The singular behavior of one-loop massive QCD amplitudes with one external soft gluon

or

calculating the real-virtual corrections at NNLO with massive quarks

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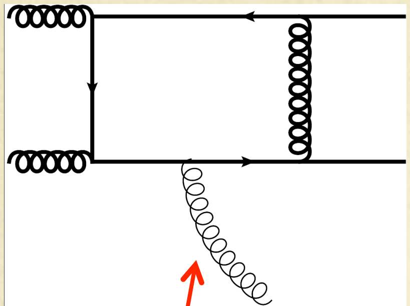
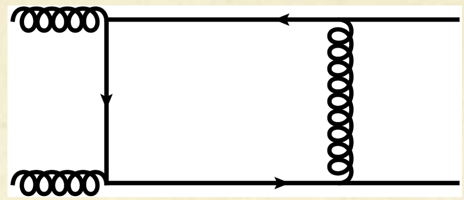
Based on arXiv:1107.4384 with
I. Bierenbaum and M. Czakon

What do we do?

Derive the last missing singular limit of a QCD amplitude needed at NNLO

 =  x J_soft(q)

The external gluon becomes soft (i.e. q → 0)

Process-independent;
Color correlated

What happens is very intuitive:

1) When the gluon's energy vanishes, it is like it was not there at all.

2) The only trace of it is in color correlations (they are large!)

Now all singular limits of massless and massive amplitudes are exactly known through NNLO:

IR singularities at 2 loops for any massive amplitude:

Czakon, Mitov, Sterman, Sung '08-09

Becher, Ferroglia, Neubert, Pecjak, Yang '09

Small mass limit of massive amplitudes (even the finite terms!)
(instrumental to figure out the 2-loop massive singularities and in the derivation of the 2-loop amplitudes)

Mitov, Moch `06
Melnikov, Becher `07

New general formalism derived for studying this at any loop:

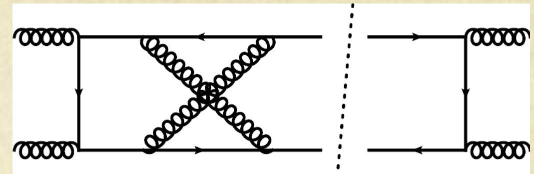
- ✓ Generalization of the non-abelian exponentiation theorem.
- ✓ Applies to Wilson lines and loops.
- ✓ Most importantly systematizes the renormalization procedure of such object (was crucial in the massive 2-loop case)

Mitov, Sterman, Sung '10
Gardi et al `10

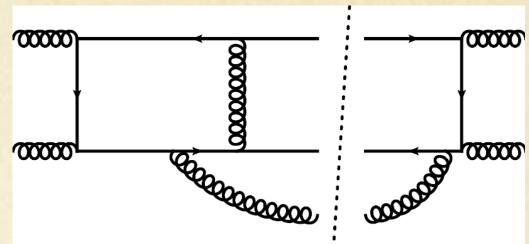
Towards NNLO

There are 3 principle contributions:

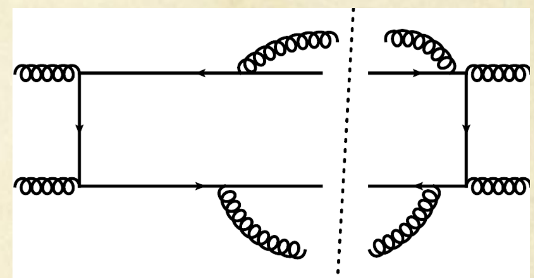
- ✓ 2-loop virtual corrections (V-V)



- ✓ 1-loop virtual with one extra parton (R-V)

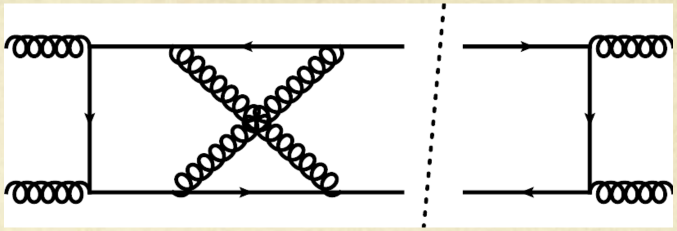


- ✓ 2 extra emitted partons at tree level (R-R)



Towards NNLO: V-V (with masses)

Required are the two loop amplitudes:
qq → QQ and gg → QQ.



