\mu^{2\epsilon})^2 \frac{1}{s_{kt}}\frac{1}{\widehat{s}_{ir}} \langle \mathcal{M}_m^{(0)}(\{\tilde{p}\})|P_{f_k f_t}^{(0)}(z_{t,k};\epsilon)P_{f_i f_r}^{(0)}(\widehat{z}_{r,i};\epsilon)|\mathcal{M}_m^{(0)}(\{\tilde{p}\})\rangle$$
$$\cdot (1-\alpha_{kt})^{2d_0-2m(1-\epsilon)}(1-\widehat{\alpha}_{kt})^{2d_0-2m(1-\epsilon)}\Theta(\alpha_0-\alpha_{kt})\Theta(\alpha_0-\widehat{\alpha}_{ir})$$

Using the discussed explicit parametrization of the factorized PS measures, we find its integral can be expressed as a linear combination of the following "MIs"

$$\mathcal{I}_{\mathcal{C}}^{(4)}(x_k,x_i;\epsilon,\alpha_0,d_0,k,l) = x_k x_i \int_0^{\alpha_0}\mathrm{d}\alpha\int_0^{\alpha_0}\mathrm{d}\beta\,\alpha^{-1-\epsilon}(1-\alpha)^{2d_0-1}$$
$$\cdot\beta^{-1-\epsilon}(1-\beta)^{2d_0-2+2\epsilon}[\alpha+(1-\alpha)(1-\beta)x_k]^{-1-\epsilon}[\beta+(1-\beta)x_i]^{-1-\epsilon}$$
$$\cdot\int_0^1\mathrm{d}v\int_0^1\mathrm{d}u\,v^{-\epsilon}(1-v)^{-\epsilon}u^{-\epsilon}(1-u)^{-\epsilon}$$
$$\cdot\left(\frac{\alpha+(1-\alpha)(1-\beta)x_k v}{2\alpha+(1-\alpha)(1-\beta)x_k}\right)^k\left(\frac{\beta+(1-\beta)x_i u}{2\beta+(1-\beta)x_i}\right)^l$$

where $k,l=-1,0,1,2$.

Examples of iterated one-particle integrals

Abelian soft-double soft counterterm from $\mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2}$ (Bolzoni, Trócsányi, GS)

$$\left(\mathcal{S}_t\mathcal{S}^{(0)}_{rt}\right)^{\mathrm{ab}} = (8\pi\alpha_\mathrm{s}\mu^{2\epsilon})^2 \sum_{i,j,k,l} \frac{1}{8}\mathcal{S}_{\hat{i}\hat{k}}(\hat{r})\mathcal{S}_{jl}(t)|\mathcal{M}^{(0)}_{m,(i,k)(j,l)}(\{\tilde{p}\})|^2$$
$$\cdot (1-y_{tQ})^{d_0'-m(1-\epsilon)}(1-y_{\hat{r}Q})^{d_0'-m(1-\epsilon)}\Theta(y_0-y_{tQ})\Theta(y_0-y_{\hat{r}Q})$$

Consider e.g. $j=i$ and $l=k$. Using the discussed explicit parametrization of the factorized PS measures, we find that $[\mathrm{S}_t\mathrm{S}^{(0)}_{rt}]_{ikik}$ is proportional to

$$\mathcal{I}^{(11)}_{\mathcal{S}}(Y_{ik,Q};\epsilon,y_0,d_0') = -\frac{4\Gamma^4(1-\epsilon)}{\pi\Gamma^2(1-\epsilon)}\frac{B_{y_0}(-2\epsilon,d_0'+1)}{\epsilon}Y_{ik,Q}\int_0^{y_0}\mathrm{d}y\,y^{-1-2\epsilon}(1-y)^{d_0'-1+\epsilon}$$
$$\cdot\int_{-1}^{1}\mathrm{d}(\cos\vartheta)\,(\sin\vartheta)^{-2\epsilon}\int_{-1}^{1}\mathrm{d}(\cos\varphi)\,(\sin\varphi)^{-1-2\epsilon}\left[f(\vartheta,\varphi;0)\right]^{-1}\left[f(\vartheta,\varphi;Y_{ik,Q})\right]^{-1}$$
$$\cdot\left[Y(y,\vartheta,\varphi;Y_{ik,Q})\right]^{-\epsilon}{}_2F_1\left(-\epsilon,-\epsilon,1-\epsilon,1-Y(y,\vartheta,\varphi;Y_{ik,Q})\right)$$

where

$$f(\vartheta,\varphi;Y_{ik,Q}) = 1-2\sqrt{Y_{ik,Q}(1-Y_{ik,Q})}\sin\vartheta\cos\varphi-(1-2Y_{ik,Q})\chi\cos\vartheta$$

$$Y(y,\vartheta,\varphi;\chi) = \frac{4(1-y)Y_{ik,Q}}{[2(1-y)+y\,f(\vartheta,\varphi;0)][2(1-y)+y\,f(\vartheta,\varphi;Y_{ik,Q})]}\,.$$

Doubly-unresolved integrals

Genuine doubly-unresolved momentum mapping

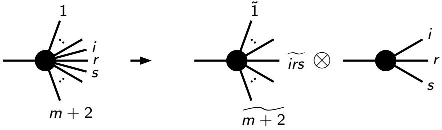
$$\{p\}_{m+2} \to \{\tilde{p}\}_m$$

Phase space factorization (schematically)

$$\mathrm{d}\phi_{m+2}(\{p\}; Q) = \mathrm{d}\phi_m(\{\tilde{p}\}^{(RS)}; Q)[\mathrm{d}p_{2,m}^{(RS)}]$$

factorized phase space measure for (e.g.) the triple collinear mapping is

$$[\mathrm{d}p_{2;m}^{(irs)}(p_r, p_s, \tilde{p}_{irs}; Q)] = \mathrm{d}\alpha(1-\alpha)^{2(m-1)(1-\epsilon)-1} \frac{s_{\widetilde{irs}\,Q}}{2\pi} \mathrm{d}\phi_3(p_i, p_r, p_s; p_{(irs)})$$



An example of a two-particle integral

Triple collinear counterterm from $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$

$$\mathcal{C}_{irs}^{(0,0)} = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{s_{irs}^2} \langle \mathcal{M}_m^{(0)}(\{\tilde{p}\})| P_{f_i f_r f_s}^{(0)}(\{z_{j,kl}, s_{jk}\};\epsilon)|\mathcal{M}_m^{(0)}(\{\tilde{p}\})\rangle$$
$$\cdot (1-\alpha_{irs})^{2d_0-2m(1-\epsilon)}\Theta(\alpha_0-\alpha_{irs})$$

