Does one need the $\mathcal{O}(\varepsilon)$- and $\mathcal{O}(\varepsilon^2)$-terms of one-loop amplitudes in an NNLO calculation ?

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The amplitudes in an NNLO calculation

$$
\begin{aligned}
|\mathcal{A}_n|^2_{\mathrm{NNLO}} &= \mathcal{A}_n^{(0)\,*} \mathcal{A}_n^{(2)} + \mathcal{A}_n^{(2)\,*} \mathcal{A}_n^{(0)} + \mathcal{A}_n^{(1)\,*} \mathcal{A}_n^{(1)}, \\
|\mathcal{A}_{n+1}|^2_{\mathrm{NNLO}} &= \mathcal{A}_{n+1}^{(0)\;*} \mathcal{A}_{n+1}^{(1)} + \mathcal{A}_{n+1}^{(1)\;*} \mathcal{A}_{n+1}^{(0)}, \\
|\mathcal{A}_{n+2}|^2_{\mathrm{NNLO}} &= \mathcal{A}_{n+2}^{(0)\;*} \mathcal{A}_{n+2}^{(0)}.
\end{aligned}
$$

The one-loop amplitude $\mathcal{A}_n^{(1)}$ has a Laurent expansion

$$
\mathcal{A}_n^{(1)} = \frac{1}{\varepsilon^2}\mathcal{A}_n^{(1,-2)} + \frac{1}{\varepsilon}\mathcal{A}_n^{(1,-1)} + \mathcal{A}_n^{(1,0)} + \varepsilon \mathcal{A}_n^{(1,1)} + \varepsilon^2 \mathcal{A}_n^{(1,2)} + O\left(\varepsilon^3\right),
$$

therefore

$$
\left|\mathcal{A}_n^{(1)}\right|^2 = \ldots + \left|\mathcal{A}_n^{(1,0)}\right|^2 + 2\,\mathrm{Re}\left(\mathcal{A}_n^{(1,-2)\,*} \mathcal{A}_n^{(1,2)} + \mathcal{A}_n^{(1,-1)\,*} \mathcal{A}_n^{(1,1)}\right) + \ldots
$$

One might expect that the $O(\varepsilon)$-term $\mathcal{A}_n^{(1,1)}$ and the $O(\varepsilon^2)$-term $\mathcal{A}_n^{(1,2)}$ are needed.

Methods for the calculation of amplitudes

- Feynman-diagram-based approach: Reduction to master integrals, solving master integrals by Mellin-Barnes, differential equations or nested sums.
  - Calculation of $\mathcal{O}(\varepsilon)$- and $\mathcal{O}(\varepsilon^2)$-terms feasible within this approach.

- Sector decomposition:
  - $\mathcal{O}(\varepsilon)$- and $\mathcal{O}(\varepsilon^2)$-terms feasible, but computer-time expensive.

- Unitarity- or cut-based approach:
  - Master integrals as in the first approach.
  - Generalisation of "rational" terms to $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\varepsilon^2)$.

- Numerical approach to loop amplitudes: Based on universal subtraction terms, finite part calculated numerically in four dimensions.
  - Calculation of $\mathcal{O}(\varepsilon)$- and $\mathcal{O}(\varepsilon^2)$-terms unknown within this approach.

Where does the one-loop amplitude occur ?

Without initial-state partons:

- In the square of the one-loop amplitude.
- In Catani's decomposition of the two-loop amplitude into a pole part and a finite remainder.
- In the integrated subtraction term for the $(n+1)$-parton real+virtual contribution.

With initial-state partons:

- Additional contribution from the integrated subtraction term for the $(n+1)$-parton real+virtual contribution.
- Collinear counterterm from factorisation.

Catani's decomposition

$$\begin{aligned}\mathcal{A}_n^{(1)} &= \mathbf{I}^{(1)}\mathcal{A}_n^{(0)} + \mathcal{F}_n^{(1)}, \\ \mathcal{A}_n^{(2)} &= \mathbf{I}^{(2)}\mathcal{A}_n^{(0)} + \mathbf{I}^{(1)}\mathcal{A}_n^{(1)} + \mathcal{F}_n^{(2)}.\end{aligned}$$

The remainder functions $\mathcal{F}_n^{(1)}$ and $\mathcal{F}_n^{(2)}$ start at $O(\varepsilon^0)$.