- ✓ Their high energy limits and their poles are known analytically

  Czakon, Mitov, Moch '07
  Czakon, Mitov, Sterman '09
  Ferroglia, Neubert, Pecjak, Yang '09

- ✓ The qq→QQ amplitude is known numerically — Czakon `07

- ✓ Numerical work underway for the gg→QQ — Czakon, Bärnreuther

- ✓ Some color structures derived analytically — Bonciani et al '10-

- ✓ Of course in the massless case all amplitudes are known analytically

  Anastasiou et al '99-

Comments about the 2-loop massive amplitudes

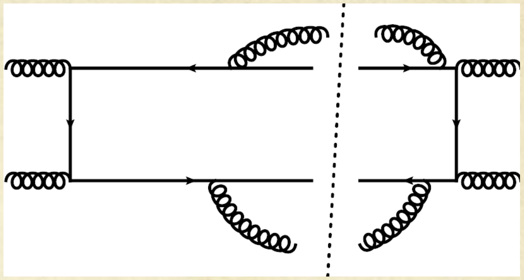
- ✓ Czakon’s calculation is in principle straightforward (highly tedious):
  - ✓ Derive a large set of masters,
  - ✓ Derive differential equations for them (2-dim pde)
  - ✓ Derive numerically boundary conditions in the high energy limit (Mellin-Barnes)
  - ✓ Solve numerically the pde’s
- ✓ That works for qq → QQ and for gg → QQ
- ✓ This will work for any 2-to-2 amplitude, obviously. But hardly beyond.

What’s the future here?

- ✓ I believe that right now this is the biggest (and perhaps only) obstacle for NNLO phenomenology on a mass scale.
- ✓ Two-loop unitary: Going beyond 2-2? New ideas around:

Mastrolia Osola ‘11
Kosower Larsen ‘11

Towards NNLO: R-R



✓ Great result! All is done ☺

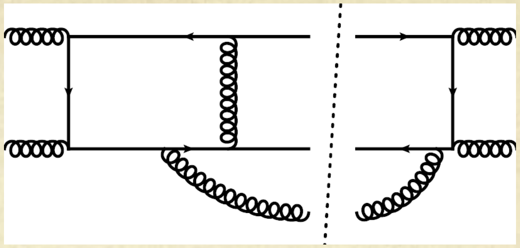
Czakon `10-11

✓ The method is general.

✓ Explicit contribution to the total cross-section given

✓ Works for massive and massless partons!

Towards NNLO: R-V



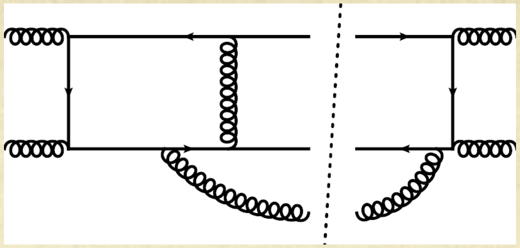
- ✓ The idea here is to get the NLO revolution to work
- ✓ All that is needed are the finite terms of the amplitude.
  No subleading terms in Epsilon needed (use counterterms - more later).
- ✓ Existing packages can handle any problem (in principle).

See also Weinzierl `11

- ✓ Speed seems to be an issue (even with OPP)
- ✓ An analytical evaluation in terms of masters might be needed.
- ✓ The poles of any 1-loop amplitude can be written analytically.
- ✓ So, in principle, all is known.

The last piece are the subtraction terms in the soft and collinear limits

Towards NNLO: R-V



- Why subtractions?

  We need to integrate over the real gluon. That generates additional divergences when the gluon is soft and/or collinear to external legs.

- Idea: devise a CT which approximates |amplitude|^2 in the soft/collinear limits:

$$|M|^2 = \underbrace{|M|^2 - CT}_{} + CT$$

- Singular phase-space integration
- Simple function

- Finite in all limits. Can be integrated numerically in 4d.
- Note: the poles of the amplitude are known. Assumed subtracted beforehand.

Counter-terms for 1-loop amplitudes

How to devise CT for any 1-loop amplitude (masses and all)?