Using an explicit parametrization of the factorized PS measure, we find its integral can be expressed as a linear combination of the following "MIs"

$$\mathcal{I}(x;\epsilon,\alpha_0,d_0,n_1,n_2,n_3,n_4,n_5,n_6,n_7) = \frac{\Gamma^2(1-\epsilon)}{\pi\Gamma(1-2\epsilon)} x \int_0^{\alpha_0} \mathrm{d}\alpha\, \alpha^{-1-2\epsilon}(1-\alpha)^{2d_0-3+2\epsilon}$$
$$\cdot [\alpha+(1-\alpha)x]^{-1-\epsilon} \int_0^1 \mathrm{d}v_r\,\mathrm{d}v_s\,\mathrm{d}t_{ir}\,\mathrm{d}t_{is}\,\mathrm{d}t_{rs}\,\delta(1-t_{ir}-t_{is}-t_{rs})(1-t_{ir})(1-t_{is})$$
$$\cdot [(t_{rs}^{+}-t_{rs})(t_{rs}-t_{rs}^{-})]^{-\frac{1}{2}-\epsilon}\Theta(t_{rs}^{+}-t_{rs})\Theta(t_{rs}-t_{rs}^{-}) t_{ir}^{n_1} t_{is}^{n_2} t_{rs}^{n_3}(1-t_{ir})^{n_4}(1-t_{is})^{n_5}$$
$$\cdot \left(\frac{\alpha+(1-\alpha)xv_r}{2\alpha+(1-\alpha)x}\right)^{n_6} \left(\frac{\alpha+(1-\alpha)xv_s}{2\alpha+(1-\alpha)x}\right)^{n_7}$$

where

$$t_{rs}^{\pm} = (1-t_{ir})(1-t_{is})\left[\sqrt{v_r(1-v_s)} \pm \sqrt{v_s(1-v_r)}\right]^2 .$$

Methods

Methods

Have explored several methods to compute the integrals

- IBP
  - reduction to master integrals via integration-by-parts identities
  - solution of MIs by differential equations
  - successfully applied to tree level and (a class of) one-loop singly-unresolved integrals and a class of iterated singly-unresolved integrals

    (Aglietti, Del Duca, Duhr, Trócsányi, GS)
- MB
  - Mellin-Barnes representations to extract pole structure
  - summation of nested series or direct numerical integration
  - successfully applied to all singly-unresolved and iterated singly-unresolved integrals

    (Bolzoni, Moch, Trócsányi, GS; Bolzoni, Trócsányi, GS)
- SD
  - sector decomposition
  - useful numerical check
  - all integrals also computed with SD

    (Trócsányi, GS; Bolzoni, Moch, Trócsányi, GS; Bolzoni, Trócsányi, GS)

Methods

Method	Analytical	Numerical
IBP	✔ singly-unresolved integrals ✘ bottleneck is the proliferation of denominators	✔ by evaluating the analytic expressions ✘ no numbers without full analytical results
MB	✔ iterated singly-unresolved integrals ✘ bottleneck is the evaluation of sums	✔ direct numerical evaluation of MB integrals possible ✔ fast and accurate
SD	✔ easy to automate ✘ only in principle, except for lowest order poles	✘ numerical behavior is generally worse than MB method (speed, accuracy)

Analytical vs. numerical results

As a matter of principle

- The rigorous proof of cancellation of IR poles requires the poles of integrated counterterms in analytical form.
- Analytical forms are fast and accurate compared to numerical ones.

However

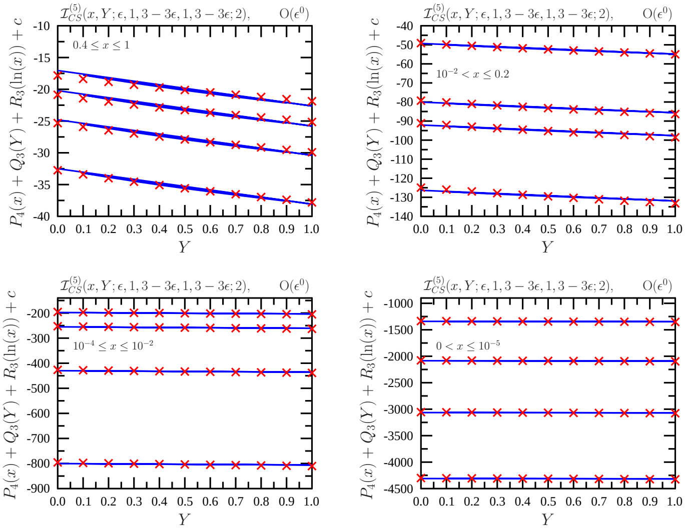
- Analytical results show (in all cases where they are available) that the integrated counterterms are smooth functions of kinematic variables.

Hence

- Numerical forms of the integrated counterterms are sufficient for practical purposes. Final results can be conveniently given by interpolating tables computed once and for all or approximating functions. Hence, an efficient implementation is possible even in cases where the full analytical calculation is not feasible or practical (e.g. finite parts of integrated counterterms).
- In particular, suitable approximating functions may be obtained by fitting.

Example of approximation by fitting

Doubly-unresolved soft-collinear master integral $\mathcal{I}_{\mathcal{CS}}^{(5)}(x, Y; \epsilon)$



Results

Structure of the results

Integrated approximate cross sections

- After summing over unobserved flavors, all integrated approximate cross sections can be written as products (in color space) of various insertion operators with lower point cross sections.

Insertion operators

- color and flavor structure of all insertion operators known
- first two leading poles of kinematical functions entering insertion operators known analytically in all cases (except $\mathbf{I}_2^{(0)}$)
- higher order expansion coefficients computed numerically

Integrated approximate cross sections - an example

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

Each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] \right\} J_m$$

- tree-level and one-loop singly-unresolved integrals
- tree-level iterated singly-unresolved integrals
- tree-level doubly-unresolved integrals

Integrated approximate cross sections - an example

Iterated singly-unresolved

$$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \mathbf{I}_{12}^{(0)}(\{p\}_m;\epsilon)$$

- structure of insertion operator in color $\otimes$ flavor space

$$\mathbf{I}_{12}^{(0)}(\{p\}_m;\epsilon) = \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \Bigg\{ \sum_i \left[\mathrm{C}_{12,f_i}^{(0)} \mathbf{T}_i^2 + \sum_k \mathrm{C}_{12,f_i f_k}^{(0)} \mathbf{T}_k^2\right] \mathbf{T}_i^2$$
$$+ \sum_{j,l} \left[\mathrm{S}_{12}^{(0),(j,l)} C_{\mathrm{A}} + \sum_i \mathrm{CS}_{12,f_i}^{(0),(j,l)} \mathbf{T}_i^2\right] \mathbf{T}_j \mathbf{T}_l$$
$$+ \sum_{i,k,j,l} \mathrm{S}_{12}^{(0),(i,k)(j,l)} \{\mathbf{T}_i \mathbf{T}_k, \mathbf{T}_j \mathbf{T}_l\} \Bigg\}$$

- $\mathrm{C}_{12,f_i}^{(0)}$, $\mathrm{C}_{12,f_i f_k}^{(0)}$, $\mathrm{S}_{12}^{(0),(j,l)}$, $\mathrm{CS}_{12,f_i}^{(0),(j,l)}$ and $\mathrm{S}_{12}^{(0),(i,k)(j,l)}$ are kinematical functions with poles up to $\mathrm{O}(\epsilon^{-4})$ (also depend on PS cut parameters)
- kinematical dependence through

$$x_i = y_{iQ} \equiv \frac{2p_i \cdot Q}{Q^2} \qquad \text{and} \qquad Y_{ik,Q} = \frac{y_{ik}}{y_{iQ} y_{kQ}}$$

