Model building using Lie-point symmetries

Damien P. George

Nikhef theory group
Amsterdam, The Netherlands

CERN LPCC Summer Institute — 18^{th} August 2011

(arXiv:1105.4604)

The Lie point symmetry method

Want to systematically find all the symmetries of a model,

$\rightarrow$ even if symmetry is spontaneously broken,

$\rightarrow$ also derive parameter relationships that give enhanced symmetries.

The Lie point symmetry method consists of finding the determining equations, whose solutions describe infinitesimal symmetries, and then solving these equations.

Point: transformations depend only on coords and fields, not on derivatives of fields.

Overview:

- The determining equations.
- Example with 2 scalars.
- Automation.
- N interacting scalars.
- Spin-1 plus N scalars.
- Spontaneous symmetry breaking.
- The standard model.

Variation of the action

Infinitesimal Lie point symmetries:

$$x^\mu \to x^\mu + \eta^\mu(x,\phi) \qquad \phi_i \to \phi_i + \chi_i(x,\phi)$$

$S \to S + \delta S$ should be unchanged.

Solve for the fields → Euler-Lagrange equations: $\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = 0.$

Form a divergence → Noether's theorem: $\partial_\mu \left[\mathcal{L}\eta^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} (\chi_i - \eta^\nu \partial_\nu \phi_i) \right] = 0.$

Solve for the infinitesimals → master determining equation:

$$\boxed{\mathcal{L}\frac{\mathrm{d}\eta^\mu}{\mathrm{d}x^\mu} + \frac{\partial \mathcal{L}}{\partial x^\mu}\eta^\mu + \frac{\partial \mathcal{L}}{\partial \phi_i}\chi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \left(\frac{\mathrm{d}\chi_i}{\mathrm{d}x^\mu} - \frac{\partial \phi_i}{\partial x^\nu}\frac{\mathrm{d}\eta^\nu}{\mathrm{d}x^\mu} \right) = 0}$$

Total derivative: $\frac{\mathrm{d}}{\mathrm{d}x^\mu} \equiv \frac{\partial}{\partial x^\mu} + \frac{\partial \phi_i}{\partial x^\mu}\frac{\partial}{\partial \phi_i}.$

Example: two scalars

Only field symmetries, $\phi_i \to \phi_i + \chi_i(\phi_i)$.
Master determining equation:

$$\frac{\partial \mathcal{L}}{\partial \phi_i}\chi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\frac{\partial \phi_j}{\partial x^\mu}\frac{\partial \chi_i}{\partial \phi_j} = 0\,.$$

Apply to Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi_1\partial_\mu\phi_1 + \frac{1}{2}\partial^\mu\phi_2\partial_\mu\phi_2 - \frac{1}{2}m_1^2\phi_1^2 - \frac{1}{2}m_2^2\phi_2^2\,.$$

Determining equation is

$$\begin{aligned} &- m_1^2\phi_1\chi_1 - m_2^2\phi_2\chi_2 + \partial^\mu\phi_1\partial_\mu\phi_1\frac{\partial\chi_1}{\partial\phi_1} \\ &\quad + \partial^\mu\phi_1\partial_\mu\phi_2\frac{\partial\chi_1}{\partial\phi_2} + \partial^\mu\phi_2\partial_\mu\phi_1\frac{\partial\chi_2}{\partial\phi_1} + \partial^\mu\phi_2\partial_\mu\phi_2\frac{\partial\chi_2}{\partial\phi_2} = 0\,. \end{aligned}$$

Equate independent terms to zero:

$$-m_1^2\phi_1\chi_1 - m_2^2\phi_2\chi_2 = 0\,,\quad \frac{\partial\chi_1}{\partial\phi_1} = 0\,,\quad \frac{\partial\chi_1}{\partial\phi_2} + \frac{\partial\chi_2}{\partial\phi_1} = 0\,,\quad \frac{\partial\chi_2}{\partial\phi_2} = 0\,.$$