✓ Collinear limits are easy: emissions off massive lines are finite; massless - known

Bern, Del Duca, Kilgore, Schmidt '98-99
Kosower, Uwer '98

✓ Soft limit calculated in this work

Bierenbaum, Czakon, Mitov '11

Soft limit of 1-loop massive amplitudes

Consider an (n+1) - point amplitude, with one external gluon (momentum q)

When the gluon becomes soft (q → 0) the amplitude becomes singular:

$$M_a(n+1;q) = J_a(q)M(n) + \mathcal{O}(\lambda^p) \qquad q \to \lambda q,\ \ \lambda \to 0$$

J : the soft-gluon (eikonal) current. It is process-independent!

The one-loop soft-gluon current

- So, the singular (soft) limit of 1-loop amplitudes is controlled by J:

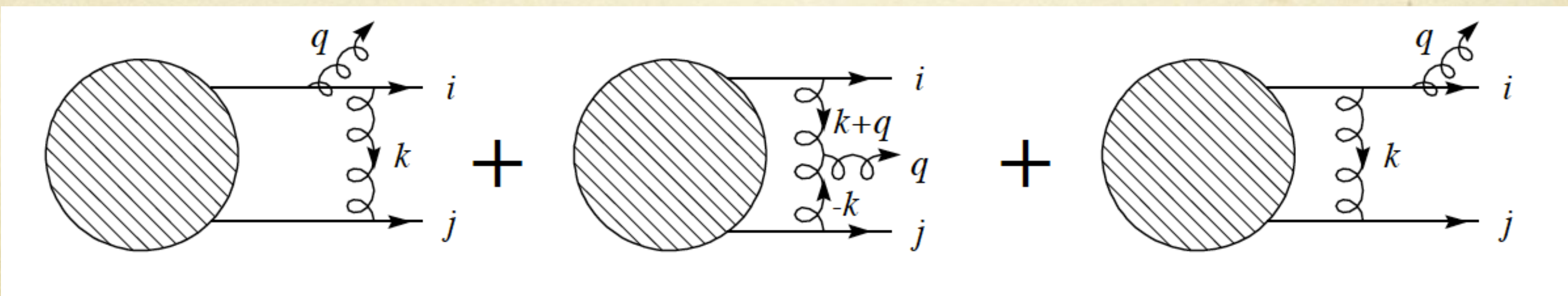
$$J_a(q) = g_S \mu^\epsilon \left(J_a^{(0)}(q) + J_a^{(1)}(q) + \mathcal{O}(\alpha_S^2) \right)$$

$$J_a^{\mu(0)}(q) = \sum_{i=1}^{n} T_i^a \frac{p_i^\mu}{p_i \cdot q} \equiv \sum_{i=1}^{n} T_i^a e_i^\mu$$

- The one-loop correction J^(1) is known in the massless case:

Bern, Del Duca, Kilgore, Schmidt '98-99
Catani, Grazzini `00

Process independent calculation of eikonal diagrams
(we follow the very nice process-independent approach of Catani and Grazzini)



The one-loop soft-gluon current

- Result expressed in terms of 3 scalar integrals (simple, complicated, harsh)

$$M_1 \equiv \Phi \int \frac{d^d k}{i(2\pi)^d} \frac{1}{[k^2][(k+q)^2][-p_j \cdot k]}$$
$$M_2 \equiv \Phi \int \frac{d^d k}{i(2\pi)^d} \frac{1}{[k^2][p_i \cdot k + p_i \cdot q][-p_j \cdot k]}$$
$$M_3 \equiv \Phi \int \frac{d^d k}{i(2\pi)^d} \frac{1}{[k^2][(k+q)^2][p_i \cdot k + p_i \cdot q][-p_j \cdot k]}$$

Note:
these are not
scaleless integrals!

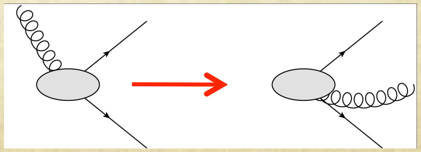
- M_1 and M_2 through Gauss hypergeometric functions.

