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Colour-friendly FKS subtraction

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- Colour algebra is a serious complication in tree-level calculations. For large multiplicities, exact sums are not an option
- At the NLO, it will be (much) worse. Cancellations of IR singularities require colour sums be 100% correlated among the various contributions
- Thanks to advances in automation, this is not an academic problem

FKS and dipole work with colour-summed matrix elements. Antenna deals with colour-ordered amplitudes, and is more colour friendly

Re-write FKS by defining subtraction terms that have a straightforward interpretation in terms of colours

FKS #1: Simplify the problem

Find parton pairs (i, j) that can give collinear singularities. Then:

$$\mathcal{M} = \sum_{ij} \mathcal{M}_{ij} \qquad \mathcal{M}_{ij} = \mathcal{S}_{ij} \mathcal{M}$$

with

$$\sum_{ij} \mathcal{S}_{ij} = 1$$

$$\sum_{j} \mathcal{S}_{ij} \longrightarrow 1 \qquad k_i \longrightarrow 0$$

$$\mathcal{S}_{ij} \longrightarrow 1 \qquad k_i \parallel k_j$$

$$\mathcal{S}_{ij} \longrightarrow 0 \qquad \text{all other singularities}$$

- $\mathcal{M}_{ij}$ has one soft and one collinear singularity at most
- Soft singularities are "split" into underlying collinear structures
- The $\mathcal{M}_{ij}$'s are independent from each other

FKS #2: Subtract

For a given $\mathcal{M}_{ij}$, the choice of variables associated with singularities is natural and essentially unique

$$\mathcal{M}_{ij} d\phi_{n+1} \longrightarrow \left(\frac{1}{E_i}\right)_+ \left(\frac{1}{1-\cos\theta_{ij}}\right)_+ E_i^2 \left(1-\cos\theta_{ij}\right) \mathcal{M}_{ij} \frac{d\phi_{n+1}}{E_i}$$

- Plus distributions understand the projections $\mathcal{P}\phi_{n+1}$. FKS defines subtraction terms *exactly* on shell, and thus eliminates the problem of recoil altogether (NO approximation is required)
- Soft and collinear counterterms can be defined so as they have the *same* kinematics $\Longrightarrow$ the subtraction term is unique

By-product: important sampling (thanks to E_i and θ_{ij}) is straightforward

FKS #3: Exploit symmetries

Collinear structure and definition of projections imply that the total number of subtraction terms scales as n^2. However for any observable O

$$O(\mathcal{M}_{ij}) = O(\mathcal{M}_{kl})$$

when flavour$_i$=flavour$_k$ and flavour$_j$=flavour$_l$, the key being the independence of $\mathcal{M}_{ij}$ and $\mathcal{M}_{kl}^{(r)}$

$\Longrightarrow$ The majority of contributions can be taken into account simply with an overall symmetry factor

Thus: for a $2 \to n$ gluon process, the total number of subtractions is 3 ($\forall n$). The presence of quarks complicates the counting, but the subtractions are never more than a few

What to take home

Pick one contribution (i.e., a (i,j) pair) at a time. i will be the soft parton, i will be collinear to j, and that's it. If one can solve this, the whole problem is solved

In order to simplify the discussion:

- Gluons only
- $0 \to n$ (Born) and $0 \to n+1$ (real) processes
- $i = n+1$, $j = n$ (thanks to invariance under relabeling)

Master formula:

$$\mathcal{M} - \mathcal{M}_{\text{SOFT}} - \mathcal{M}_{\text{COLL}} + \mathcal{M}_{\text{SC}}$$

Amplitudes

Use e.g. Mangano and Parke representation

$$\mathcal{A}^{(n)}(a_1,\dots a_n) = \sum_{\sigma\in P'_n} \mathrm{Tr}\Big(\lambda^{a_{\sigma(1)}}\dots\lambda^{a_{\sigma(n)}}\Big)\widehat{\mathcal{A}}^{(n)}(\sigma(1),\dots\sigma(n))$$

$$\equiv \sum_{\sigma\in P'_n} \Lambda\left(\{a_i\},\sigma\right)\widehat{\mathcal{A}}^{(n)}(\sigma)$$

Colour configuration:

$$\{a_i\}_{i=1}^n\ , \qquad a_i \in \left\{1,\dots N^2-1\right\}$$

Colour flow:

$$(\sigma(1),\dots\sigma(n))\ , \qquad \sigma(1)=1$$

Use orthonormal colour basis:

$$|a_1,\dots a_n\rangle\ , \qquad \langle b_1,\dots b_n|a_1,\dots a_n\rangle = \prod_{i=1}^n \delta_{a_i b_i}$$