Example: two scalars

Determining equations:

$$-m_1^2\phi_1\chi_1 - m_2^2\phi_2\chi_2 = 0\,, \quad \frac{\partial\chi_1}{\partial\phi_1} = 0\,, \quad \frac{\partial\chi_1}{\partial\phi_2} + \frac{\partial\chi_2}{\partial\phi_1} = 0\,, \quad \frac{\partial\chi_2}{\partial\phi_2} = 0\,.$$

General solution to last three equations:

$$\chi_1(\phi_2) = \alpha_1 + \beta\phi_2\,, \qquad \chi_2(\phi_1) = \alpha_2 - \beta\phi_1\,.$$

Symmetries:

- α_1: shift of ϕ_1.
- α_2: shift of ϕ_2.
- β: rotation between ϕ_1 and ϕ_2.

Final determining equation is

$$\alpha_1 m_1^2\phi_1 + \alpha_2 m_2^2\phi_2 + \beta(m_1^2 - m_2^2)\phi_1\phi_2 = 0\,.$$

$\rightarrow$ *the model parameters dictate the symmetries.*

Automation of LPS method

Two (massive) scalars have algebraic determining equation

$$\alpha_1 m_1^2 \phi_1 + \alpha_2 m_2^2 \phi_2 + \beta(m_1^2 - m_2^2)\phi_1\phi_2 = 0\,.$$

Gaussian elimination (with branching) to find null space of

$$\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_1^2 - m_2^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta \end{pmatrix} = 0\,.$$

Differential equations $\rightarrow$ generalised Gaussian elimination.

Define ordering on η^μ and χ_i. Sort terms. Arrange as rows.

Perform "row reduction" to "diagonal" form.

$$c_1(\lambda_i)\,\partial_i f + X_1(f) = 0\,,$$
$$c_2(\lambda_i)\,\partial_{i+j} f + X_2(f) = 0\,.$$

- $c_1(\lambda_i) = 0$: remove $\partial_i f$ term.
- $c_1(\lambda_i) \neq 0$: use $\partial_i f$ to eliminate $\partial_{i+j} f$.

N interacting scalar fields

Symmetries dictated by structure of interactions between fields.

General Lagrangian for N spin-0 fields

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu \phi_i \partial_\mu \phi_i - V(\phi)\,.$$

Determining equations

$$V\partial_\mu \eta^\mu + \frac{\partial V}{\partial \phi_i}\chi_i = 0\,,$$

$$\partial^\mu \chi_i - V\frac{\partial \eta^\mu}{\partial \phi_i} = 0 \qquad \forall\mu\ \forall i\,, \qquad (\chi = \chi(\phi))$$

$$\partial^\mu \eta^\nu + \partial^\nu \eta^\mu = 0 \qquad \forall\mu\ \forall\nu,\ \mu \neq \nu\,, \qquad \text{(Poincaré)}$$

$$\frac{\partial \chi_i}{\partial \phi_j} + \frac{\partial \chi_j}{\partial \phi_i} = 0 \qquad \forall i\ \forall j,\ i \neq j\,, \qquad \text{(shift, rot.)}$$

$$\frac{1}{2}\partial_\sigma \eta^\sigma - \partial_{\bar{\mu}}\eta^{\bar{\mu}} + \frac{\partial \chi_{\bar{i}}}{\partial \phi_{\bar{i}}} = 0 \qquad \forall\bar{\mu}\ \forall\bar{i}\,, \qquad \text{(scaling)}$$

$$\frac{\partial \eta^\mu}{\partial \phi_i} = 0 \qquad \forall\mu\ \forall i\,. \qquad (\eta = \eta(x))$$

N interacting scalar fields

General Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial^{\mu}\phi_i\partial_{\mu}\phi_i - V(\phi)\,.$$

For $D \neq 2$ the general coordinate symmetries are ($b^{\mu\nu}$ anti-symm)

$$\eta^{\mu}(x) = a^{\mu} + b^{\mu}{}_{\nu}x^{\nu} + c\,x^{\mu}\,.$$

General field symmetries are (β_{ij} anti-symm)

$$\chi_i(\phi) = \alpha_i + \beta_{ij}\phi_j + \frac{2-D}{2}c\,\phi_i\,.$$

Remaining determining equation is

$$DcV + \frac{\partial V}{\partial \phi_i}\left(\alpha_i + \beta_{ij}\phi_j + \frac{2-D}{2}c\,\phi_i\right) = 0\,.$$

Form of V $\leftrightarrow$ allowed symmetries.