- M_3 is very hard. Involves multiple polylogs (also appear in one-loop squared)

$$F_c(x_1, x_2) = \int_0^1 dt \frac{\ln(1-t)\ln\left(1 - t\frac{x_2}{x_1}\right)}{\frac{1}{x_2} - t}$$

- Calculate 3 kinematical configurations:
  - (i,j) massive/massless, or
  - (i,j) incoming/outgoing

- Understood the analytical continuation spacelike → timelike.



- Number of checks (small mass limit; poles; numerical checks)

The one-loop soft-gluon current

✓ Here is the result for the IM parts of the one-mass case (simplest thing to show):

$$g_{ij}^{(1)}(Case\ 1) \;=\; R_{ij}^{[C1]} + i\pi I_{ij}^{[C1]} \equiv \underbrace{\left(\frac{2(p_i\cdot p_j)\mu^2}{2(p_i\cdot q)2(p_j\cdot q)}\right)^{\epsilon}}_{\text{The massless case}} \sum_{n=-2}^{2} G_{ij}^{(n)[C1]}\epsilon^n$$

The leading q-dependence factors out in d-dimensions; the rest is homogeneous!

$$\begin{aligned}
I_{ij}^{(-2)[C1]} &= 0\,,\\
I_{ij}^{(-1)[C1]} &= -\frac{1}{2}\,,\\
R_S\, I_{ij}^{(0)[C1]} &= 2m_i^2(p_j\cdot q)\ln\left(\frac{\alpha_i}{2}\right)\,,\\
R_S\, I_{ij}^{(1)[C1]} &= 4\left[(p_i\cdot p_j)(p_i\cdot q)-m_i^2(p_j\cdot q)\right]\mathrm{Li}_2\left(1-\frac{\alpha_i}{2}\right)+m_i^2(p_j\cdot q)\ln^2\left(\frac{\alpha_i}{2}\right)\\
&\quad+\pi^2\frac{-2(p_i\cdot p_j)(p_i\cdot q)+m_i^2(p_j\cdot q)}{2}\,,\\
R_S\, I_{ij}^{(2)[C1]} &= 4\left[(p_i\cdot p_j)(p_i\cdot q)-m_i^2(p_j\cdot q)\right]\left[\mathrm{Li}_3\left(1-\frac{\alpha_i}{2}\right)+\mathrm{Li}_3\left(\frac{\alpha_i}{2}\right)\right]-\zeta_3\frac{40(p_i\cdot p_j)(p_i\cdot q)-26m_i^2(p_j\cdot q)}{3}\\
&\quad+2\left[(p_i\cdot p_j)(p_i\cdot q)-m_i^2(p_j\cdot q)\right]\ln\left(1-\frac{\alpha_i}{2}\right)\ln^2\left(\frac{\alpha_i}{2}\right)+\frac{m_i^2(p_j\cdot q)}{3}\ln^3\left(\frac{\alpha_i}{2}\right)\\
&\quad+\ln\left(\frac{\alpha_i}{2}\right)\left(\pi^2\frac{-4(p_i\cdot p_j)(p_i\cdot q)+m_i^2(p_j\cdot q)}{6}+4\left[(p_i\cdot p_j)(p_i\cdot q)-m_i^2(p_j\cdot q)\right]\mathrm{Li}_2\left(1-\frac{\alpha_i}{2}\right)\right)
\end{aligned}$$

$$R_S = 4\left[m_i^2(p_j\cdot q)-2(p_i\cdot p_j)(p_i\cdot q)\right]\;,\;\;\alpha_i=\frac{m_i^2(p_j\cdot q)}{(p_i\cdot q)(p_i\cdot p_j)}$$

Derivation of CT

The current reads: $$J_a^{\mu(1)}(q) = i f_{abc} \sum_{i\neq j=1}^{n} T_i^b T_j^c \left(e_i^\mu - e_j^\mu\right) g_{ij}^{(1)}(\epsilon, q, p_i, p_j)$$ where: $$g_{ij}^{(1)} \equiv R_{ij} + i\pi I_{ij}$$