Integrated approximate cross sections - an example

Iterated singly-unresolved

- example: $e^+e^- \to 3$ jets (momentum assignment is $1_q, 2_{\bar{q}}, 3_g$)

$$
\begin{aligned}
\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_\mathrm{s}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \Bigg\{ & \frac{C_\mathrm{A}^2 + 2C_\mathrm{A} C_\mathrm{F} + 6C_\mathrm{F}^2}{\epsilon^4} + \Bigg[\frac{11 C_\mathrm{A}^2}{2} + \frac{50 C_\mathrm{A} C_\mathrm{F}}{3} \\
& + 12 C_\mathrm{F}^2 - \frac{C_\mathrm{A} T_\mathrm{R} n_\mathrm{f}}{3} - \frac{C_\mathrm{A}^2 T_\mathrm{R} n_\mathrm{f}}{C_\mathrm{F}} - 4 C_\mathrm{F} T_\mathrm{R} n_\mathrm{f} + \left(\frac{5 C_\mathrm{A}^2}{2} - C_\mathrm{A} C_\mathrm{F} - 8 C_\mathrm{F}^2\right) \ln y_{12} \\
& - \frac{C_\mathrm{A}(5 C_\mathrm{A} + 8 C_\mathrm{F})}{2} (\ln y_{13} + \ln y_{23}) + (C_\mathrm{A}^2 + 6 C_\mathrm{A} 2 C_\mathrm{F} - 4 C_\mathrm{F}^2) \Sigma(y_0, D_0') \\
& + 4 C_\mathrm{F}(C_\mathrm{A} - C_\mathrm{F}) \Sigma(y_0, D_0' - 1)\Bigg] \frac{1}{\epsilon^3} + \mathrm{O}(\epsilon^{-2}) \Bigg\}
\end{aligned}
$$

- notice x and Y dependence combine to produce just y_{ik} dependence, as expected
- dependence on PS cut parameters through

$$\Sigma(z, N) = \ln z - \sum_{k=1}^{N} \frac{1-(1-z)^k}{k}$$

Integrated approximate cross sections - an example

Iterated singly-unresolved

- example: $e^+e^- \to 3$ jets (momentum assignment is $1_q, 2_{\bar{q}}, 3_g$)

$$
\begin{aligned}
\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \Bigg\{ & \frac{C_A^2 + 2C_A C_F + 6C_F^2}{\epsilon^4} + \Bigg[\frac{11C_A^2}{2} + \frac{50C_A C_F}{3} \\
& + 12C_F^2 - \frac{C_A T_R n_f}{3} - \frac{C_A^2 T_R n_f}{C_F} - 4C_F T_R n_f + \left(\frac{5C_A^2}{2} - C_A C_F - 8C_F^2\right) \ln y_{12} \\
& - \frac{C_A(5C_A + 8C_F)}{2}(\ln y_{13} + \ln y_{23}) + (C_A^2 + 6C_A 2C_F - 4C_F^2)\Sigma(y_0, D_0') \\
& + 4C_F(C_A - C_F)\Sigma(y_0, D_0' - 1)\Bigg] \frac{1}{\epsilon^3} + \mathrm{O}(\epsilon^{-2}) \Bigg\}
\end{aligned}
$$

- notice x and Y dependence combine to produce just y_{ik} dependence, as expected
- dependence on PS cut parameters through

$$\Sigma(z, N) = \ln z - \sum_{k=1}^{N} \frac{1-(1-z)^k}{k}$$

Integrated approximate cross sections - an example

Iterated singly-unresolved

- example: $e^+e^- \to 3$ jets (momentum assignment is $1_q, 2_{\bar{q}}, 3_g$)
- higher order expansion coefficients can be computed numerically

$$\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i=-4}^{0} \sum_{\text{color}} \frac{\text{Col}}{\epsilon^i} \mathcal{I}_{12,3j}^{(\text{Col},i)}(p_1, p_2, p_3) + \text{O}(\epsilon^1)$$

- kinematical point parametrized by y_{ij}

$$y_{12} = 0.333333, \qquad y_{13} = 0.333333, \qquad y_{23} = 0.333333$$

Col	$\text{O}(\epsilon^{-4})$	$\text{O}(\epsilon^{-3})$	$\text{O}(\epsilon^{-2})$	$\text{O}(\epsilon^{-1})$	$\text{O}(\epsilon^{0})$
C_F^2	6	34.12	82.98	34.59	-543.8
$C_A C_F$	2	9.721	1.209	-142.2	-696.6
C_A^2	1	6.497	12.80	15.87	-47.92
$C_F T_R n_f$	0	$-\frac{13}{3}$	-32.40	-127.9	-355.2
$C_A T_R n_f$	0	$-\frac{3}{2}$	-12.01	-46.90	-104.1

Integrated approximate cross sections - an example

Iterated singly-unresolved

- example: $e^+e^- \to 3$ jets (momentum assignment is $1_q, 2_{\bar{q}}, 3_g$)
- higher order expansion coefficients can be computed numerically

$$\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i=-4}^{0} \sum_{\text{color}} \frac{\text{Col}}{\epsilon^i} \mathcal{I}_{12,3j}^{(\text{Col},i)}(p_1, p_2, p_3) + \text{O}(\epsilon^1)$$

- kinematical point parametrized by y_{ij}

$$y_{12} = 0.238667, \qquad y_{13} = 0.758153, \qquad y_{23} = 0.003180$$

Col	$\text{O}(\epsilon^{-4})$	$\text{O}(\epsilon^{-3})$	$\text{O}(\epsilon^{-2})$	$\text{O}(\epsilon^{-1})$	$\text{O}(\epsilon^{0})$
C_F^2	6	36.79	106.0	120.6	-431.0
$C_A C_F$	2	25.38	143.6	537.3	1505
C_A^2	1	15.24	119.5	660.5	2902
$C_F T_R n_f$	0	$-\frac{13}{3}$	-31.30	-121.7	-346.0
$C_A T_R n_f$	0	$-\frac{3}{2}$	-17.72	-109.1	-470.9

Integrated approximate cross sections - an example

Iterated singly-unresolved

- example: $e^+e^- \to 3$ jets (momentum assignment is $1_q, 2_{\bar{q}}, 3_g$)
- higher order expansion coefficients can be computed numerically

$$\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i=-4}^{0} \sum_{\text{color}} \frac{\text{Col}}{\epsilon^i} \mathcal{I}_{12,3j}^{(\text{Col},i)}(p_1, p_2, p_3) + \text{O}(\epsilon^1)$$