One can thus define amplitudes as vectors in colour space

$$|\mathcal{A}^{(n)}(a_1,\dots a_n)\rangle = \mathcal{A}^{(n)}(a_1,\dots a_n)|a_1,\dots a_n\rangle$$
$$|\mathcal{A}^{(n)}(\sigma)\rangle = \sum_{\{a_i\}_{i=1}^{n}} \Lambda\left(\{a_i\},\sigma\right)\widehat{\mathcal{A}}^{(n)}\left(\sigma\right)|a_1,\dots a_n\rangle$$

"Physical" amplitude:

$$|\mathcal{A}^{(n)}\rangle = \sum_{\{a_i\}_{i=1}^{n}} |\mathcal{A}^{(n)}(a_1,\dots a_n)\rangle$$
$$|\mathcal{A}^{(n)}\rangle = \sum_{\sigma\in P_n'} |\mathcal{A}^{(n)}(\sigma)\rangle$$

The amplitude squared is what is relevant to cross sections

$$\mathcal{M}^{(n)} = \langle\mathcal{A}^{(n)}|\mathcal{A}^{(n)}\rangle$$

This is the colour-summed quantity we use to work with

Fixed colour configurations

$$\mathcal{M}^{(n)} = \sum_{\{a_i\}_{i=1}^{n}} \mathcal{M}^{(n)}(a_1, \ldots a_n)$$

with:

$$\mathcal{M}^{(n)}(a_1, \ldots a_n) = \langle \mathcal{A}^{(n)}(a_1, \ldots a_n) | \mathcal{A}^{(n)}(a_1, \ldots a_n) \rangle$$
$$\equiv \left| \mathcal{A}^{(n)}(a_1, \ldots a_n) \right|^2$$

Colour configurations are orthogonal, hence the same on the left and on the right of the cut

The absence of interference is what makes them appealing at the LO

- Drawback: no straightforward way to do $1/N$ expansion and to work out colour connections to give to MCs

Fixed colour flows

$$\mathcal{M}^{(n)}(\sigma',\sigma) = \langle \mathcal{A}^{(n)}(\sigma') | \mathcal{A}^{(n)}(\sigma) \rangle$$

With scalar quantities:

$$\mathcal{M}^{(n)}(\sigma',\sigma) = \widehat{\mathcal{A}}^{(n)}(\sigma')^{\star}\, C(\sigma',\sigma)\, \widehat{\mathcal{A}}^{(n)}(\sigma)$$

Colour-flow matrix (real and symmetric):

$$C(\sigma',\sigma) = \sum_{\{a_i\}_{i=1}^{n}} \Lambda\left(\{a_i\},\sigma'\right)^{\star} \Lambda\left(\{a_i\},\sigma\right)$$

- Obvious, MC-friendly, physical interpretation
- Directly related to $1/N$ expansion. The leading power of N in each $C(\sigma',\sigma)$ does not require any calculation, but only the knowledge of the *closed flow* (σ',σ)
- Orthogonality only at leading colour

The problem: find local subtraction counterterms for

$$\mathcal{M}^{(n+1)}(a_1, \ldots a_n) \qquad\qquad \mathcal{M}^{(n)}(\sigma', \sigma)$$

This can be done by starting from the colour-summed counterterms

Start by writing them down as done in standard FKS

Soft, colour-summed

$$\mathcal{M}_{\mathrm{SOFT}}^{(n+1)} = \frac{1}{2} g_S^2 \sum_{k,l=1}^{n} \Big[k,l\Big] \mathcal{M}_{kl}^{(n)}$$

with:

$$\begin{aligned} \Big[k,l\Big] &= \frac{k_k \cdot k_l}{k_k \cdot k_{n+1}\, k_l \cdot k_{n+1}} (1 - \delta_{kl}) \\ \mathcal{M}_{kl}^{(n)} &= -2 \langle \mathcal{A}^{(n)} | \sum_b Q^b(k) Q^b(l) \, | \mathcal{A}^{(n)} \rangle \end{aligned}$$

Colour operators:

$$\begin{aligned} \langle a_k a_{n+1} | Q^b(k) | c_k \rangle &= \delta_{b a_{n+1}} \Big(Q^b(k) \Big)_{a_k c_k} = \delta_{b a_{n+1}} \Big(T^b \Big)_{a_k c_k} \\ \Big(T^b \Big)_{ac} &= -i f^{bac} \\ \sum_b Q^b(k) Q^b(k) &= C_A \, I \end{aligned}$$

Collinear, colour-summed

$$\mathcal{M}^{(n+1)}_{\text{COLL}} = \frac{g_S^2}{k_n \cdot k_{n+1}} \left\{ P_{gg}(z) \mathcal{M}^{(n)} + Q_{gg^\star}(z) \, \Re\left(\frac{\langle k_n k_{n+1} \rangle}{[k_n k_{n+1}]} \mathcal{M}^{(n)}_{-+} \right) \right\}$$