Then, the square of the Born diagram becomes:

$$\langle M_a^{(0)}(n+1;q)|M_a^{(0)}(n+1;q)\rangle =$$
$$-4\pi\alpha_S\mu^{2\epsilon}\left\{\sum_{i\neq j=1}^{n} e_{ij}\langle M^{(0)}(n)|T_i\cdot T_j|M^{(0)}(n)\rangle \;+\; \sum_{i=1}^{n}\mathcal{C}_i e_{ii}\langle M^{(0)}(n)|M^{(0)}(n)\rangle\right\} + \mathcal{O}(\lambda^p)$$

... and the one-loop amplitude reads:

Note!

$$\langle M_a^{(0)}(n+1;q)|M_a^{(1)}(n+1;q)\rangle + c.c. = -4\pi\alpha_S\mu^{2\epsilon}\Bigg\{$$
$$2C_A\sum_{i\neq j=1}^{n}\left(e_{ij}-e_{ii}\right)R_{ij}\langle M^{(0)}(n)|T_i\cdot T_j|M^{(0)}(n)\rangle - 4\pi\sum_{i\neq j\neq k=1}^{n} e_{ik}I_{ij}\langle M^{(0)}(n)|f^{abc}T_i^aT_j^bT_k^c|M^{(0)}(n)\rangle$$
$$+\left(\sum_{i\neq j=1}^{n} e_{ij}\langle M^{(0)}(n)|T_i\cdot T_j|M^{(1)}(n)\rangle + c.c.\right) + \left(\sum_{i=1}^{n}\mathcal{C}_i e_{ii}\langle M^{(0)}(n)|M^{(1)}(n)\rangle + c.c.\right)\Bigg\} + \mathcal{O}(\lambda^p)\,, \quad (14)$$

Comment on subtractions (dipole or otherwise)

- ✓ So, CT are derived. What's next?
- ✓ Integrate over Phase Space. Most natural approach seems to be FKS.
- ✓ No details at that point, just few comments:
  - ✓ Dipoles are not enough beyond NLO (known from many places now)
  - ✓ FKS: no shift in variables; no divergent integration.
    Works similarly to sector decomposition.
  - ✓ FKS does not really care about the color structure (which is very involved);
    It is all about the kinematics.

Summary and Conclusions

Abstract QCD amplitudes

- We have derived the last remaining singular limit of massive QCD amplitudes at NNLO
- We know their collinear limit as well as the singularities and high energy limits of massive 2-loop amplitudes.

- 3-loop massless case is still an open problem (lots of partial results).

Collider applications

- Required for a numerical calculation of NNLO observables with masses (subtraction approach)

- Applications are tT – fully exclusive, including Afb.

- Applications for dijets at NNLO and even heavy flavor production in DIS at NNLO.

- Methods are fully exclusive and numeric.

- Down the road – produce partonic Monte Carlo at NNLO.

Backup Slides

Singularities of Massive Gauge Theory Amplitudes

Amplitudes: the basics

- Gauge theory amplitudes: UV renormalized, S-matrix elements
- The amplitudes are not observables:
  - UV renormalized gauge amplitudes are not finite due to IR singularities.
  - Assume they are regulated dimensionally $d=4-2\varepsilon$

What was known before (massive case):

✓ Explicit expression for the IR poles of any one-loop amplitude derived

Catani, Dittmaier, Trocsanyi '00

✓ The small mass limit is proportional to the massless amplitude

Mitov, Moch '06
Becher, Melnikov '07

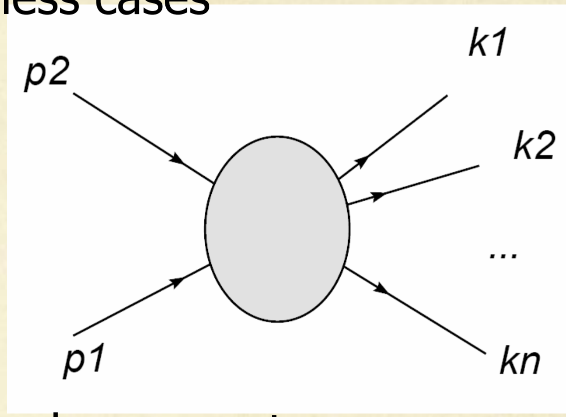
Note: predicts not just the poles but the finite parts too (for $m \to 0$)!