- kinematical point parametrized by y_{ij}

$y_{12} = 0.937044, \quad y_{13} = 0.024207, \quad y_{23} = 0.038749$

Col	$\text{O}(\epsilon^{-4})$	$\text{O}(\epsilon^{-3})$	$\text{O}(\epsilon^{-2})$	$\text{O}(\epsilon^{-1})$	$\text{O}(\epsilon^{0})$
C_F^2	6	25.85	34.59	−84.25	−566.8
$C_A C_F$	2	27.79	136.8	330.6	46.20
C_A^2	1	21.02	195.4	1174	5355
$C_F T_R n_f$	0	$-\frac{13}{3}$	−57.59	−405.2	−2120
$C_A T_R n_f$	0	$-\frac{3}{2}$	−24.07	−194.7	−1083

Overview

Counterterm	Types of integrals	Done
$\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}$	tree level singly-unresolved	✔
$\int_1 \mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1}$	one-loop singly-unresolved	✔
$\int_1 (\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1})^{\mathrm{A}_1}$	tree level iterated singly-unresolved (1)	✔
$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$	tree level iterated singly-unresolved (2)	✔
$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$	tree level doubly-unresolved	✔/✘

Overview

NNLO correction is the sum of three terms

$$\sigma^{\rm NNLO} = \sigma^{\rm RR}_{m+2} + \sigma^{\rm RV}_{m+1} + \sigma^{\rm VV}_{m} = \sigma^{\rm NNLO}_{m+2} + \sigma^{\rm NNLO}_{m+1} + \sigma^{\rm NNLO}_{m}$$

Each integrable in four dimensions

$$\sigma^{\rm NNLO}_{m+2} = \int_{m+2} \left\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m - \left[{\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\rm NNLO}_{m+1} = \int_{m+1} \left\{ \left[{\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right] J_{m+1} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] J_m \right\}$$

$$\sigma^{\rm NNLO}_{m} = \int_{m} \left\{ {\rm d}\sigma^{\rm VV}_{m} + \int_2 \left[{\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \right] + \int_1 \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] \right\} J_m$$

- tree-level and one-loop singly-unresolved integrals
- tree-level iterated singly-unresolved integrals
- tree-level doubly-unresolved integrals

Conclusions and outlook

Conclusions and outlook

Summing up

- set up a general, explicit and local subtraction scheme for computing NNLO jet cross sections, for processes with no colored particles in the initial state
- investigated various methods to compute the integrated counterterms: IBP's, MB, SD
- integration of all singly-unresolved and iterated singly-unresolved counterterms finished
- integration of doubly-collinear and double soft-collinear counterterms finished

Next steps

- finish doubly-unresolved integrals: only triply-collinear and double soft counterterms left
- consolidate numerical evaluation of integrals: e.g. through approximating functions obtained by fitting

Backup slides

More integrated approximate cross sections 1

NNLO correction is the sum of three terms

$$\sigma^{\rm NNLO} = \sigma^{\rm RR}_{m+2} + \sigma^{\rm RV}_{m+1} + \sigma^{\rm VV}_{m} = \sigma^{\rm NNLO}_{m+2} + \sigma^{\rm NNLO}_{m+1} + \sigma^{\rm NNLO}_{m}$$

Each integrable in four dimensions

$$\sigma^{\rm NNLO}_{m+2} = \int_{m+2} \left\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m - \left[{\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\rm NNLO}_{m+1} = \int_{m+1} \left\{ \left[{\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right] J_{m+1} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] J_m \right\}$$

$$\sigma^{\rm NNLO}_{m} = \int_{m} \left\{ {\rm d}\sigma^{\rm VV}_{m} + \int_2 \left[{\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \right] + \int_1 \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] \right\} J_m$$

- tree-level and one-loop singly-unresolved integrals
- tree-level iterated singly-unresolved integrals
- tree-level doubly-unresolved integrals

More integrated approximate cross sections 1

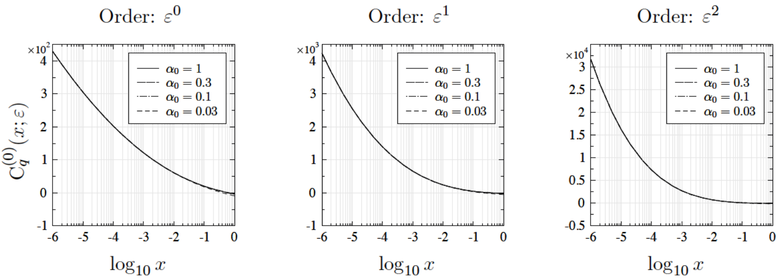
Tree-level singly-unresolved

$$\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} = \mathrm{d}\sigma^{\mathrm{R}}_{m+1} \otimes \mathbf{I}^{(0)}_1(\{p\}_{m+1};\epsilon)$$

- The insertion operator has the following structure

$$\mathbf{I}^{(0)}_1(\{p\}_{m+1};\epsilon) = \frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon \sum_i \left[\mathrm{C}^{(0)}_{1,f_i}\mathbf{T}^2_i + \sum_k \mathrm{S}^{(0),(i,k)}_1 \mathbf{T}_i\mathbf{T}_k\right]$$

  - Pole structure of $\mathbf{I}^{(0)}_1$ coincides with known result
  - In this case higher order expansion coefficients known analytically



More integrated approximate cross sections 1

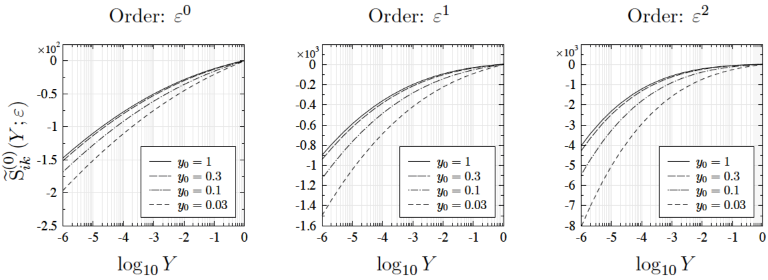
Tree-level singly-unresolved

$$\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_1} = \mathrm{d}\sigma_{m+1}^{\mathrm{R}} \otimes \mathbf{I}_1^{(0)}(\{p\}_{m+1};\epsilon)$$

- The insertion operator has the following structure

$$\mathbf{I}_1^{(0)}(\{p\}_{m+1};\epsilon) = \frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon \sum_i \left[\mathrm{C}_{1,f_i}^{(0)} \mathbf{T}_i^2 + \sum_k \mathrm{S}_1^{(0),(i,k)} \mathbf{T}_i \mathbf{T}_k\right]$$

  - Pole structure of $\mathbf{I}_1^{(0)}$ coincides with known result
  - In this case higher order expansion coefficients known analytically



More integrated approximate cross sections 2

NNLO correction is the sum of three terms

$$\sigma^{\rm NNLO} = \sigma_{m+2}^{\rm RR} + \sigma_{m+1}^{\rm RV} + \sigma_{m}^{\rm VV} = \sigma_{m+2}^{\rm NNLO} + \sigma_{m+1}^{\rm NNLO} + \sigma_{m}^{\rm NNLO}$$

