Perturbative and Non-Perturbative Insights into Deformed JT gravity from Random Matrix Ensembles

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Outline:

Based on work with Felipe Rosso

CVJ + FR: 2011.06026

- Motivation
- Random Matrix Ensembles — Double Scaling Limit
- Application 1: JT gravity
  - Perturbative
  - Non-Perturbative
- Application 2: Deformations of JT gravity
  - Example A   Example B
  - Perturbative + Non-perturbative
- Summary

Supported in part by DoE + NSF.

Key discovery: (Saad, Shenker, Stanford 1903.11115)

JT gravity $=$ Random Matrix Model

Generalized to various other JT gravity variants and cousins by Stanford + Witten 1907.03363

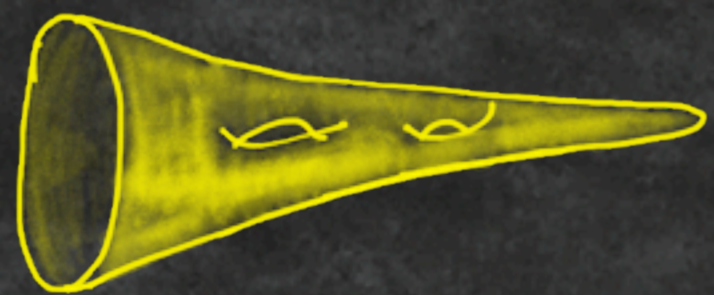
- Instructive examples of "fully" quantum gravity
- Key examples of the "averaging" phenomenon.

Action:

$$I = -S_0\chi - \frac{1}{2}\int d^2x\sqrt{g}\,\phi(R+2) + \text{boundary terms}$$

Jackiw - Teitelboim gravity '83, '85

β



$$\chi = 2-2g-b$$

disc: $$Z_0(\beta) = \frac{e^{S_0}e^{\pi^2/\beta}}{4\pi^{1/2}\beta^{3/2}}$$

$$= \int \rho_0(E)e^{-\beta E}dE$$

Maldacena + Stanford
1604.07818
⋮
(Many other teams)

$$\rho_0(E) = \frac{\sinh(2\pi\sqrt{E})}{4\pi^2\hbar}$$

$$\boxed{\hbar = e^{-S_0}}$$

$$Z(\beta) = \sum_g Z_g\, \hbar^{2g-2+\overset{1}{\overset{\shortparallel}{b}}} \quad + \underbrace{\text{non-perturbative}}_{\text{extremely important!}}$$

$$\Downarrow$$

$$\rho(E) = \rho_0(E) + \rho_1(E) + \cdots + \text{non-perturbative}$$

$$\rho_0(E) = \frac{\sinh 2\pi\sqrt{E}}{4\pi^2 \hbar}$$

Will use a different formulation of the double-scaling limit from SSS.

Recent work by Witten, and by Maxfield + Turiaci (and very recently by Turiaci, Usatyuk, Weng) has shown:

$$\boxed{\begin{array}{c}\text{various deformations}\\ \text{of JT gravity}\\ \equiv \text{ random matrix models.}\end{array}}$$

But: There is strange behaviour ($\rho_0(E) < 0$) in parts of parameter space.

- What does it mean? How to resolve?

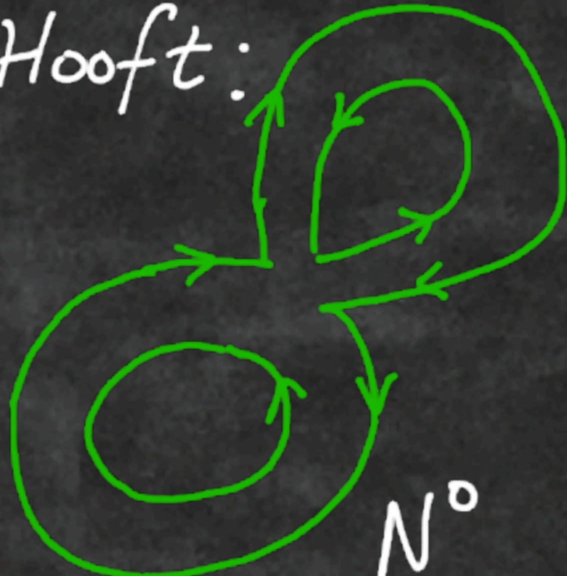
Here: $\boxed{\begin{array}{l}\text{Will identify root cause}\\ \text{and find new phenomena.}\end{array}}$

Random matrix models + Double-scaling limit.

$$\tilde{Z}(g_4) = \int DM \exp\left\{-N \operatorname{Tr} V(M)\right\} \qquad M \text{ is } N \times N \text{ HM}$$

$$V(M) = \frac{1}{2} M^2 + g\, M^4 + \dots\dots$$

't Hooft:



vs

N^0 $\qquad$ N^{-2}

vs

etc...

- large N expansion captures topology.
- DSL (BK, DS, GM)

Smooth universal physics if $N \to \infty$ and $g \to g_c$ where large surfaces dominate.