Spin-1 plus N scalars

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi_i\partial_\mu\phi_i - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J_i A^\mu\partial_\mu\phi_i + K_{ij}A^\mu\phi_i\partial_\mu\phi_j - V(\phi, A^2)$$

General solution for infinitesimals:

$$\eta^\mu(x) = a^\mu + b^\mu{}_\nu x^\nu + c\,x^\mu + 2d_\nu x^\nu x^\mu - d^\mu x^\nu x_\nu$$
$$\chi_i(x,\phi) = \alpha_i(x) + \beta_{ij}(x)\phi_j + (2-D)(\tfrac{1}{2}c + d_\nu x^\nu)\phi_i$$
$$\xi^\mu(x,A) = \partial^\mu\Lambda(x) + (b^\mu{}_\nu + 2d_\nu x^\mu - 2d^\mu x_\nu)A^\nu + (2-D)(\tfrac{1}{2}c + d_\nu x^\nu)A^\mu$$

E.g. massive $\mathrm{U}(1)$: when solving rest of determining equations, demand:

- gauge symmetry: $\Lambda(x)$ is arbitrary,
- massive vector: $\frac{\partial V}{\partial A^\mu} = m^2 A_\mu + \ldots$.

$\rightarrow$ derive allowed form of $\mathcal{L}$ and relations between parameters.

1 field: Stückelberg ($J = m$), 2 fields: Higgs.

Spontaneously broken symmetries

Spontaneously broken scale symmetry:

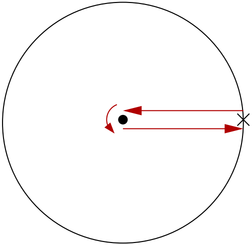
$V = \lambda\phi^4$ has scale symmetry.

$V = \lambda(\phi + v)^4$ has shift-scale-shift symmetry.

$V = \lambda(\phi_1^2 + \phi_2^2 - v^2)^2$ has $\mathrm{U}(1)$.

Define $\phi_2 = v + \varphi$.

$V = \lambda(\phi_1^2 + \varphi^2 + 2v\varphi)^2$ has shift-$\mathrm{U}(1)$-shift.



LPS method will find symmetry, no matter how broken/hidden it may be.

For example, solve for relationships between c_i in

$$V = c_1 + c_2\phi_1 + c_3\phi_2 + c_4\phi_1^2 + c_5\phi_1\phi_2 + c_6\phi_2^2 + c_7\phi_1^3 + c_8\phi_1^2\phi_2 + c_9\phi_1\phi_2^2 + c_{10}\phi_2^3 + c_{11}\phi_1^4 + c_{12}\phi_1^3\phi_2 + c_{13}\phi_1^2\phi_2^2 + c_{14}\phi_1\phi_2^3 + c_{15}\phi_2^4 \,.$$

The standard model

Schematic structure of the standard model:

$$\mathcal{L}_{\mathsf{SM}} \sim (\partial\phi)^2 + \phi^2\partial\phi + \phi^2 + \phi^4 + \psi\partial\psi + \phi\psi^2 \,.$$

- $N = 244$ real degrees of freedom (with RH neutrinos and Higgs).
- About 10^7 terms in $\mathcal{L}_{\mathsf{SM}}$.
- Maximum number of determining equations: 2.5×10^6 (but many are duplicated, and many are single term).