Each integrable in four dimensions

$$\sigma_{m+2}^{\rm NNLO} = \int_{m+2} \left\{ {\rm d}\sigma_{m+2}^{\rm RR} J_{m+2} - {\rm d}\sigma_{m+2}^{\rm RR,A_2} J_m - \left[{\rm d}\sigma_{m+2}^{\rm RR,A_1} J_{m+1} - {\rm d}\sigma_{m+2}^{\rm RR,A_{12}} J_m \right] \right\}$$

$$\sigma_{m+1}^{\rm NNLO} = \int_{m+1} \left\{ \left[{\rm d}\sigma_{m+1}^{\rm RV} + \int_1 {\rm d}\sigma_{m+2}^{\rm RR,A_1} \right] J_{m+1} - \left[{\rm d}\sigma_{m+1}^{\rm RV,A_1} + \left(\int_1 {\rm d}\sigma_{m+2}^{\rm RR,A_1} \right)^{\rm A_1} \right] J_m \right\}$$

$$\sigma_{m}^{\rm NNLO} = \int_{m} \left\{ {\rm d}\sigma_{m}^{\rm VV} + \int_2 \left[{\rm d}\sigma_{m+2}^{\rm RR,A_2} - {\rm d}\sigma_{m+2}^{\rm RR,A_{12}} \right] + \int_1 \left[{\rm d}\sigma_{m+1}^{\rm RV,A_1} + \left(\int_1 {\rm d}\sigma_{m+2}^{\rm RR,A_1} \right)^{\rm A_1} \right] \right\} J_m$$

- tree-level and one-loop singly-unresolved integrals
- tree-level iterated singly-unresolved integrals
- tree-level doubly-unresolved integrals

More integrated approximate cross sections 2

One-loop singly-unresolved

$$\int_1 \mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1} = \mathrm{d}\sigma_m^{\mathrm{V}} \otimes \mathbf{I}_1^{(0)}(\{p\}_m;\epsilon) + \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \mathbf{I}_1^{(1)}(\{p\}_m;\epsilon)$$

- Notice $\mathbf{I}_1^{(0)}(\{p\}_m;\epsilon)$ is the same insertion operator as at tree level
- The one-loop insertion operator has the following structure

$$\mathbf{I}_1^{(1)}(\{p\}_m;\epsilon) = \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \frac{\Gamma^2(1+\epsilon)\Gamma^4(1-\epsilon)}{\Gamma(1+2\epsilon)\Gamma^2(1-2\epsilon)}$$

$$\cdot \sum_i \left[\mathrm{C}_{1,f_i}^{(1)} \mathbf{T}_i^2 + \sum_k \mathrm{S}_1^{(1),(i,k)} \mathbf{T}_i \mathbf{T}_k + \sum_{k,l} \mathrm{S}_1^{(1),(i,k,l)} f_{abc} T_i^a T_k^b T_l^c\right]$$

  - Pole structure of $\mathbf{I}_1^{(1)}$ known analytically up to $\mathrm{O}(\epsilon^{-1})$
  - Higher order expansion coefficients computed numerically

More integrated approximate cross sections 2

Approximation to integrated singly-unresolved

$$\int_1 \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}} \right)^{\mathrm{A_1}} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \left[\frac{1}{2} \left\{ \mathbf{I}_1^{(0)}(\{p\}_m;\epsilon), \mathbf{I}_1^{(0)}(\{p\}_m;\epsilon) \right\} + \mathbf{I}_1^{R\times(0)}(\{p\}_m;\epsilon) \right]$$

- Notice $\mathbf{I}_1^{(0)}(\{p\}_m;\epsilon)$ is the same insertion operator as at tree level
- $\mathbf{I}_1^{R\times(0)}(\{p\}_m;\epsilon)$ part has the same structure as $\mathbf{I}_1^{(0)}(\{p\}_m;\epsilon)$

$$\mathbf{I}_1^{R\times(0)}(\{p\}_m;\epsilon) = \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2} \right)^\epsilon \right]^2 \sum_i \left[\mathrm{C}_{1,f_i}^{R\times(0)} \mathbf{T}_i^2 + \sum_k \mathrm{S}_1^{R\times(0),(i,k)} \mathbf{T}_i \mathbf{T}_k \right]$$

  - Pole structure of $\mathbf{I}_1^{R\times(0)}$ known analytically up to $\mathrm{O}(\epsilon^{-1})$
  - Higher order expansion coefficients computed numerically

More integrated approximate cross sections 3

Doubly-unresolved

$$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \mathbf{I}_2^{(0)}(\{p\}_m;\epsilon)$$

- The insertion operator has the same structure as $\mathbf{I}_{12}^{(0)}(\{p\}_m;\epsilon)$

$$\mathbf{I}_2^{(0)}(\{p\}_m;\epsilon) = \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \Bigg\{ \sum_i \left[\mathrm{C}_{2,f_i}^{(0)} \mathbf{T}_i^2 + \sum_k \mathrm{C}_{2,f_i f_k}^{(0)} \mathbf{T}_k^2\right] \mathbf{T}_i^2 + \sum_{j,l} \left[\mathrm{S}_2^{(0),(j,l)} C_{\mathrm{A}} + \sum_i \mathrm{CS}_{2,f_i}^{(0),(j,l)} \mathbf{T}_i^2\right] \mathbf{T}_j \mathbf{T}_l + \sum_{i,k,j,l} \mathrm{S}_2^{(0),(i,k)(j,l)} \{\mathbf{T}_i \mathbf{T}_k, \mathbf{T}_j \mathbf{T}_l\} \Bigg\}$$

- Work in progress

More on methods: IBP

1. Algebraic reduction of the integrand by means of partial fractioning

$$\frac{1}{x(1-x)(1-xyz)} = \frac{1}{x} + \frac{1}{1-yz}\frac{1}{1-x} + \frac{y^2z^2}{1-yz}\frac{1}{1-xyz}$$

   Note the appearance of a new denominator: $1 - yz$. With increasing numbers of variables, the number of new denominators grows very rapidly.

2. Reduction to master integrals by means of IBP identities. We can use the standard Laporta algorithm to solve the IBP relations, but we find the occurrence of surface terms in the IBPs, consisting of integrals of lower dimensionality than the original ones.

3. Analytical evaluation of the master integrals. We obtain the ϵ expansion of the MIs by solving systems of differential equations, expanded in ϵ. The final results contain one- and two-dimensional harmonic polylogarithms. For some MIs, a nontrivial basis extension of 2dHPLs is necessary.

More on methods: MB

1. Convert sums into products in the integrand

$$\frac{1}{(a+b)^\nu} = \frac{1}{\Gamma(\nu)} \int_{q-\mathrm{i}\infty}^{q+\mathrm{i}\infty} \frac{\mathrm{d}z}{2\pi\mathrm{i}} a^{-\nu-z} b^z \Gamma(\nu+z)\Gamma(-\nu)$$

2. Integrate over the real variables to obtain MB integrals

$$(1-x)^p = \int_0^1 \mathrm{d}y\, y^p \delta(1-x-y)$$

$$\int_0^1 \mathrm{d}x\, \mathrm{d}y\, x^{p_1} y^{p_2} \delta(1-x-y) = \frac{\Gamma(p_1)\Gamma(p_2)}{\Gamma(p_1+p_2)}$$

3. Resolve the pole structure by shifting integration contours.
4. Compute the MB integrals, converting them into sums over residua.
5. Perform the sums.