DSL cont'd

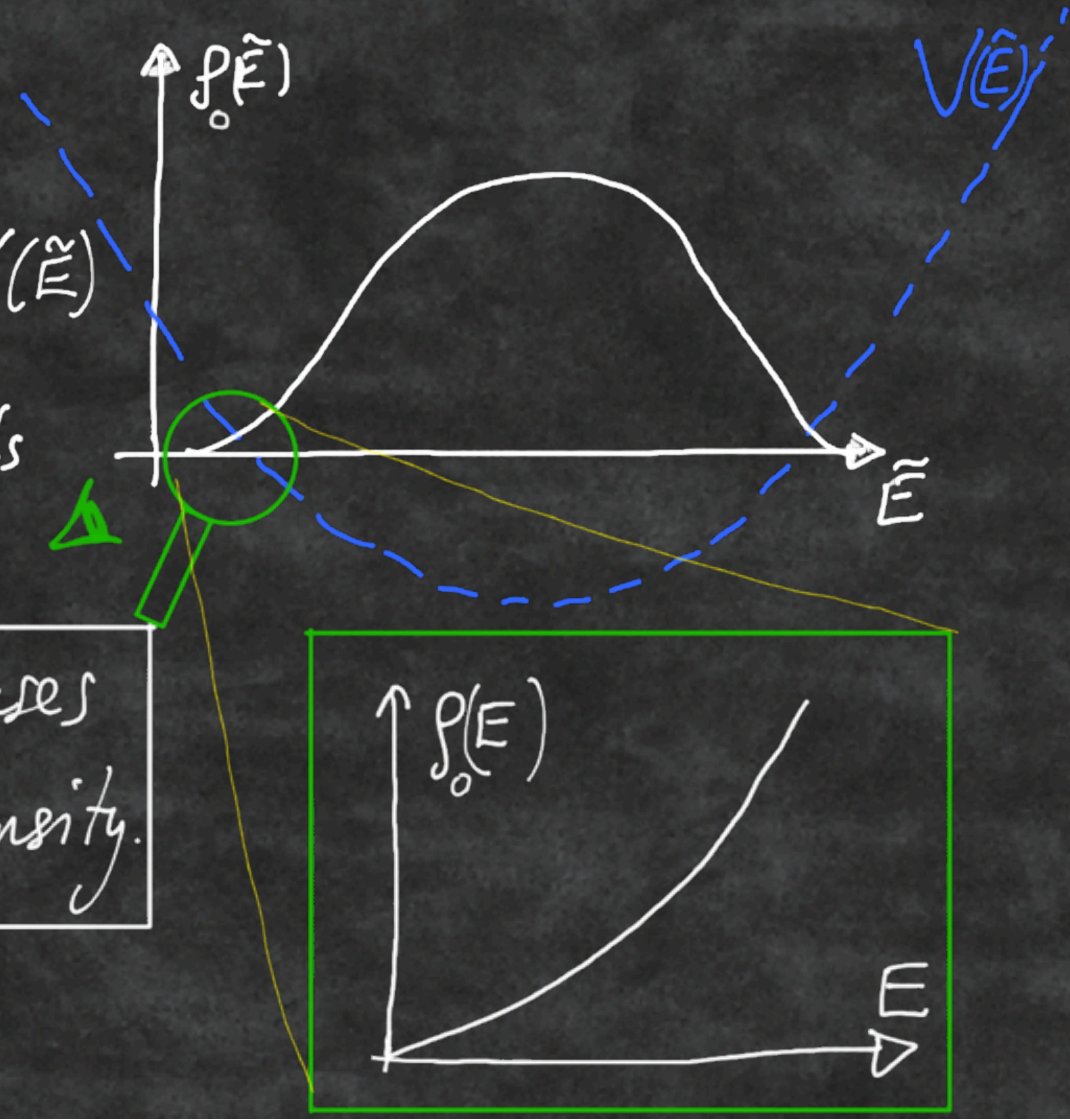
Diagonalize: $M = U \Lambda U^{+}$

$\Lambda = \mathrm{diag}\{\hat{E}_1, \hat{E}_2, \hat{E}_3 \ldots\}$

$$\tilde{Z} = \int \prod_i d\hat{E}_i \underbrace{\prod_{i<j} (\hat{E}_i - \hat{E}_j)^2}_{\mathrm{VdM}} \exp\{-N \sum_i V(\hat{E}_i)\}$$

$\rho(\tilde{E})$ before DSL, at large N:

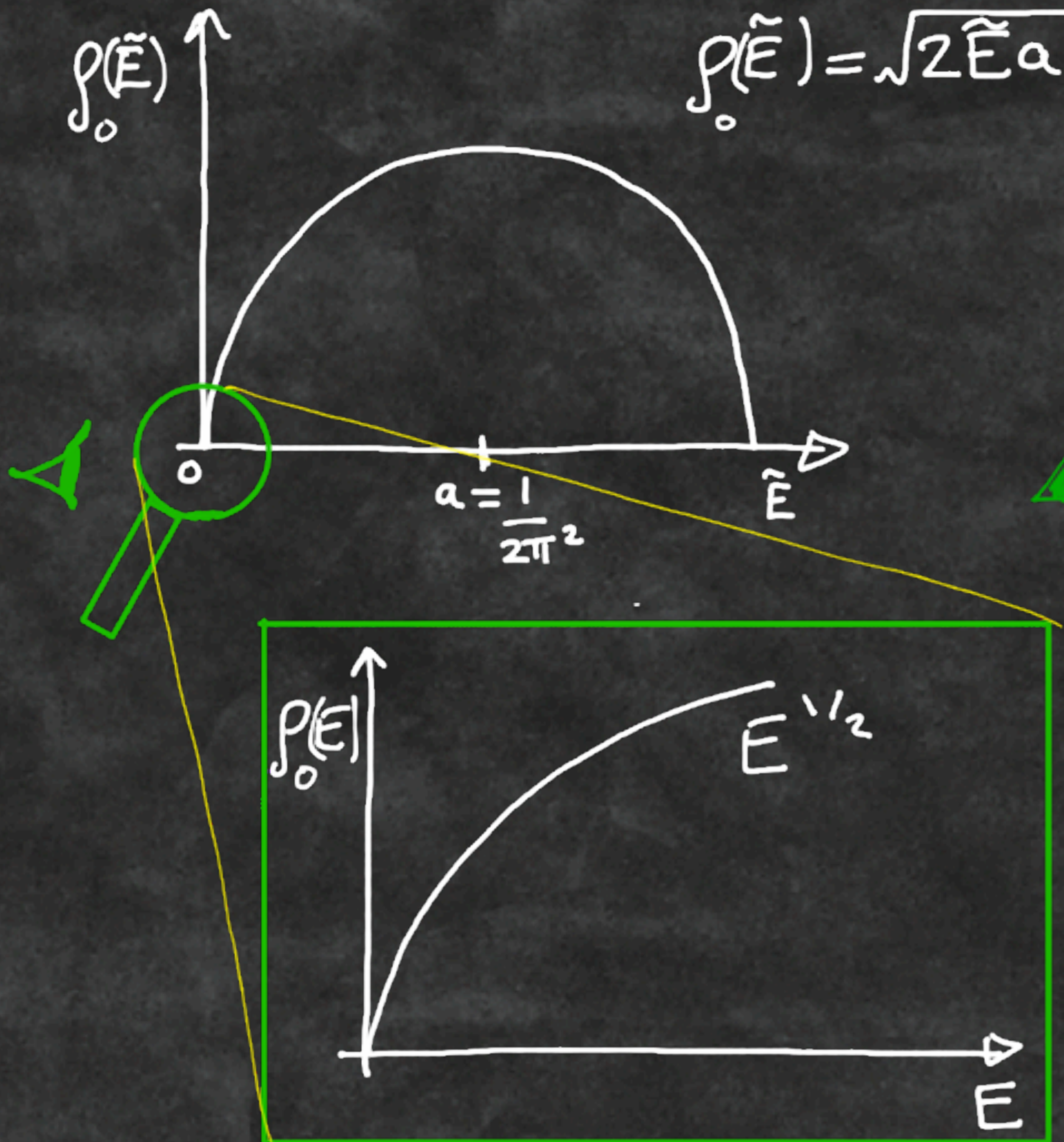
- Dyson gas confined by $V(\tilde{E})$
- Van der Monde gives repulsion between levels



$\boxed{\text{Double scaling limit focuses on an endpoint of the density.}}$

e.g. the Gaussian case

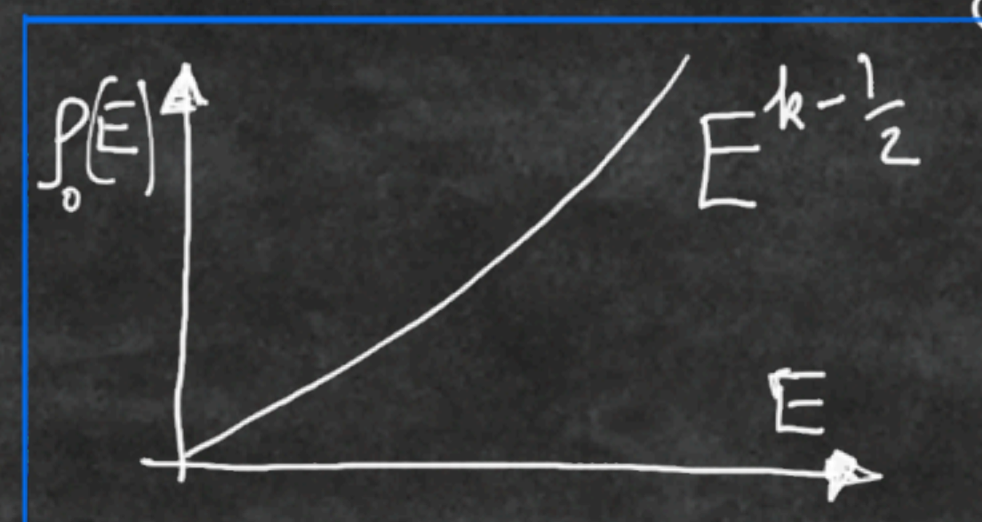
$$\rho_0(\tilde{E}) = \sqrt{2\tilde{E}a - \tilde{E}^2}$$



$$\tilde{E} = 0 + E_{\text{small}}$$

(label this "$k=1$")