The one-loop insertion operator $\mathbf{I}^{(1)}$ is given in the massless case by

$$\mathbf{I}^{(1)} = \frac{\alpha_s}{2\pi}\frac{1}{2}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\sum_i\sum_{j\neq i}\mathbf{T}_i\mathbf{T}_j\left(\frac{1}{\varepsilon^2}+\frac{\gamma_i}{\mathbf{T}_i^2}\frac{1}{\varepsilon}\right)\left(\frac{-2p_ip_j}{\mu^2}\right)^{-\varepsilon}.$$

Note that the insertion operators $\mathbf{I}^{(1)}$ and $\mathbf{I}^{(2)}$ contain also terms of $O(\varepsilon^0)$ and higher.

Scheme dependence of the insertion operators

One might prefer a "minimal" scheme, in which the insertion operators contain exactly the pole terms and nothing else.

$$\mathbf{Z}^{(1)} = R\left(\mathbf{I}^{(1)}\right), \qquad \mathbf{Z}^{(2)} = R\left(\mathbf{I}^{(2)} + \mathbf{I}^{(1)} R\left(\mathbf{I}^{(1)}\right)\right),$$

where R denotes the projection onto the pole part. Changing the insertion operators to the minimal scheme will also change the finite remainder functions. One has

$$\begin{aligned} \mathcal{A}_n^{(1)} &= \mathbf{Z}^{(1)} \mathcal{A}_n^{(0)} + \mathcal{F}_{n,\mathrm{minimal}}^{(1)}, \\ \mathcal{A}_n^{(2)} &= \left(\mathbf{Z}^{(2)} - \mathbf{Z}^{(1)} \mathbf{Z}^{(1)}\right) \mathcal{A}_n^{(0)} + \mathbf{Z}^{(1)} \mathcal{A}_n^{(1)} + \mathcal{F}_{n,\mathrm{minimal}}^{(2)}. \end{aligned}$$

The single unresolved limit of amplitudes

At NLO we need to know the soft and collinear limit of $\mathcal{A}_{n+1}^{(0)}$:

$$\lim_{n+1\to n} \mathcal{A}_{n+1}^{(0)} \quad = \quad \mathrm{Sing}^{(0)} \circ \mathcal{A}_n^{(0)}$$

At NNLO we also need the soft and the collinear limit of $\mathcal{A}_{n+1}^{(0)}$:

$$\lim_{n+1\to n} \mathcal{A}_{n+1}^{(1)} \quad = \quad \mathrm{Sing}^{(0)} \circ \mathcal{A}_n^{(1)} + \mathrm{Sing}^{(1)} \circ \mathcal{A}_n^{(0)}.$$

The functions $\mathrm{Sing}^{(0)}$ and $\mathrm{Sing}^{(1)}$ describing the singular limit are universal.

The subtraction method

We need a subtraction term $d\alpha^{\text{loop}}$ for the singular behaviour of $2\,\text{Re}\,{\mathcal{A}_{n+1}^{(0)}}^* \mathcal{A}_{n+1}^{(1)}$ in the soft and collinear limits. Because of

$$\lim_{n+1 \to n} \mathcal{A}_{n+1}^{(1)} = \text{Sing}^{(0)} \circ \mathcal{A}_n^{(1)} + \text{Sing}^{(1)} \circ \mathcal{A}_n^{(0)}.$$

decompose the subtraction term into

$$d\alpha^{\text{loop}} = d\alpha^{\text{loop},a} + d\alpha^{\text{loop},b},$$

with

$$\lim_{n+1 \to n} d\alpha^{\text{loop},a} = \left|\text{Sing}^{(0)}\right|^2 \circ d\sigma_n^{(1)},$$
$$\lim_{n+1 \to n} d\alpha^{\text{loop},b} = 2\,\text{Re}\,\text{Sing}^{(0)*}\,\text{Sing}^{(1)} \circ d\sigma_n^{(0)}.$$

The integrated subtraction term

$$\int d\alpha_n^{\text{loop},a} = \int \left(\mathbf{I}^{(1)}_{\text{real}} + \mathbf{F}^{(1)}_{\text{real}} \right) \circ d\sigma_n^{(1)}$$

$\mathbf{I}^{(1)}_{\text{real}}$: contains all the poles in the dimensional regularisation parameter ε, the poles are independent of the subtraction scheme.

$\mathbf{F}^{(1)}_{\text{real}}$: finite remainder, depends on the subtraction scheme.

$\mathbf{I}^{(1)}_{\text{real}}$ is given in massless QCD by

$$\mathbf{I}^{(1)}_{\text{real}} = -\frac{\alpha_s}{2\pi} \frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)} \sum_i \sum_{j\neq i} \mathbf{T}_i \mathbf{T}_j \left(\frac{1}{\varepsilon^2} + \frac{\gamma_i}{\mathbf{T}_i^2} \frac{1}{\varepsilon} \right) \left(\frac{|2p_i p_j|}{\mu^2} \right)^{-\varepsilon}$$

$\mathbf{I}^{(1)}_{\text{real}}$ differs from $\mathbf{I}^{(1)}$ by a factor (-2) and by $(-2p_i p_j) \to |2p_i p_j|$.