Apply the LPS method:

- Find all (continuous) symmetries and *prove* that there are no more.
- Use know values of parameters, and run them.
- Find approximate symmetries.
- Add new degrees of freedom looking for new symmetries (e.g. GUT).
- Given measurements of new particles/interactions, can they form part of a new symmetry?

Conclusions

Coordinate variation η^μ, field variation χ_i.
Master determining equation:

$$\mathcal{L}\frac{\mathrm{d}\eta^\mu}{\mathrm{d}x^\mu} + \frac{\partial \mathcal{L}}{\partial x^\mu}\eta^\mu + \frac{\partial \mathcal{L}}{\partial \phi_i}\chi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\left(\frac{\mathrm{d}\chi_i}{\mathrm{d}x^\mu} - \frac{\partial \phi_i}{\partial x^\nu}\frac{\mathrm{d}\eta^\nu}{\mathrm{d}x^\mu}\right) = 0$$

The Lie point symmetry method:

- Counterpart to the Euler-Lagrange equations.
- Finds all possible symmetries.
- Finds all interesting relationships between parameters.
- Works even for spontaneously broken symmetries.
- Can be automated; crucial for large systems.

Future work:

- Find all symmetries of the standard model.
- Allow for discrete symmetries [Hydon (1998)].
- Extend to supersymmetry [Grundland, Hariton, Snobl (2008)].

References

Text book:

- Olver, *Applications of Lie Groups to Differential Equations*, 1986.

Reduction to standard form:

- Reid, J. Phys. A: Math. and General, 23 (1990) L853.
- Reid, Eur. J. of Appl. Math., 2 (1991) 293.
- Reid, Proc. ISSAC '92 (1992).

LPS method and computation:

- Hereman, CRC Handbook of Lie Group Analysis of Differential Equations, (1996) 367.

Previous work using LPS for field theories:

- Hereman, Marchildon & Grundland, Proc. XIX Intl. Colloq. Spain, (1992) 402.
- Marchildon, J. Group Theor. Phys., 3 (1995) 115.
- Marchildon, J. Nonlin. Math. Phys., 5 (1998) 68.

DPG, *A systematic approach to model building*, arXiv:1105.4604.

Symmetries of one scalar

Specialise to $N = 1$:

$$-d\gamma V + \frac{\mathrm{d}V}{\mathrm{d}\phi}(\alpha + \gamma\phi) = 0\,.$$

Four distinct cases:

$V = 0$: α and γ free. Independent shift and scale symmetries.
Rank associated with field is $R_\chi = (2)$.

$V =$ const: $\gamma = 0$ but α is free.
Field rank $R_\chi = (1)$.

$V = \lambda(\phi + v)^d$: Solve above differential equation.
Given v, relationship between shift and scale symmetry is fixed by $v = \alpha/\gamma$.
Field rank $R_\chi = (1)$.

V arbitrary: $\alpha = \gamma = 0$. No shift or scale symmetry.
Field rank $R_\chi = (0)$.

Symmetries of two scalars

$$-d\gamma V + \frac{\partial V}{\partial \phi_1}(\alpha_1 + \beta\phi_2 + \gamma\phi_1) + \frac{\partial V}{\partial \phi_2}(\alpha_2 - \beta\phi_1 + \gamma\phi_2) = 0\,.$$

Go to polar field variables, $\phi_1 = r\cos\theta$, $\phi_2 = r\sin\theta$:

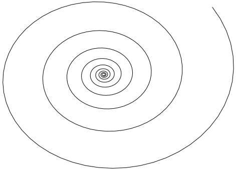
$$\mathcal{L} = \frac{1}{2}\partial^\mu r \partial_\mu r + r^2 \frac{1}{2}\partial^\mu\theta\partial_\mu\theta - V(r,\theta)\,.$$

Determining equation is

$$-d\gamma V + \frac{\partial V}{\partial r}(\alpha_1\cos\theta + \alpha_2\sin\theta + \gamma r) - \frac{\partial V}{\partial\theta}\left(\alpha_1\frac{\sin\theta}{r} - \alpha_2\frac{\cos\theta}{r} + \beta\right) = 0.$$

A solution:

$$V(r,\theta) = \lambda\left(r^k - v\mathrm{e}^{l\theta}\right)^m\,.$$



k and m related by $mk = d$. Relationship between scale and rotation symmetry fixed by $k\gamma = l\beta$. Action of the symmetry is $r \to \mathrm{e}^\gamma r$, $\theta \to \theta - k\gamma/l$ and $x^\mu \to \mathrm{e}^{-d\gamma/D}x^\mu$.