- More generally (Kazakov '89) can tune $V(M)$ to get:



- Theories of smooth surfaces called "minimal string theories" (Sieberg)

More tools:

Matrix model: (recall) $\tilde{Z} = \int \prod_i d\tilde{E}_i \prod_{i<j} (\hat{E}_i - \hat{E}_j)^2 \exp\{-N \sum_i V(\tilde{E}_i)\}$

Brezin, Itzykson, Parisi, Zuber '78

Bessis, Itzykson, Zuber '80

- Can write everything in terms of N orthogonal polynomials $P_n(\tilde{E}) = \tilde{E}^n + \dots$

$$\int P_n(\tilde{E}) P_m(\tilde{E})\, d\tilde{E}\, e^{-NV(\hat{E})} = h_n \delta_{mn}$$

$$\langle m|n\rangle = \delta_{mn} \qquad |n\rangle = P_n / \sqrt{h_n}$$

$$\tilde{E} P_n(\tilde{E}) = P_{n+1}(\tilde{E}) + A_n P_{n-1}(\tilde{E})$$

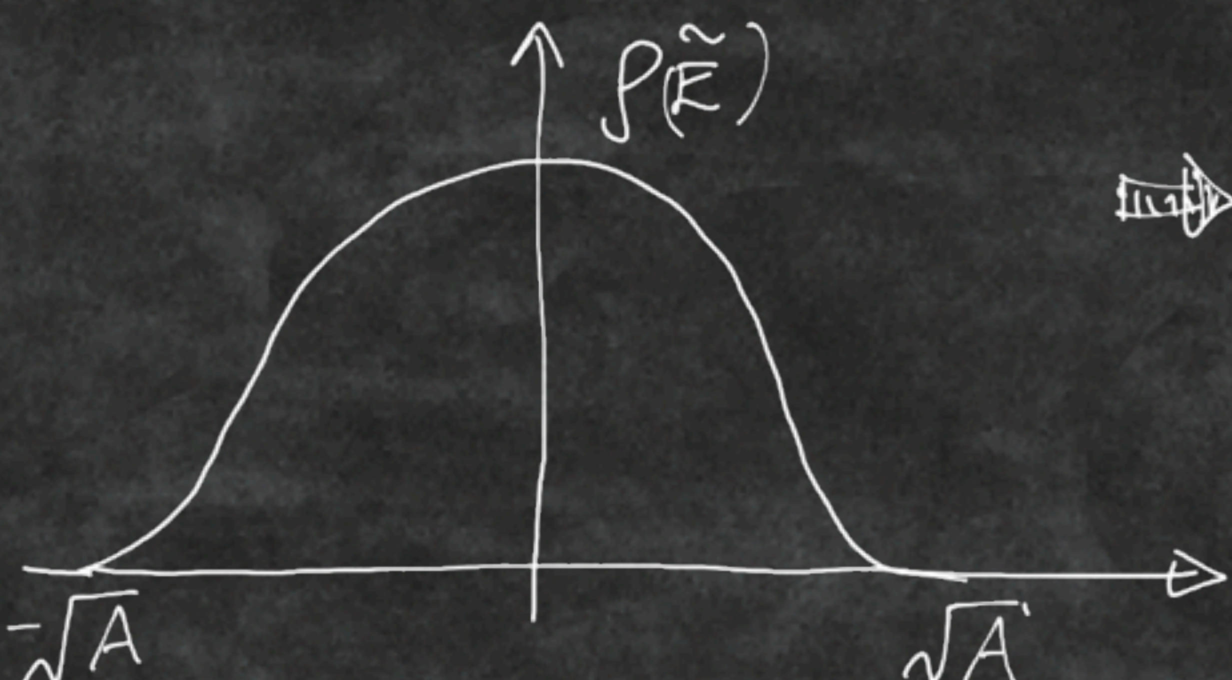
- $\tilde{Z}$ can be written in terms of the A_n

There are many details that must be skipped here, but:

- At large N, write $x = \frac{n}{N}$, $\epsilon = \frac{1}{N}$

  and, e.g. $A_n \longrightarrow A(x)$

- $$\rho_0(\tilde{E}) = \frac{1}{2\pi}\int_0^1 \frac{dx}{\sqrt{A(x)-\tilde{E}^2}} \Longrightarrow f(\tilde{E})\sqrt{A-\tilde{E}^2}$$



$\Longrightarrow$ endpoints at $x = 0, 1$

- Double scaling limit $\equiv$ Zoom in...

$X = \frac{n}{N}$, $\epsilon = \frac{1}{N}$ $\qquad A_n \longrightarrow A(X)$

DSL:

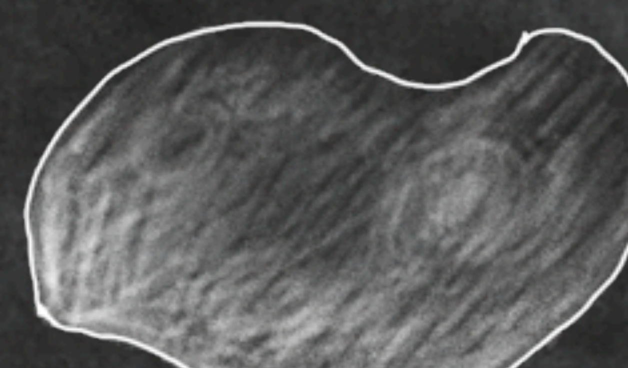
$$\left.\begin{aligned} X &= -x\,\delta^{\#_1} \qquad \epsilon = \hbar\,\delta^{\#_3} \\ A(X) &= A_0 - u(x)\,\delta^{\#_2} \end{aligned}\right\} \text{Then send } \delta \to 0$$

Brezin-Kazakov
Gross-Migdal '89
Douglas-Shenker

δ?