Factorization: "divide and conquer"

Structure of amplitudes becomes transparent thanks to factorization th.

$$M_I(\varepsilon,\mu_R,s_{ij},m_i) = J(\varepsilon,\mu_R,\mu_F,m_i)\cdot S_{IJ}(\varepsilon,\mu_R,\mu_F,s_{ij},m_i)\cdot H_J(\varepsilon,\mu_R,\mu_F,s_{ij},m_i)$$

Note: applicable to both massive and massless cases



I,J – color indexes.

J(…) – "jet" function. Absorbs all the collinear enhancement.

S(…) – "soft" function. All soft non-collinear contributions.

H(…) – "hard" function. Insensitive to IR.

Factorization: the Jet function

$$\boxed{M_I(Q,m,\epsilon) = J(m,\epsilon)\cdot S_{IJ}(Q,m,\epsilon)\cdot H_J(Q,m)}$$

For an amplitude with n-external legs, J(…) is of the form:

$$J(m,\epsilon) = \prod_{i=1}^{n} J_i(m,\epsilon)$$

i.e. we associate a jet factor to each external leg.

Some obvious properties:
- Color singlets,
- Process independent; i.e. do not depend on the hard scale Q.

J_i not unique (only up to sub-leading soft terms).

A natural scheme: J_i = square root of the space-like QCD formfactor.

Sterman and Tejeda-Yeomans '02

Scheme works in both the massless and the massive cases.

The massive form-factor's exponentiation known through 2 loops

Mitov, Moch '06

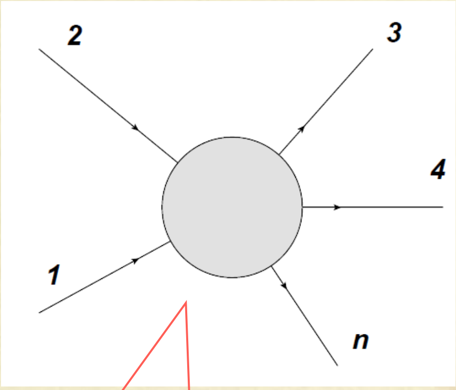
Factorization: the Soft function

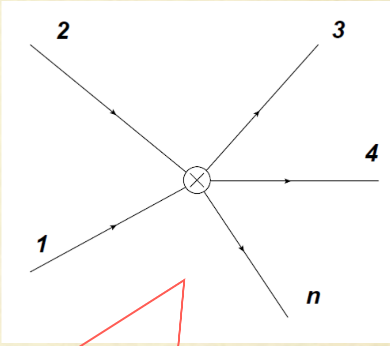
$$M_I(Q,m,\epsilon) = J(m,\epsilon)\cdot S_{IJ}(Q,m,\epsilon)\cdot H_J(Q,m)$$

Soft function is the most non-trivial element
(recall: it contains only soft poles).

But we know that the soft limit is reproduced by the eikonal approximation.

➔ Extract S(…) from the eikonalized amplitude:





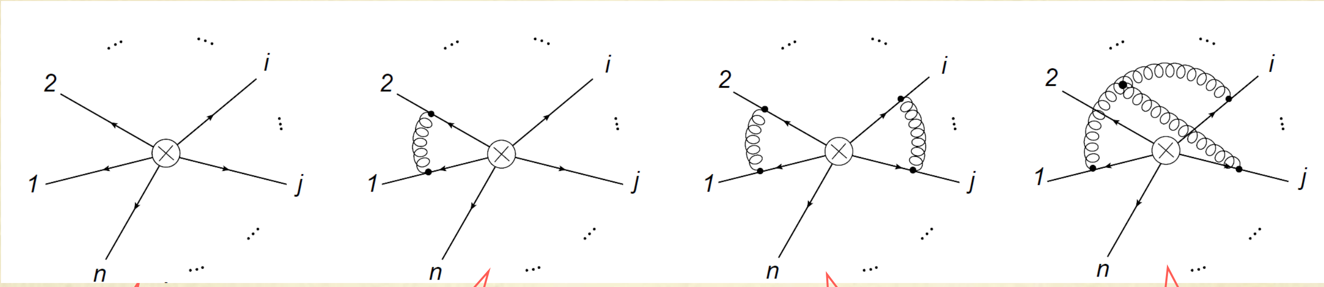
The LO amplitude M(…)

The eikonal version of the amplitude.
(the blob is replaced by an effective n-point vertex)