Non-linear symmetries

Field (no coordinate) symmetries of

$$\mathcal{L} = \phi^m (\partial^\mu \phi \partial_\mu \phi)^n \,.$$

m and $n \neq 0$ are constant exponents.

Determining equation

$$m\phi^{m-1}\chi + 2n\phi^m \frac{\mathrm{d}\chi}{\mathrm{d}\phi} = 0 \,.$$

Solve for χ:

$$\chi = a\phi^{-m/2n} \qquad a \text{ is integration constant} \,.$$

Non-linear symmetry acts by $\bar{\phi}' = a\bar{\phi}^{-m/2n}$, solution

$$\phi \to (\phi^p + pa\epsilon)^{1/p} \qquad \text{with} \quad p = 1 + m/2n \,.$$

The action versus the equations of motion

Distinction between the symmetries of action and symmetries of corresponding equations of motion.

G a symmetry of an action $\implies$ G also a symmetry of the Euler-Lagrange equations. Converse not necessarily true.

Denote the system by $\Delta_j(x^\mu, \phi_i, \partial\phi_i) = 0$.

1. Construct the prolonged symmetry operator $\mathrm{pr}^{(k)}\,\boldsymbol{\alpha}$.

$$\boldsymbol{\alpha} = \eta^\mu \frac{\partial}{\partial x^\mu} + \chi_i \frac{\partial}{\partial \phi_i}\ .$$

   Prolongation extends $\boldsymbol{\alpha}$ to include all possible combinations of derivatives of ϕ, to order k.

2. Apply $\mathrm{pr}^{(k)}\,\boldsymbol{\alpha}$ to the system: $(\mathrm{pr}^{(k)}\,\boldsymbol{\alpha}\cdot\Delta)|_{\Delta=0} = 0$.
3. Equate all independent coefficients to zero $\rightarrow$ determining equations.

Equations of motion example

System defined by Euler-Lagrange equation $\ddot{\phi} - \phi'' + m^2\phi = 0$.

What are its symmetries?

- $m = 0$ has

$$\begin{aligned} \eta^t(t,x) &= F_+(t+x) + F_-(t-x)\,, \\ \eta^x(t,x) &= F_+(t+x) - F_-(t-x) + f\,, \\ \chi(t,x,\phi) &= G_+(t+x) + G_-(t-x) + g\,\phi(t,x)\,. \end{aligned}$$

- $m \neq 0$ has

$$\begin{aligned} \eta^t(x) &= a^t + bx\,, \\ \eta^x(t) &= a^x + bt\,, \\ \chi(t,x,\phi) &= \int_{-\infty}^{+\infty} dk \left[H_+(k)\, \mathrm{e}^{i(\omega t + kx)} + H_-(k)\, \mathrm{e}^{i(\omega t - kx)} \right] + g\,\phi(t,x)\,, \end{aligned}$$

where $\omega = \sqrt{k^2 + m^2}$.

Degrees of freedom in the standard model

$N = 244$ real degrees of freedom (with RH neutrinos):

- gauge $=$ 4 real components $\times$ (1 hyp $+$ 3 weak $+$ 8 strong) $=$ 48,
- leptons $=$ 8 real components $\times$ 3 gens $\times$ (ν $+$ e) $=$ 48,
- quarks $=$ 8 real components $\times$ 3 gens $\times$ 3 cols $\times$ (u $+$ d) $=$ 144,
- and Higgs $=$ 2 real components $\times$ weak-doublet $=$ 4.