discrete

$\delta \to 0$



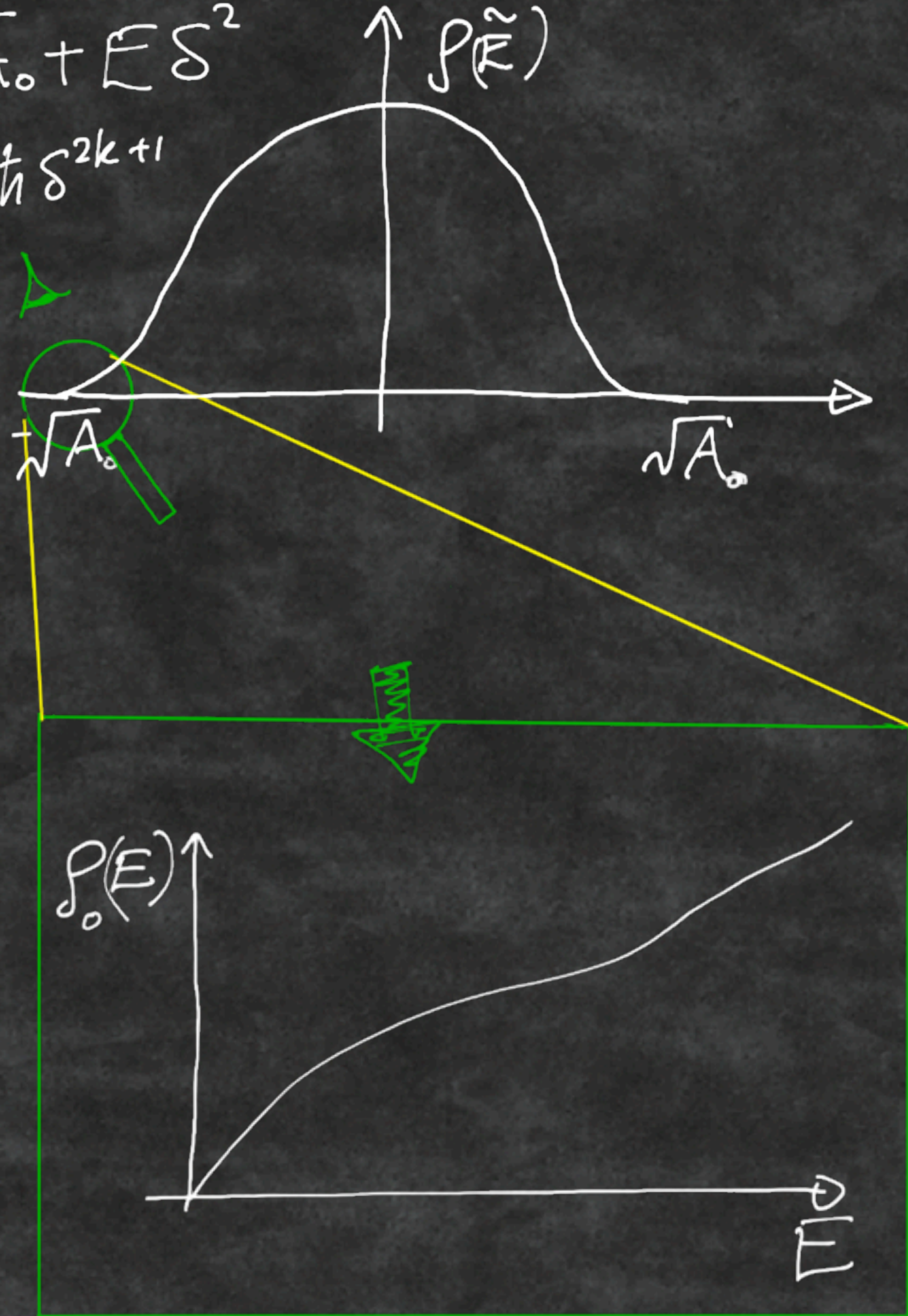
smooth

$$X = -x\,\delta^{2k} \qquad \tilde{E} = -\sqrt{A_0} + E\,\delta^2$$

$$A(X) = A_0 - u_0(x)\,\delta^2 \qquad \frac{1}{N} = \hbar\,\delta^{2k+1}$$

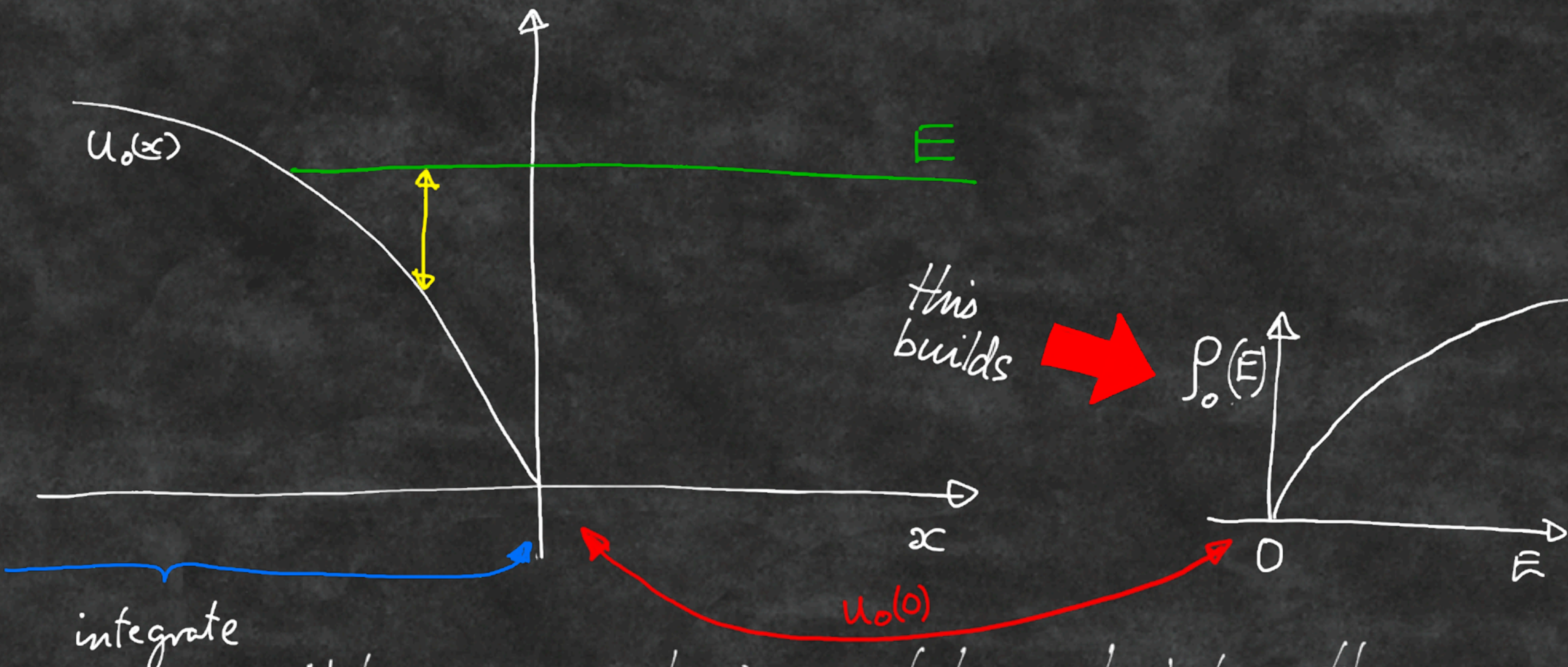
$$\rho_0(\tilde{E}) = \frac{1}{2\pi}\int_0^1 \frac{dX}{\sqrt{A(X) - \tilde{E}^2}}$$

$$\rho_0(E) = \frac{1}{2\pi\hbar}\int_{-\infty}^0 \frac{dx}{\sqrt{E - u_0(x)}}$$



Picture:

$$\rho_0(E) = \frac{1}{2\pi\hbar}\int_{-\infty}^{0} \frac{dx}{\sqrt{\underbrace{E - u_0(x)}}}$$