More on methods: SD

Example

1. Transform the integral so that the range of integration is the unit hypercube, and all singularities are at the borders.

$$I = \int_0^1 \mathrm{d}x\,\mathrm{d}y\,x^{-1-\epsilon}y^{-\epsilon}[x+(1-x)y]^{-1}$$

2. Decompose into "sectors" using $1 = [\Theta(x-y)+\Theta(y-x)]$
3. Remap each integration region to the unit hypercube: for $x \geq y$ set $y \to xt$, for $y \geq x$ set $x \to yt$.

$$I = \int_0^1 \mathrm{d}x\,\mathrm{d}t\,x^{-1-2\epsilon}t^{-\epsilon}[1+(1-x)t]^{-1} + \int_0^1 \mathrm{d}t\,\mathrm{d}y\,t^{-1-\epsilon}y^{-1-2\epsilon}[1+(1-y)t]^{-1}$$

4. Resolve the pole structure using simple residuum subtraction. This gives a finite integral representation for the expansion coefficients.
5. Integrate these representations.

Spinoff - angular integrals in d dimensions

Consider the d dimensional angular integral with n denominators

$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1 \cdot q)^{j_1} \cdots (p_n \cdot q)^{j_n}}$$

We find (with $j = j_1 + \ldots + j_n$)

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon} \pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$$

where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables.

Spinoff - angular integrals in d dimensions

Consider the d dimensional angular integral with n denominators

$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1 \cdot q)^{j_1} \cdots (p_n \cdot q)^{j_n}}$$

We find (with $j = j_1 + \ldots + j_n$)

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon} \pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_s}]$$

where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables. We have

$$\mathbf{v} = (v_{11}, v_{12}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{n-1n}, v_{nn}), \qquad v_{kl} \equiv \begin{cases} \dfrac{p_k \cdot p_l}{2} & ; \quad k \neq l \\ \dfrac{p_k^2}{4} & ; \quad k = l \end{cases}$$

$$\boldsymbol{\alpha} = (\mathbf{0}_N, j_1, \ldots, j_n, 1 - j - \epsilon), \qquad \boldsymbol{\beta} = (j_1, \ldots, j_n, 2 - j - 2\epsilon)$$

and $\mathbf{L_s} = L_{s_1} \times \ldots L_{s_N}$, where L_{s_k} is an infinite contour in the complex s_k-plane running from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$.

Spinoff - angular integrals in d dimensions

Consider the d dimensional angular integral with n denominators

$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1 \cdot q)^{j_1} \cdots (p_n \cdot q)^{j_n}}$$

We find (with $j = j_1 + \ldots + j_n$)

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon} \pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_s}]$$

where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables. We have

$$\mathbf{A} = \left[\begin{array}{c} -\mathbf{1}_{N\times N} \\ \hline \mathbf{M}_{n\times N} \\ \hline -1 \cdots -1 \end{array} \right], \qquad \mathbf{B} = [(0)_{(n+1)\times N}]$$

i.e. $\mathbf{B}$ is zero, while the $n \times N$ dimensional matrix $\mathbf{M}$ has the following block form:

$$\mathbf{M}_{n\times N} = \left[\; \mathbf{m}_{n\times n} \; \middle| \; \mathbf{m}_{n\times(n-1)} \; \middle| \; \cdots \; \middle| \; \mathbf{m}_{n\times 1} \; \right] \quad \text{with} \quad \mathbf{m}_{n\times p} = \left[\begin{array}{c|c} 0 & (0)_{(n-p)\times(p-1)} \\ \hline 2 & 1\cdots 1 \\ \hline 0 & \\ \vdots & \mathbf{1}_{(p-1)\times(p-1)} \\ 0 & \end{array} \right]$$

Spinoff - angular integrals in d dimensions

Consider the d dimensional angular integral with n denominators

$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1\cdot q)^{j_1}\cdots(p_n\cdot q)^{j_n}}$$

We find (with $j = j_1 + \ldots + j_n$)

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v};(\boldsymbol{\alpha},\mathbf{A});(\boldsymbol{\beta},\mathbf{B});\mathbf{L_s}]$$

where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables. We have

$$\Omega_{j_1,\ldots,j_n}(\{v_{kl}\};\epsilon) = 2^{2-j-2\epsilon}\pi^{1-\epsilon}\frac{1}{\prod_{k=1}^{n}\Gamma(j_k)\Gamma(2-j-2\epsilon)} \times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left[\prod_{k=1}^{n}\prod_{l=k}^{n}\frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}}\Gamma(-z_{kl})\,(v_{kl})^{z_{kl}}\right]\left[\prod_{k=1}^{n}\Gamma(j_k+z_k)\right]\Gamma(1-j-\epsilon-z)\,.$$

where

$$z = \sum_{k=1}^{n}\sum_{l=k}^{n} z_{kl}\,, \qquad \text{and} \qquad z_k = \sum_{l=1}^{k} z_{lk} + \sum_{l=k}^{n} z_{kl}\,.$$

Angular Integral with n Denominators – Definition

Motivated by the previous example, we define d dimensional angular integrals with n denominators as follows

Definition (angular integral with n denominators)

Let $p_1^\mu, \dots, p_n^\mu$ be some fixed vectors in d dimensional Minkowski space. Then

$$\Omega_{j_1,\dots,j_n} \equiv \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1 \cdot q)^{j_1} \cdots (p_n \cdot q)^{j_n}}$$

where $\mathrm{d}\Omega_{d-1}(q)$ is the angular measure in d dimensions for the massless vector q^μ.

Note that $\Omega_{j_1,\dots,j_n}$ is clearly rotationally invariant and the overall normalization of the p_i^μ and q^μ plays no essential role.