Putting it all together

Consider first the case with no initial partons:

Express one- and two-loop amplitudes according to Catani's decomposition, then

$$2\,\mathrm{Re}\,\mathcal{A}_n^{(0)\,*}\,\mathcal{A}_n^{(2)} + \mathcal{A}_n^{(1)\,*}\,\mathcal{A}_n^{(1)} + 2\,\mathrm{Re}\,\mathcal{A}_n^{(0)\,*}\left(\mathbf{I}_{\mathrm{real}}^{(1)} + \mathbf{F}_{\mathrm{real}}^{(1)}\right)\mathcal{A}_n^{(1)} =$$

$$\ldots + \underbrace{\mathcal{F}_n^{(1)\,*}\,\mathcal{F}_n^{(1)}}_{\text{finite}} + \underbrace{2\,\mathrm{Re}\,\mathcal{A}_n^{(0)\,*}\,\mathbf{F}_{\mathrm{real}}^{(1)}\mathcal{F}_n^{(1)}}_{\text{finite}} + 2\,\mathrm{Re}\,\mathcal{A}_n^{(0)\,*}\underbrace{\left(\mathbf{I}^{(1)} + \mathbf{I}^{(1)\,*} + \mathbf{I}_{\mathrm{real}}^{(1)}\right)}_{\text{finite}}\mathcal{F}_n^{(1)} + \ldots$$

In Catani's decomposition we need only the $O(\varepsilon^0)$-term of $\mathcal{F}_n^{(1)}$.

Initial state partons

- With an initial state parton, the integration over the subtraction term $d\alpha^{\text{loop,a}}$ will give additional pole terms proportional to the Altarelli-Parisi splitting functions $P^{(1)}(z)$ and the one-loop amplitude $\mathcal{A}_n^{(1)}$.

- Cancelled by the collinear counterterm from factorisation:

$$-\frac{\alpha_s}{2\pi} S_\varepsilon \int_0^1 dz \left[-\frac{1}{\varepsilon} \left(\frac{\mu_F^2}{\mu^2} \right)^{-\varepsilon} P^{(1)}(z) \right] 2\,\mathrm{Re}\, \mathcal{A}_n^{(0)\,*} \mathcal{A}_n^{(1)},$$

Net result:

$$\int_0^1 dz \left[\mathbf{P}^{(1)}(z) + \mathbf{K}^{(1)}(z) \right] 2\,\mathrm{Re}\, \mathcal{A}_n^{(0)\,*} \mathcal{A}_n^{(1)}$$

The colour-charge operators $\mathbf{P}^{(1)}$ and $\mathbf{K}^{(1)}$ are finite, therefore only the $O(\varepsilon^0)$-terms of the one-loop amplitude are needed.

Take-home message

For an NNLO calculation we need to know as far as amplitudes are concerned:

- The tree-level amplitudes $\mathcal{A}_n^{(0)}$, $\mathcal{A}_{n+1}^{(0)}$ and $\mathcal{A}_{n+2}^{(0)}$ in four-dimensions.
- The insertion operators $\mathbf{I}^{(1)}$ and $\mathbf{I}^{(2)}$.
- The finite remainder functions $\mathcal{F}_n^{(1)}$, $\mathcal{F}_{n+1}^{(1)}$ and $\mathcal{F}_n^{(2)}$ at $\mathcal{O}(\varepsilon^0)$.

Theorem: The $\mathcal{O}(\varepsilon)$- and $\mathcal{O}(\varepsilon^2)$-terms of the one-loop finite remainder function $\mathcal{F}_n^{(1)}$ drop out from the final result and are therefore not needed.

Remark 1: Holds also in a minimal scheme, where $\mathbf{I}^{(1)}$ and $\mathbf{I}^{(2)}$ are replaced by $\mathbf{Z}^{(1)}$ and $\mathbf{Z}^{(2)}$, and $\mathcal{F}_n^{(1)}$, $\mathcal{F}_{n+1}^{(1)}$ and $\mathcal{F}_n^{(2)}$ are replaced by $\mathcal{F}_{n,\mathrm{minimal}}^{(1)}$, $\mathcal{F}_{n+1,\mathrm{minimal}}^{(1)}$ and $\mathcal{F}_{n,\mathrm{minimal}}^{(2)}$.

Remark 2: Need to know the $\mathcal{O}(\varepsilon^0)$-terms of $\mathcal{F}_n^{(2)}$, not the ones of $\mathcal{A}_n^{(2)}$.