- Note: $x<0$ physics controls perturbation theory.
- $x>0$ physics will govern non-perturbative sector

Important: Can also accommodate:

$$u_0(x=0) = E_0$$

Useful to write as:

$$\boxed{\rho_0(E) = \frac{1}{2\pi\hbar}\int_{-\infty}^{0} \frac{dx}{\sqrt{E-U(x)}} = \frac{1}{2\pi\hbar}\int_{0}^{E} \frac{f(U_0)\, dU_0}{\sqrt{E-U_0}}}$$

The disc level equation for $U_0(x)$ will be of the form: $R_0(U_0, x) = 0$

$$\boxed{f(U_0) \equiv -\frac{\partial x}{\partial U_0}}$$

$$R_0(U_0, x) \equiv G(U_0) + x = 0$$

Including non-zero threshold E_0, write:

$$\boxed{\rho_0(E) = \frac{1}{2\pi\hbar}\int_{E_0}^{E} \frac{\partial_{U_0} R_0\, dU_0}{\sqrt{E-U_0}}}$$

A Quantum Mechanics

The orthog. poly. description yields a QM

$\frac{P_n}{\sqrt{h_n}} \equiv |n\rangle \longrightarrow |x\rangle$

Gross + Migdal '89

$\langle \mathcal{O}(M) \rangle \longrightarrow \langle x|\hat{\mathcal{O}}|y\rangle$ etc

$\hat{E} \longrightarrow \boxed{\mathcal{H} = -\hbar^2 \frac{\partial^2}{\partial x^2} + u(x)}$

- Note: Defined on all of x.

$u(x)$ supplied by a non-linear ODE "string equation". It is the continuum limit of identities expressing the content of the matrix model.

Key Observable

(Banks Douglas Seiberg Shenker '89)
"Macroscopic Loop"

Hold at fixed length β

$$\langle Z(\beta)\rangle \xrightarrow{\text{DSL}} \int_{-\infty}^{0} dx\, \langle x|\, e^{-\beta\mathcal{H}}\, |x\rangle$$

$$= \mathrm{Tr}\left(e^{-\beta\mathcal{H}}\, \mathbb{P}\right)$$

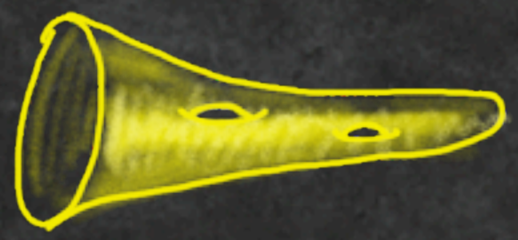
$$\mathbb{P} \Downarrow \int_{-\infty}^{0} dx\, |x\rangle\langle x|$$

$$= \int_{-\infty}^{0} dx \int d\psi\, \langle x|\, e^{-\beta\mathcal{H}}\, |\psi\rangle\langle\psi|x\rangle$$

$$= \int_{-\infty}^{0} dx \int dE\, \langle x|\psi_E\rangle\langle\psi_E|x\rangle\, e^{-\beta E}$$

$$= \int dE\, \rho(E)\, e^{-\beta E} \qquad \rho(E) = \int_{-\infty}^{0} \psi(x,E)\,\psi^{*}(x,E)\, dx$$

So $\langle Z(\beta)\rangle = \mathrm{Tr}\left(e^{-\beta \mathcal{H}_{\mathbb{P}}}\right)$
is our gravity theory



$$\boxed{\mathcal{H} \equiv -\hbar^2 \frac{\partial^2}{\partial x^2} + u(x)}$$

- solve: $\mathcal{H}\,\psi(x,E) = E\,\psi(x,E)$
- Construct: $\rho(E) = \int_{-\infty}^{0} \psi(x,E)\,\psi^*(x,E)\,dx$

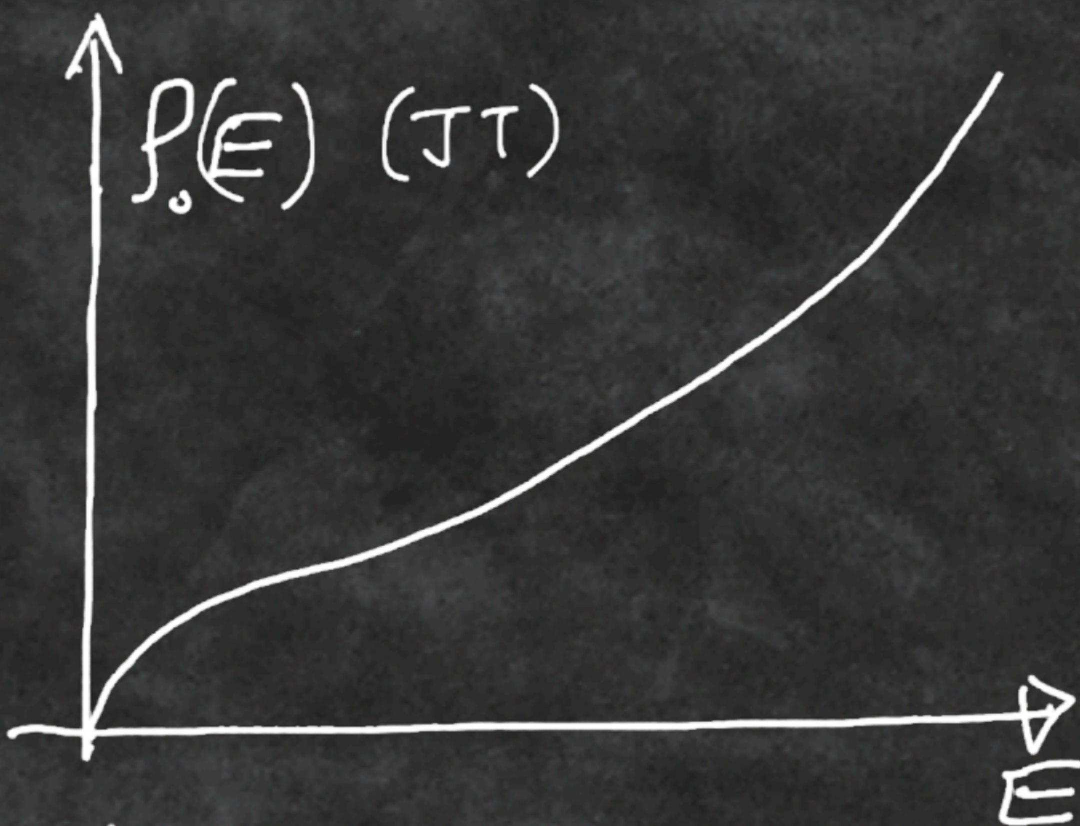
- The challenge is to find the correct $u(x)$
- Depends upon which JT gravity
- Depends upon which matrix model

$\hookrightarrow$ $u(x)$ solves a non-linear ODE (see later)