Angular Integral with n Denominators – Definition (the small print)

More precisely (but less to the point), we may choose a Lorentz frame and normalization such that

$$p_1^\mu = (1, \mathbf{0}_{d-2}, \beta_1)\,, \qquad p_2^\mu = (1, \mathbf{0}_{d-3}, \beta_2 \sin\chi_2^{(1)}, \beta_2\cos\chi_2^{(1)})$$

$$\vdots$$

$$p_n^\mu = (1, \mathbf{0}_{d-1-n}, \beta_n \prod_{k=1}^{n-1}\sin\chi_n^{(k)}, \beta_n\cos\chi_n^{(n-1)}\prod_{k=1}^{n-2}\sin\chi_n^{(k)}, \ldots, \beta_n\cos\chi_n^{(2)}\sin\chi_n^{(1)}, \beta_n\cos\chi_n^{(1)})$$

$$q^\mu = (1, \text{..`angles'..}, \cos\vartheta_n\prod_{k=1}^{n-1}\sin\vartheta_k, \cos\vartheta_{n-1}\prod_{k=1}^{n-2}\sin\vartheta_k, \ldots, \cos\vartheta_2\sin\vartheta_1, \cos\vartheta_1)$$

Definition (angular integral with n denominators)

$$\Omega_{j_1,\ldots,j_n} \equiv \int \mathrm{d}\Omega_{d-1-n}(q)\int_{-1}^{1}\left[\prod_{k=1}^{n}\mathrm{d}(\cos\vartheta_k)\,(\sin\vartheta_k)^{-k+1-2\epsilon}\right]$$

$$\times\prod_{k=1}^{n}\left\{1-\beta_k\sum_{l=1}^{k}\left[\left(\delta_{lk}+(1-\delta_{lk})\cos\chi_k^{(l)}\right)\cos\vartheta_l\prod_{m=1}^{l-1}\left(\sin\chi_k^{(m)}\sin\vartheta_m\right)\right]\right\}^{-j_k}$$

where we used

$$\mathrm{d}\Omega_{d-1}(q) = \prod_{k=1}^{n}\mathrm{d}(\cos\vartheta_k)\,(\sin\vartheta_k)^{-k+1-2\epsilon}\mathrm{d}\Omega_{d-1-n}(q)$$

Angular Integral with n Denominators – Definition

Motivated by the previous example, we define d dimensional angular integrals with n denominators as follows

Definition (angular integral with n denominators)

Let $p_1^\mu, \ldots, p_n^\mu$ be some fixed vectors in d dimensional Minkowski space. Then

$$\Omega_{j_1,\ldots,j_n} \equiv \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1 \cdot q)^{j_1} \cdots (p_n \cdot q)^{j_n}}$$

where $\mathrm{d}\Omega_{d-1}(q)$ is the angular measure in d dimensions for the massless vector q^μ.

Note that the definition is more general than it seems at first sight, e.g.

$$p_1^\mu = (1, \mathbf{0}_{d-2}, \beta_1), \qquad q^\mu = (1, ..\text{'angles'}.., \cos\vartheta)$$

then

$$(p_1 + q)^2 = p_1^2 + 2p_1 \cdot q = (1 - \beta_1^2) + 2(1 - \beta_1 \cos\vartheta) = (3 - \beta_1^2)\left[1 - \frac{2\beta_1}{3 - \beta_1^2}\cos\vartheta\right]$$

$$= (3 - \beta_1^2)(p_1' \cdot q)$$

where

$$p_1'^{\,\mu} = \left(1, \mathbf{0}_{d-2}, \frac{2\beta_1}{3 - \beta_1^2}\right), \qquad q^\mu = (1, ..\text{'angles'}.., \cos\vartheta)$$

Angular Integral with n Denominators – General Result 1

The d dimensional angular integrals with n denominators can be evaluated in terms of the so-called H-function of several variables.

Result

Let $j = j_1 + \ldots + j_n$. Then we have

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon}H[\mathbf{v};(\boldsymbol{\alpha},\mathbf{A});(\boldsymbol{\beta},\mathbf{B});\mathbf{L_S}]$$

where $H[\mathbf{v};(\boldsymbol{\alpha},\mathbf{A});(\boldsymbol{\beta},\mathbf{B});\mathbf{L_S}]$ is the H-function of the following $N = \frac{n(n+1)}{2}$ variables

$$\mathbf{v} = (v_{11}, v_{12}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{n-1n}, v_{nn}), \qquad v_{kl} \equiv \begin{cases} \dfrac{p_k \cdot p_l}{2} & ; \quad k \neq l \\ \dfrac{p_k^2}{4} & ; \quad k = l \end{cases}$$

We are assuming the general case, i.e. $v_{kl} > 0$, $\forall\, k, l = 1, \ldots, n$.

(The parameters $(\boldsymbol{\alpha},\mathbf{A})$, $(\boldsymbol{\beta},\mathbf{B})$ as well as $\mathbf{L_S}$ will be discussed momentarily.)

The result looks tidy enough, but you might be wondering what exactly an H-function of several variables is.

The H-function of several variables

Definition (H-function of N variables)

$$H[\mathbf{x};(\boldsymbol{\alpha},\mathbf{A});(\boldsymbol{\beta},\mathbf{B});\mathbf{L_s}] \equiv (2\pi\mathrm{i})^{-N}\int_{\mathbf{L_s}} \Theta(\mathbf{s})\,\mathbf{x}^{\mathbf{s}}\,\mathbf{ds}$$

where

$$\Theta(\mathbf{s}) = \frac{\prod_{j=1}^{m}\Gamma\left(\alpha_j+\sum_{k=1}^{N} a_{j,k}s_k\right)}{\prod_{j=1}^{n}\Gamma\left(\beta_j+\sum_{k=1}^{N} b_{j,k}s_k\right)}$$

Here $\mathbf{s}=(s_1,\ldots,s_N)$, $\mathbf{x}=(x_1,\ldots,x_N)$, $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_m)$ and $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_n)$ denote vectors of complex numbers; while

$$\mathbf{A}=(a_{j,k})_{m\times N} \qquad \text{and} \qquad \mathbf{B}=(b_{j,k})_{n\times N}$$

are matrices of real numbers. Also

$$\mathbf{x}^{\mathbf{s}}=\prod_{k=1}^{N}(x_k)^{s_k}\,; \qquad \mathbf{ds}=\prod_{k=1}^{N}\mathrm{d}s_k\,; \qquad \mathbf{L_s}=L_{s_1}\times\ldots\times L_{s_N}\,,$$

where L_{s_k} is an infinite contour in the complex s_k-plane running from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$ such that $\Theta(\mathbf{s})$ has no singularities for $\mathbf{s}\in\mathbf{L_s}$.

[N. T. Hai, H. M. Srivastava, Computers Math. Applic. **29**, 17 (1995)]

The H-function of several variables

Some comments

- The H-function of several variables generalizes nearly all known special functions of N variables, e.g. Lauricella functions $F_A^{(N)}$, $F_B^{(N)}$, $F_C^{(N)}$ and $F_D^{(N)}$; the G-function of N variables; the special H-function of N variables, etc.
- For the specific cases of $N = 1$ and 2, it essentially reduces to the known Fox's H-function of one variable and the H-function of two variables defined by various authors scattered in the literature.
- The H-function of several variables satisfies various contiguous relations, i.e. algebraic relations between functions $H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$ with the vectors of parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ shifted by vectors of integers. These relations may be used to reduce H-functions to a set of basis functions with parameters differing form the original values by integer shifts.

  [O. P. Tandon, Indian J. Pure Appl. Math. **11**, 321 (1980)]
  [C. M. Joshi, J. P. Arya, Indian J. Pure Appl. Math. **12**, 826 (1981)]
  [O. P. Tandon, Indian J. Math. **24**, 55 (1982)]
- The definition given above is different form the H-function considered by Hai and Srivastava only in the replacement of $\mathbf{x}^{-\mathbf{S}}$ by $\mathbf{x}^{\mathbf{S}}$. We have made this replacement for convenience in our applications.