The azimuthal correlation term features a *universal* kernel Q, and the reduced matrix element:

$$\mathcal{M}^{(n)}_{-+} = \langle \mathcal{A}^{(n)}_{-} | \mathcal{A}^{(n)}_{+} \rangle$$

Consistency between soft and collinear counterterms is a direct consequence of colour conservation:

$$\sum_{k=1}^{n} Q^b(k) |\mathcal{A}^{(n)}\rangle = 0 \qquad \Longrightarrow \qquad \sum_{k=1}^{n-1} Q^b(k) |\mathcal{A}^{(n)}\rangle = -Q^b(n) |\mathcal{A}^{(n)}\rangle$$

It can be proved by only knowing the MP representation

Interlude: azimuthal kernels

$$Q_{gg^\star}(z) = -4C_A\, z(1-z) \qquad Q_{g^\star g}(z) = -4C_A\, \frac{1-z}{z}$$

$$Q_{qg^\star}(z) = 4T_F\, z(1-z) \qquad Q_{q^\star g}(z) = 0$$

$$Q_{gq^\star}(z) = 0 \qquad Q_{g^\star q}(z) = -4C_F\, \frac{1-z}{z}$$

$$Q_{qq^\star}(z) = 0 \qquad Q_{q^\star q}(z) = 0$$

Same kind of universality as Altarelli-Parisi kernels

Timelike and spacelike branchings kernels differ already at the LO
(the off-shell parton is denoted by a star)

Soft, fixed colour configurations

$$\mathcal{M}^{(n+1)}_{\text{SOFT}}(a_1, \dots a_{n+1}) = \frac{1}{2} g_S^2 \sum_{k,l=1}^{n} \Big[k, l \Big] \mathcal{M}^{(n)}_{kl}(a_1, \dots a_{n+1})$$

with:

$$\mathcal{M}^{(n)}_{kl}(a_1, \dots a_{n+1}) = -2 \sum_{b'_k b_l} \mathcal{A}^{(n)}(a_1, \dots b'_k \dots a_l \dots a_n)^\star$$
$$\times \mathcal{Q}_{gg}(a_{n+1}; a_k, a_l)_{b'_k b_l} \mathcal{A}^{(n)}(a_1, \dots a_k \dots b_l \dots a_n)$$

$$\mathcal{Q}_{gg}(a_{n+1}; a_k, a_l)_{bc} = \left(T^{a_{n+1}}\right)_{b a_k} \left(T^{a_{n+1}}\right)_{a_l c} = f^{a_{n+1} a_k b} f^{a_{n+1} a_l c}$$

- There are 512 $\mathcal{Q}_{gg}$ matrices, 8×8
- Lots of zeros, but for each real colour configurations one will need to compute at least a few Born-level ones

Collinear, fixed colour configurations

$$\mathcal{M}^{(n+1)}_{\rm COLL}(a_1,\ldots a_{n+1}) \;=\; \frac{g_S^2}{k_n\cdot k_{n+1}}\Bigg\{\hat{P}_{gg}(z)\overline{\mathcal{M}}^{(n)}(a_1,\ldots a_{n+1}) + \hat{Q}_{gg^\star}(z)\,\Re\left(\frac{\langle k_n k_{n+1}\rangle}{[k_n k_{n+1}]}\overline{\mathcal{M}}^{(n)}_{-+}(a_1,\ldots a_{n+1})\right)\Bigg\}$$

with $\hat{P} = P/C_A$, $\hat{Q} = Q/C_A$, and:

$$\overline{\mathcal{M}}^{(n)}(a_1,\ldots a_{n+1}) = \sum_{b'_n b_n}\mathcal{A}^{(n)}(a_1,\ldots a_{n-1},b'_n)^\star\mathcal{Q}_{gg}(a_{n+1};a_n,a_n)_{b'_n b_n}\mathcal{A}^{(n)}(a_1,\ldots a_{n-1},b_n)$$

$$\overline{\mathcal{M}}^{(n)}_{-+}(a_1,\ldots a_{n+1}) = \sum_{b'_n b_n}\mathcal{A}^{(n)}_{-}(a_1,\ldots a_{n-1},b'_n)^\star\mathcal{Q}_{gg}(a_{n+1};a_n,a_n)_{b'_n b_n}\mathcal{A}^{(n)}_{+}(a_1,\ldots a_{n-1},b_n)$$

When considering the subtraction at fixed flows, one realizes that there are two possible ways of doing it: either by fixing flows at the real-emission level, or at the Born level