Application 1: JT gravity

Okuyama + Sakai 1911.01659
SSS

$$\rho_0(E) = \frac{\sinh(2\pi\sqrt{E})}{4\pi^2\hbar} = \frac{1}{2\hbar}\left(\frac{\sqrt{E}}{\pi} + \frac{2}{3}\pi E^{3/2} + \frac{2}{15}\pi^3 E^{5/2} + \cdots\right)$$



$$E^{k-\frac{1}{2}} \qquad k = 1, 2, \ldots \text{etc}$$

characteristic of minimal string models

$$\rho_0(E) = \frac{1}{2\pi\hbar}\int_0^E \frac{\sum_k k t_k u_0^{k-1}\, du_0}{\sqrt{E - u_0}}$$

$$\sum_{k=1}^{\infty} t_k u_0^k = -x$$

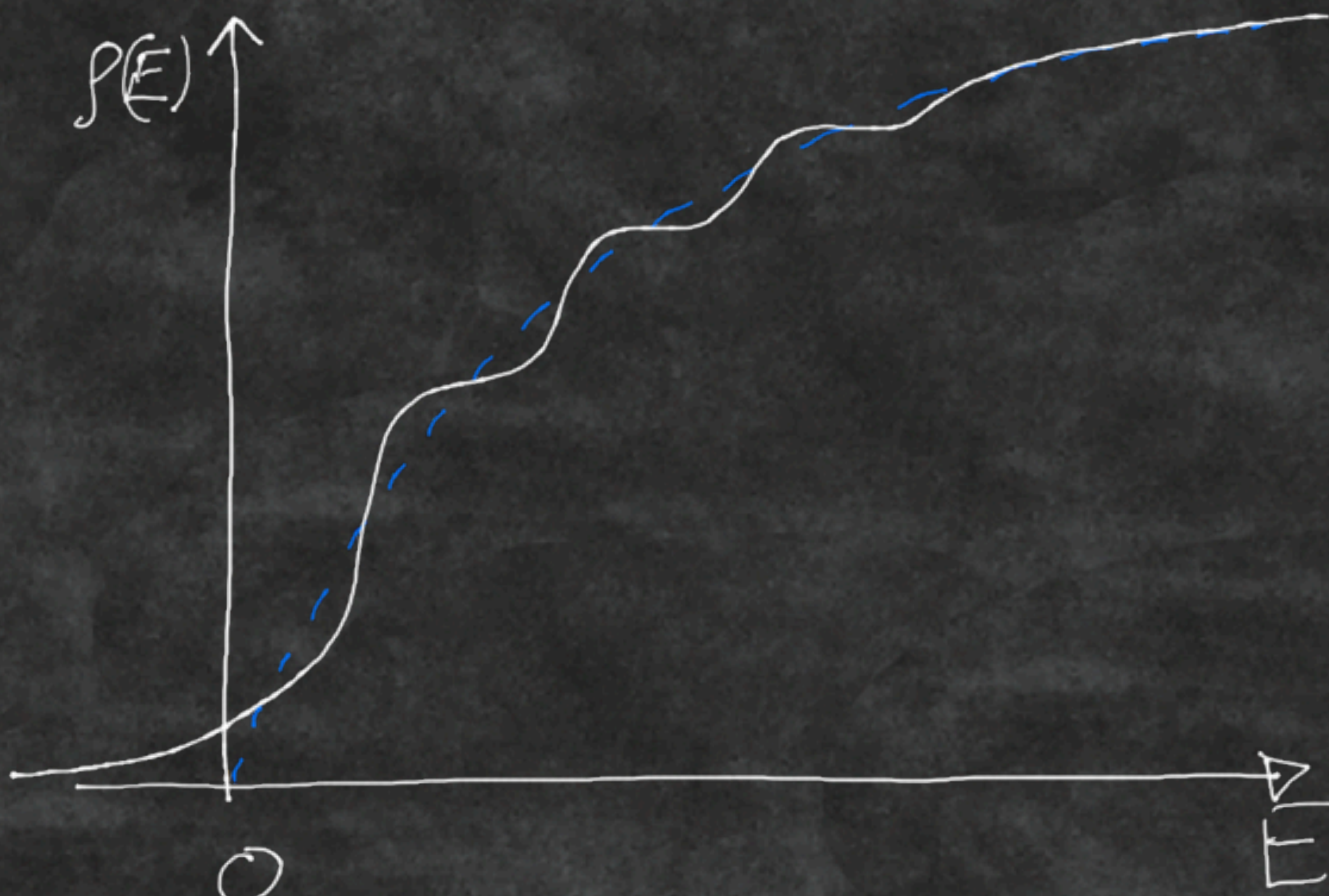
$$t_k = \frac{\pi^{2(k-1)}}{2k!(k-1)!} \quad \Longrightarrow \quad \frac{\sqrt{u_0}}{2\pi} I_1(2\pi\sqrt{u_0}) + x = 0$$

cvj 2005.01893

Non-Perturbative Physics

- The constituent minimal models of the Hermitian matrix model definition have a non-perturbative instability.

- Eigenvalues tunnel out of the system.