Angular Integral with n Denominators – General Result 2

The d dimensional angular integrals with n denominators can be evaluated in terms of the so-called H-function of several variables.

Result

Let $j = j_1 + \ldots + j_n$. Then we have

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$$

where $H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$ is the H-function of the following $N = \frac{n(n+1)}{2}$ variables

$$\mathbf{v} = (v_{11}, v_{12}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{n-1n}, v_{nn}), \qquad v_{kl} \equiv \begin{cases} \dfrac{p_k \cdot p_l}{2} & ; \quad k \neq l \\ \dfrac{p_k^2}{4} & ; \quad k = l \end{cases}$$

We are assuming the general case, i.e. $v_{kl} > 0$, $\forall\, k, l = 1, \ldots, n$.

Angular Integral with n Denominators – General Result 2

The d dimensional angular integrals with n denominators can be evaluated in terms of the so-called H-function of several variables.

Result

Let $j = j_1 + \ldots + j_n$. Then we have

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$$

$\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the following vectors of parameters

$$\boldsymbol{\alpha} = (\mathbf{0}_N, j_1, \ldots, j_n, 1-j-\epsilon)\,, \qquad \boldsymbol{\beta} = (j_1, \ldots, j_n, 2-j-2\epsilon)$$

Angular Integral with n Denominators – General Result 2

The d dimensional angular integrals with n denominators can be evaluated in terms of the so-called H-function of several variables.

Result

Let $j = j_1 + \ldots + j_n$. Then we have

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_s}]$$

A and **B** are $\frac{(n+1)(n+2)}{2} \times N$ and $(n+1) \times N$ matrices of parameters, respectively

$$\mathbf{A} = \left[\begin{array}{c} -\mathbf{1}_{N\times N} \\ \hline \mathbf{M}_{n\times N} \\ \hline -1 \cdots -1 \end{array}\right], \qquad \mathbf{B} = [(0)_{(n+1)\times N}]$$

i.e. **B** is zero, while the $n \times N$ dimensional matrix **M** has the following block form

$$\mathbf{M}_{n\times N} = \left[\ \mathbf{m}_{n\times n} \ \middle|\ \mathbf{m}_{n\times(n-1)} \ \middle|\ \cdots \ \middle|\ \mathbf{m}_{n\times 1}\ \right] \quad \text{with} \quad \mathbf{m}_{n\times p} = \left[\begin{array}{c|c} 0 & (0)_{(n-p)\times(p-1)} \\ \hline 2 & 1 \cdots 1 \\ \hline 0 & \\ \vdots & \mathbf{1}_{(p-1)\times(p-1)} \\ 0 & \end{array}\right]$$

Angular Integral with n Denominators – General Result 2

The d dimensional angular integrals with n denominators can be evaluated in terms of the so-called H-function of several variables.

Result

Let $j = j_1 + \ldots + j_n$. Then we have

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_s}]$$

$\mathbf{A}$ and $\mathbf{B}$ are $\frac{(n+1)(n+2)}{2} \times N$ and $(n+1) \times N$ matrices of parameters, respectively e.g.

$$\mathbf{A}_1 = \left[\begin{array}{c} -1 \\ \hline 2 \\ \hline -1 \end{array}\right], \quad \mathbf{A}_2 = \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ \hline 2 & 1 & 0 \\ 0 & 1 & 2 \\ \hline -1 & -1 & -1 \end{array}\right], \quad \mathbf{A}_3 = \left[\begin{array}{ccc|cc|c} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ \hline -1 & -1 & -1 & -1 & -1 & -1 \end{array}\right]$$

Angular Integral with n Denominators – General Result 2

The d dimensional angular integrals with n denominators can be evaluated in terms of the so-called H-function of several variables.

Result

Let $j = j_1 + \ldots + j_n$. Then we have

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$$

Hence we find the Mellin–Barnes representation

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} \frac{1}{\prod_{k=1}^{n} \Gamma(j_k)\Gamma(2-j-2\epsilon)}$$

$$\times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \left[\prod_{k=1}^{n}\prod_{l=k}^{n} \frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}} \Gamma(-z_{kl})(v_{kl})^{z_{kl}}\right] \left[\prod_{k=1}^{n} \Gamma(j_k + z_k)\right] \Gamma(1-j-\epsilon-z)\,.$$

where the contours of integration for z_{kl} are chosen such that poles with $\Gamma(\ldots + z_{kl})$ dependence are to the left of the contour and poles with $\Gamma(\ldots - z_{kl})$ dependence are to the right of it.

Analytical results - one denominator

- One denominator, massless ($p_1^2 = 0$)

$$\Omega_j(0;\epsilon) = \int \mathrm{d}\Omega_{d-1} \frac{1}{(p_1 \cdot q)^j} = 2^{2-j-2\epsilon} \pi^{1-\epsilon} \frac{\Gamma(1-j-\epsilon)}{\Gamma(2-j-2\epsilon)}$$

- One denominator, massive ($p_1^2 = 4v_{11} \neq 0$)

$$\Omega_j(v_{11};\epsilon) = \int \mathrm{d}\Omega_{d-1} \frac{1}{(p_1 \cdot q)^j} = 2^{2-2\epsilon} \pi^{1-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}$$
$$\times {}_2F_1\left(\frac{j}{2}, \frac{j+1}{2}, \frac{3}{2} - \epsilon, 1 - 4v_{11}\right)$$

Analytical results – two denominators

- Two denominators, massless ($p_1^2 = p_2^2 = 0$, $p_1 \cdot p_2 = 2v_{12} \neq 0$)

$$\Omega_{j,k}(v_{12}, 0, 0; \epsilon) = \int \mathrm{d}\Omega_{d-1} \frac{1}{(p_1 \cdot q)^j (p_2 \cdot q)^k} = 2^{2-j-k-2\epsilon} \pi^{1-\epsilon} \frac{\Gamma(1-j-\epsilon)\Gamma(1-k-\epsilon)}{\Gamma(1-\epsilon)\Gamma(2-j-k-2\epsilon)} \times {}_2F_1(j, k, 1-\epsilon, 1-v_{12})$$

- Two denominators, one mass ($p_1^2 = 4v_{11} \neq 0$, $p_2^2 = 0$, $p_1 \cdot p_2 = 2v_{12} \neq 0$)

$$\Omega_{j,k}(v_{12}, v_{11}, 0; \epsilon) = \int \mathrm{d}\Omega_{d-1} \frac{1}{(p_1 \cdot q)^j (p_2 \cdot q)^k} = 2^{2-j-k-2\epsilon} \pi^{1-\epsilon} \frac{\Gamma(1-k-\epsilon)}{\Gamma(2-k-2\epsilon)} v_{12}^{-j}$$
$$\times F_1\left(j, 1-k-\epsilon, 1-k-\epsilon, 2-k-2\epsilon, \frac{2v_{12}-1-\sqrt{1-4v_{11}}}{2v_{12}}, \frac{2v_{12}-1+\sqrt{1-4v_{11}}}{2v_{12}}\right)$$