Within one given FKS sector, it is easy to find a map between the two, since one knows which parton is soft. For example:

$$(\sigma(1), \ldots \sigma(j), \sigma(j+1), \ldots \sigma(n)) \qquad \text{Born}$$

$$(\sigma(1), \ldots \sigma(j), n+1, \sigma(j+1), \ldots \sigma(n)) \equiv$$
$$(\Sigma(1), \ldots \Sigma(j), \Sigma(j+1), \Sigma(j+2), \ldots \Sigma(n+1)) \qquad \text{real}$$

Still, the counterterms in the two formulations are different

I'll present only the fixed real flow case

Soft, fixed real flows

$$
\begin{aligned}
\mathcal{M}^{(n+1)}_{\text{SOFT}}(\Sigma', \Sigma) &= -g_S^2 \Big\{ \Big[\Sigma'({\Sigma'}^{-1}(n+1)-1), \Sigma(\Sigma^{-1}(n+1)-1) \Big] \\
&\quad - \Big[\Sigma'({\Sigma'}^{-1}(n+1)-1), \Sigma(\Sigma^{-1}(n+1)+1) \Big] \\
&\quad - \Big[\Sigma'({\Sigma'}^{-1}(n+1)+1), \Sigma(\Sigma^{-1}(n+1)-1) \Big] \\
&\quad + \Big[\Sigma'({\Sigma'}^{-1}(n+1)+1), \Sigma(\Sigma^{-1}(n+1)+1) \Big] \Big\} \mathcal{M}^{(n)}_{\text{EXT}}(\Sigma', \Sigma) \\
\mathcal{M}^{(n)}_{\text{EXT}}(\Sigma', \Sigma) &= \widehat{\mathcal{A}}^{(n)}(\Sigma'_{\cancel{n+1}})^{\star}\, C(\Sigma', \Sigma)\, \widehat{\mathcal{A}}^{(n)}(\Sigma_{\cancel{n+1}})
\end{aligned}
$$

- The colour-linked Borns have disappeared!
- "Replaced" by a linear combination of eikonals, computed with the momenta of the first neighbours of the soft parton, at the left and right of the cut (which need not coincide)

Collinear, fixed real flows

$$\mathcal{M}^{(n+1)}_{\text{COLL}}(\Sigma',\Sigma) = \frac{g_S^2}{k_n\cdot k_{n+1}}\,\delta(\Sigma',\Sigma)\Bigg\{\hat{P}_{gg}(z)\mathcal{M}^{(n)}_{\text{EXT}}(\Sigma',\Sigma)$$

$$+\frac{1}{2}\hat{Q}_{gg^\star}(z)\left(\frac{\langle k_n k_{n+1}\rangle}{[k_n k_{n+1}]}\mathcal{M}^{(n)}_{\text{EXT}-+}(\Sigma',\Sigma)+\frac{[k_n k_{n+1}]}{\langle k_n k_{n+1}\rangle}\mathcal{M}^{(n)}_{\text{EXT}+-}(\Sigma',\Sigma)\right)\Bigg\},$$

with:

$$\delta(\Sigma',\Sigma) = \sum_{\alpha=-1,1}\sum_{\beta=-1,1}\alpha\beta\,\delta\Big({\Sigma'}^{-1}(n),{\Sigma'}^{-1}(n+1)+\alpha\Big)\,\delta\Big(\Sigma^{-1}(n),\Sigma^{-1}(n+1)+\beta\Big)$$

$$\mathcal{M}^{(n)}_{\text{EXT}\lambda\bar{\lambda}}(\Sigma',\Sigma) = \widehat{\mathcal{A}}^{(n)}_{\lambda}(\Sigma'_{\not{n+1}})^\star\,C(\Sigma',\Sigma)\,\widehat{\mathcal{A}}^{(n)}_{\bar{\lambda}}(\Sigma_{\not{n+1}})$$

- The collinear partons must be next to each other on *both* sides of the cut
- Same reduced matrix elements as for the soft limit (up to spin)

Subtraction at fixed real flows

- The colour is completely factorized in $C(\Sigma', \Sigma)$.
  Hence, subtraction is colourless
- Can be used to perform an expansion in $1/N$
- Kernel-wise, there are obvious similarities with antenna subtraction
  (the kinematics, and in particular the phase space, is very different)
- The same techniques could be used to construct colour-free dipoles
  (useful only with a unique recoil kinematics)

Conclusions

- FKS subtraction has been reorganized in order to "commute" with colours
- A computation can be done by fixing colour configurations, Born level colour flows, or real-emission level colour flows
- It appears that fixing colour configurations is much less convenient at the NLO than at the LO
- My bet is that fixed real flows are the way to go for processes of intermediate complexity (say, less than 10 partons), especially when matched to parton showers