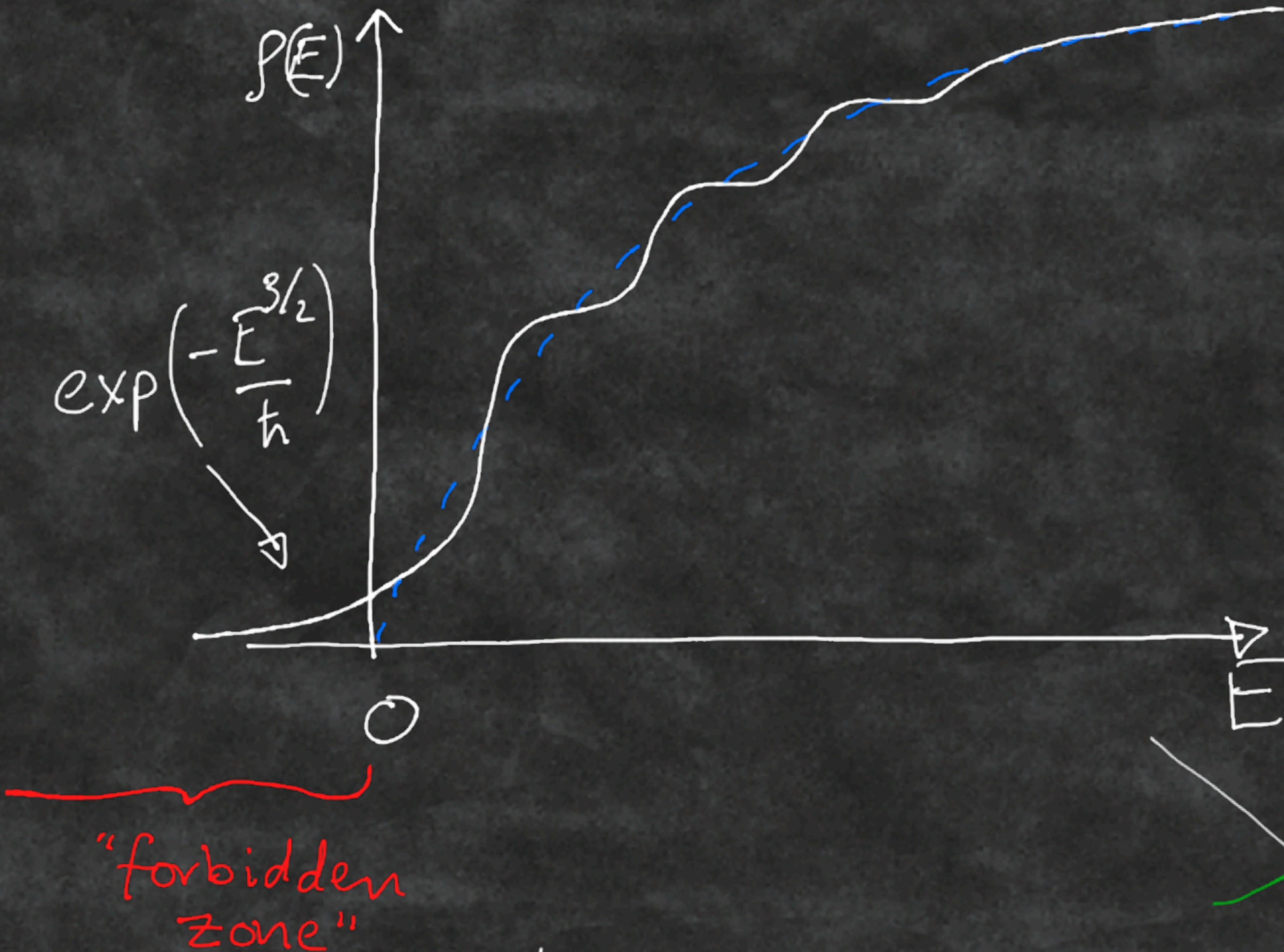


e.g. $k=1$

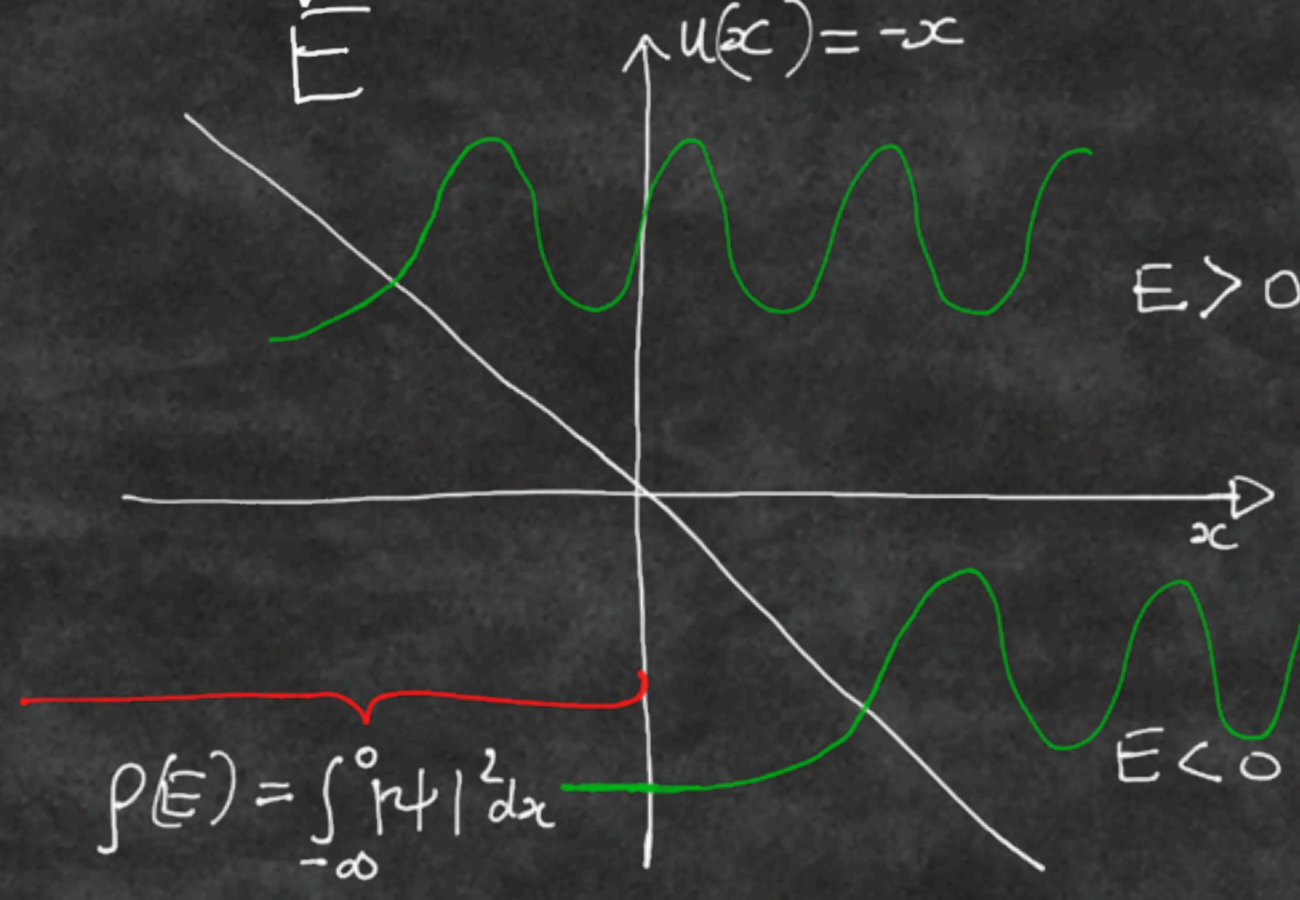
(tail of Schwarzian)

$$u = u_0 = -x$$

$$\psi(x,E) = \hbar^{-2/3} A_i\left(-\hbar^{-2/3}(x+E)\right)$$



exponential tail to $E<0$ comes from wavefunctions tunneling to $-x$ from $+$ve x