Factorization: the Soft function

Calculation of the eikonal amplitude:

consider all soft exchanges between the external (hard) partons



LO

1-loop

2-loops

2-loops

The fixed order expansion of the soft function takes the form:

$$
\begin{aligned}
S^{(1)}_{IJ}(\varepsilon, s_{ij}, m_i) &= \frac{1}{\varepsilon}\Gamma^{(1)}_{IJ}(s_{ij}, m_i) + O(\varepsilon^0)\,, \\
S^{(2)}_{IJ}(\varepsilon, s_{ij}, m_i) &= -\frac{\beta_0}{4\varepsilon^2}\Gamma^{(1)}_{IJ}(s_{ij}, m_i) + \frac{1}{2}\left(S^{(1)}_{IJ}(\varepsilon, s_{ij}, m_i)\right)^2 + \frac{1}{\varepsilon}\Gamma^{(2)}_{IJ}(s_{ij}, m_i) + O(\varepsilon^0)\,.
\end{aligned}
$$

… as follows from the usual RG equation:

$$
\left(\mu\frac{\partial}{\partial\mu} + \beta(g,\varepsilon)\frac{\partial}{\partial g}\right) S_{IJ}(\varepsilon, s_{ij}, m_i) = -\Gamma_{IK}(\varepsilon, s_{ij}, m_i)\, S_{KJ}(\varepsilon, s_{ij}, m_i)
$$

➔ All information about S(…) is contained in the anomal's dimension matrix Γ_{IJ}

the Soft function at 1 loop

Here is the result for the anomalous dim. matrix at one loop

$$\underbrace{\mathbf{\Gamma}_S^{(1)} = \frac{1}{2}\sum_{(i\neq j)=1}^{n} T_i\cdot T_j \ln\left(-\frac{\mu^2}{\sigma_{ij}}\right)}_{\text{The massless case}} + \underbrace{\frac{1}{2}\sum_{(i\neq j)\in\mathcal{N}_m} T_i\cdot T_j\left[\ln\left(1+x_{ij}^2\right)+\frac{2x_{ij}^2}{1-x_{ij}^2}\ln(x_{ij})\right]}_{\text{O(m) corrections in the massive case}}$$

where:
- all masses are taken equal,
- written for space-like kinematics (everything is real).

$$\frac{m^2}{s_{ij}} = -\frac{x_{ij}}{(1-x_{ij})^2} \quad , \quad x_{ij} = \frac{\sqrt{1-\frac{4m^2}{s_{ij}}}-1}{\sqrt{1-\frac{4m^2}{s_{ij}}}+1}$$

$$s_{ij} = (p_i+p_j)^2 \text{ and } \sigma_{ij} = 2p_i\cdot p_j = s_{ij}-m_i^2-m_j^2$$

The Soft function at 2 loops

The simplest approach is the following. Start with the Ansatz:

$$\Gamma_S^{(2)} = \underbrace{\frac{1}{2} \sum_{(i \neq j)=1}^{n} T_i \cdot T_j \, \frac{K}{2} \, \ln\left(-\frac{\mu^2}{\sigma_{ij}}\right)}_{\text{Reproduces the massless case}} + \underbrace{\frac{1}{2} \sum_{(i \neq j) \in \mathcal{N}_m} T_i \cdot T_j \, P_{ij}^{(2)}}_{\text{Parametrizes the O(m) corrections to the massless case}} + 3E \text{ terms}$$

Then note: the function $P^{(2)}_{ij}$ depends on (i,j) only through s_{ij}

$\boxed{\rightarrow P^{(2)}_{ij} = P^{(2)}(s_{ij})}$

This single function can be extracted from the known n=2 amplitude: the massive two-loop QCD formfactor.

Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi `04
Gluza, Mitov, Moch, Riemann '09

The Soft function at 2 loops

The complete result for the 2E reads: $$P^{(2)} = \frac{K}{2} P^{(1)} + P^{(2),\mathrm{m}}$$

$$\begin{aligned} P^{(2),\mathrm{m}}(x) \;=\; & \frac{C_A}{(1-x^2)^2} \left\{ -\frac{(1+x^2)^2}{2} \mathrm{Li}_3(x^2) + \left(\frac{(1+x^2)^2}{2} \ln(x) - \frac{1-x^4}{2} \right) \mathrm{Li}_2(x^2) \right. \\ & + \frac{x^2(1+x^2)}{3} \ln^3(x) + x^2(1-x^2) \ln^2(x) \\ & \left. + \left(-(1-x^4) \ln\left(1-x^2\right) + x^2(1+x^2)\zeta_2 \right) \ln(x) + x^2(1-x^2)\zeta_2 + 2x^2\zeta_3 \right\}, \end{aligned}$$

This term breaks the simple relation $\Gamma^{(2)}_{S_{\mathrm{f}}} = \frac{K}{2} \Gamma^{(1)}_{S_{\mathrm{f}}}$ from the massless case!

Aybot, Dixon, Sterman '06

Above result derived by 3 different groups:

Kidonakis '09
Becher, Neubert '09
Czakon, Mitov, Sterman '09

Kidonakis derived the massive eikonal formfactor;
Becher, Neubert used old results of Korchemsky, Radushkin

The Soft function at 2 loops

What about the 3E contributions in the massive case?

Until recently there existed no indication if they were non-zero!

In particular, the following squared two-loop amplitudes are insensitive to it:

Czakon, Mitov, Sterman '09

Known numerically

$$\langle M^{(2)}|M^{(0)}\rangle(q\bar{q} \to Q\overline{Q})$$

Czakon '07

Poles reported

$$\langle M^{(2)}|M^{(0)}\rangle(gg \to Q\overline{Q})$$

Czakon, Bärnreuther '09

3E correlators not vanish if at least two legs are massive – direct position-space calculation for Euclidean momenta (numerical results)

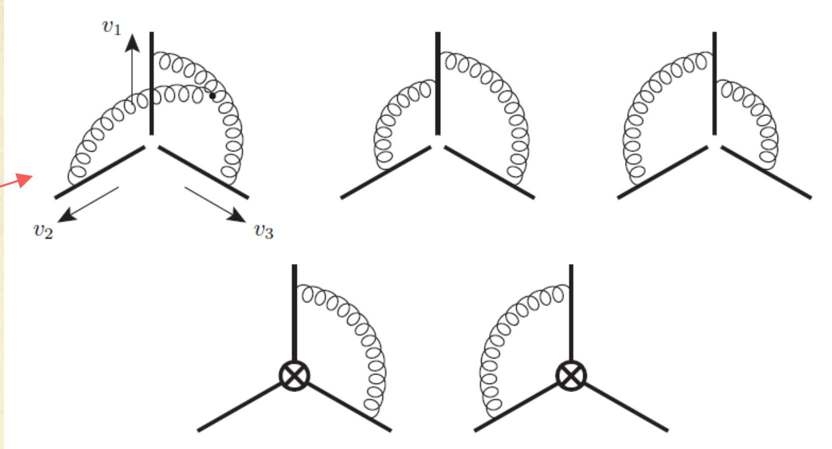
Mitov, Sterman, Sung '09

Exact result computed analytically

Ferroglia, Neubert, Pecjak, Yang '09

The Soft function at 2 loops. Massive case.

The types of contributing diagrams:



The analytical result is very simple:

Ferroglia, Neubert, Pecjak, Yang '09

$$F^{(3g)} \sim \sum_{ijk} \epsilon^{ijk} \ \ln^2(x_{ij})\, r(x_{ik})$$

where:

$$r(x) = -\frac{1+x^2}{1-x^2} \ln(x)$$

The calculation of the double exchange diagrams is very transparent.
Agrees in both momentum and position spaces

A.M., Sterman, Sung '10

Massive gauge amplitudes: Summary

- The results I presented can be used to predict the poles of any massive 2-loop amplitude with:
  - *n* external colored particles (plus arbitrary number of colorless ones),
  - arbitrary values of the masses (usefull for SUSY).

- Results checked in the 2-loop amplitudes:

$$\langle M^{(2)}|M^{(0)}\rangle(q\bar{q}\to Q\overline{Q})$$
$$\langle M^{(2)}|M^{(0)}\rangle(gg\to Q\overline{Q})$$

- Needed in jet subtractions with massive particles at 2-loops

- Input for NNLL resummation (next slides)