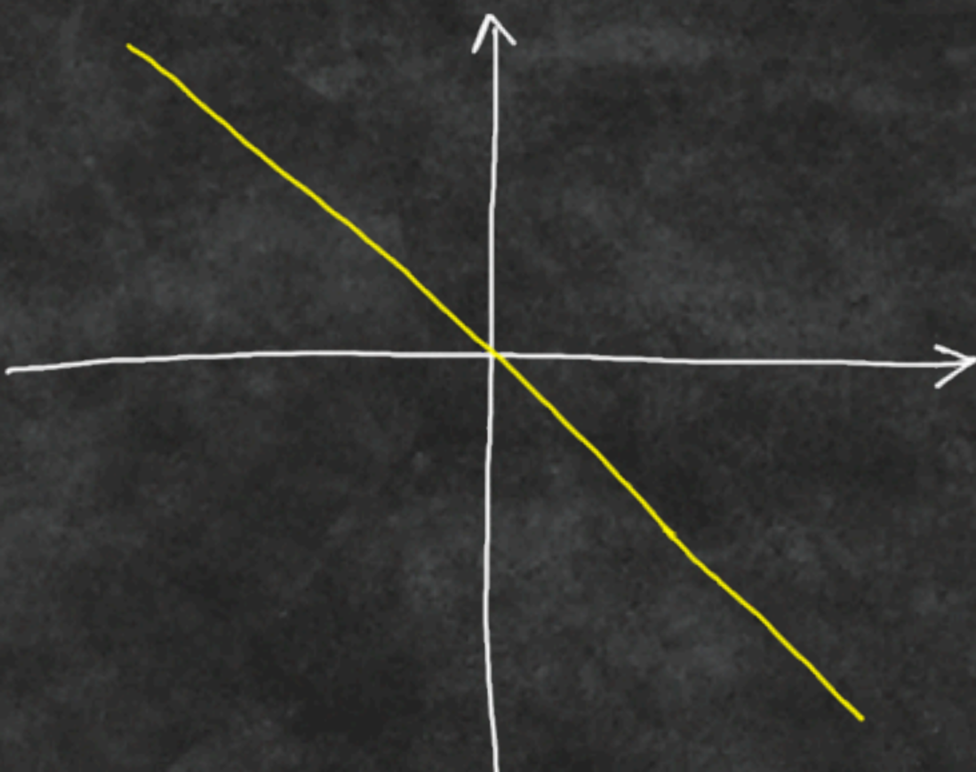


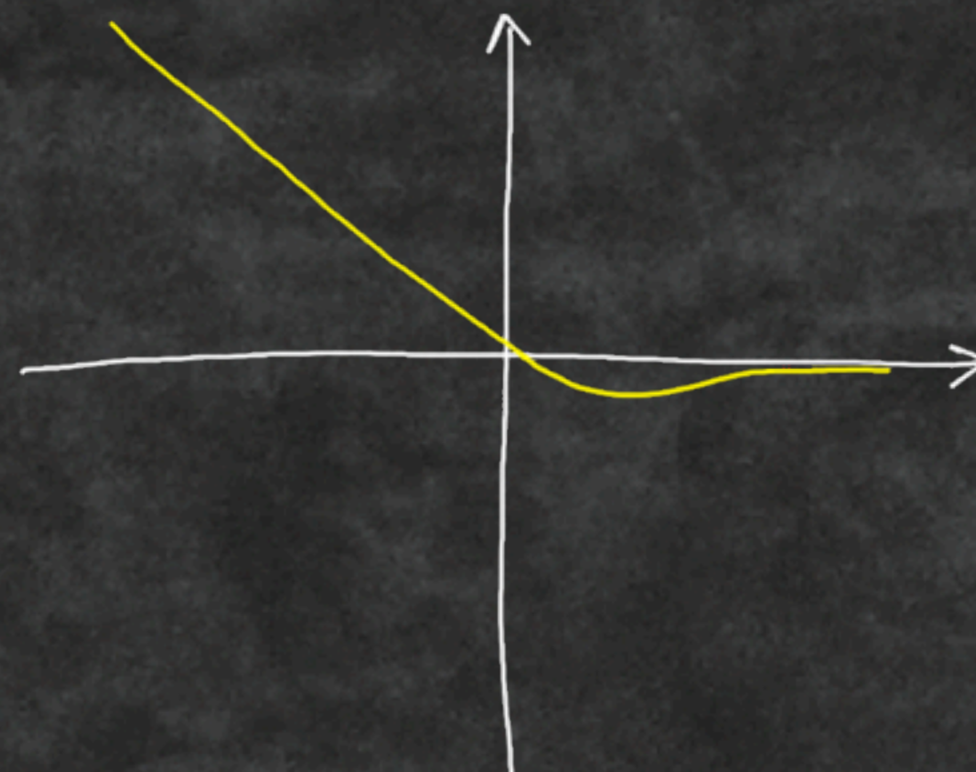
Non-Perturbative Physics

- The constituent minimal models of the Hermitian matrix model definition have a non-perturbative instability.
- Eigenvalues tunnel out of the system.
- This can be cured. Use minimal models with the same leading $x<0$ physics but better $x>0$ physics.
- These can be obtained from $\boxed{\text{complex matrix ensembles.}}$

Dalley, CVJ and Morris 91/92

e.g. $k=1$ HMM vs CMM





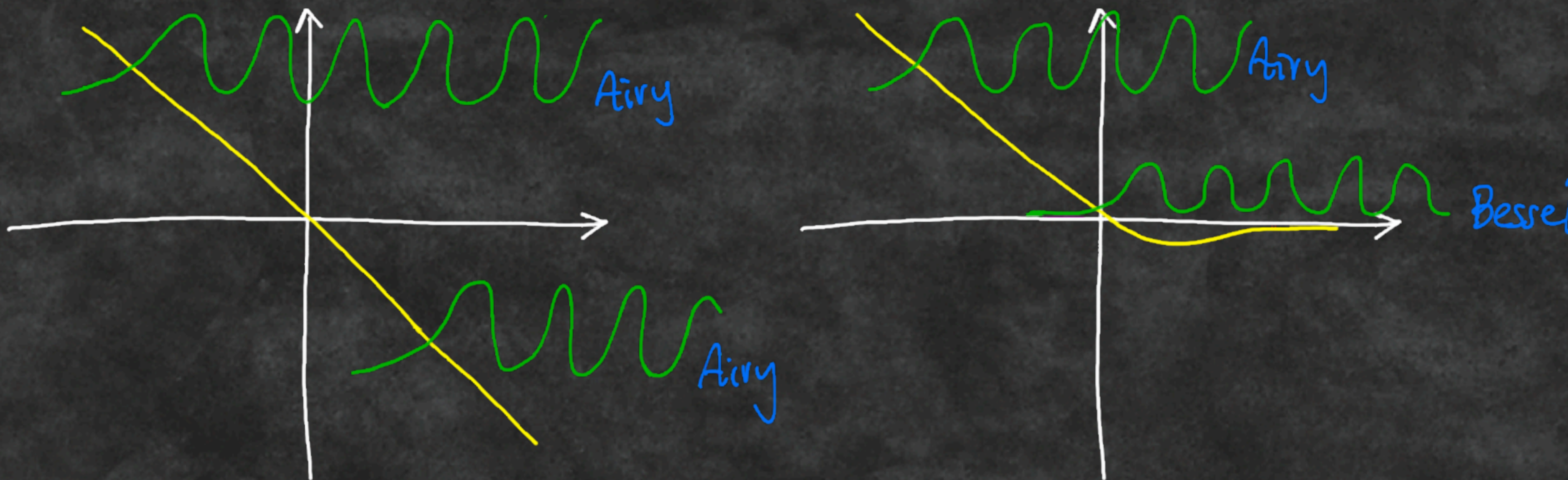
$R \equiv u + x$

HMM: $R = 0 \rightsquigarrow \boxed{u = -x}$

CMM $uR^2 - \frac{\hbar^2}{2} R R'' + \frac{\hbar^2}{4} (R')^2 = 0$

$$\rightsquigarrow \boxed{u(u+x)^2 - \frac{\hbar^2}{2} u''(u+x) + \frac{\hbar^2}{4}(u'+1)^2 = 0}$$

e.g. $k=1$ HMM vs CMM



CVJ 1912.03637

The full string equations

$$\mathcal{R}(u,x) = \sum_{k=1}^{\infty} t_k R_k[u] + x$$

$$R_k[u] = u^k + \underbrace{\cdots\cdots}_{\text{mixed terms}} + u^{\overbrace{''''}^{2k-2}}$$

$$' \equiv \hbar \frac{d}{dx}$$

$$\boxed{\text{HMM:} \quad \mathcal{R} = 0}$$

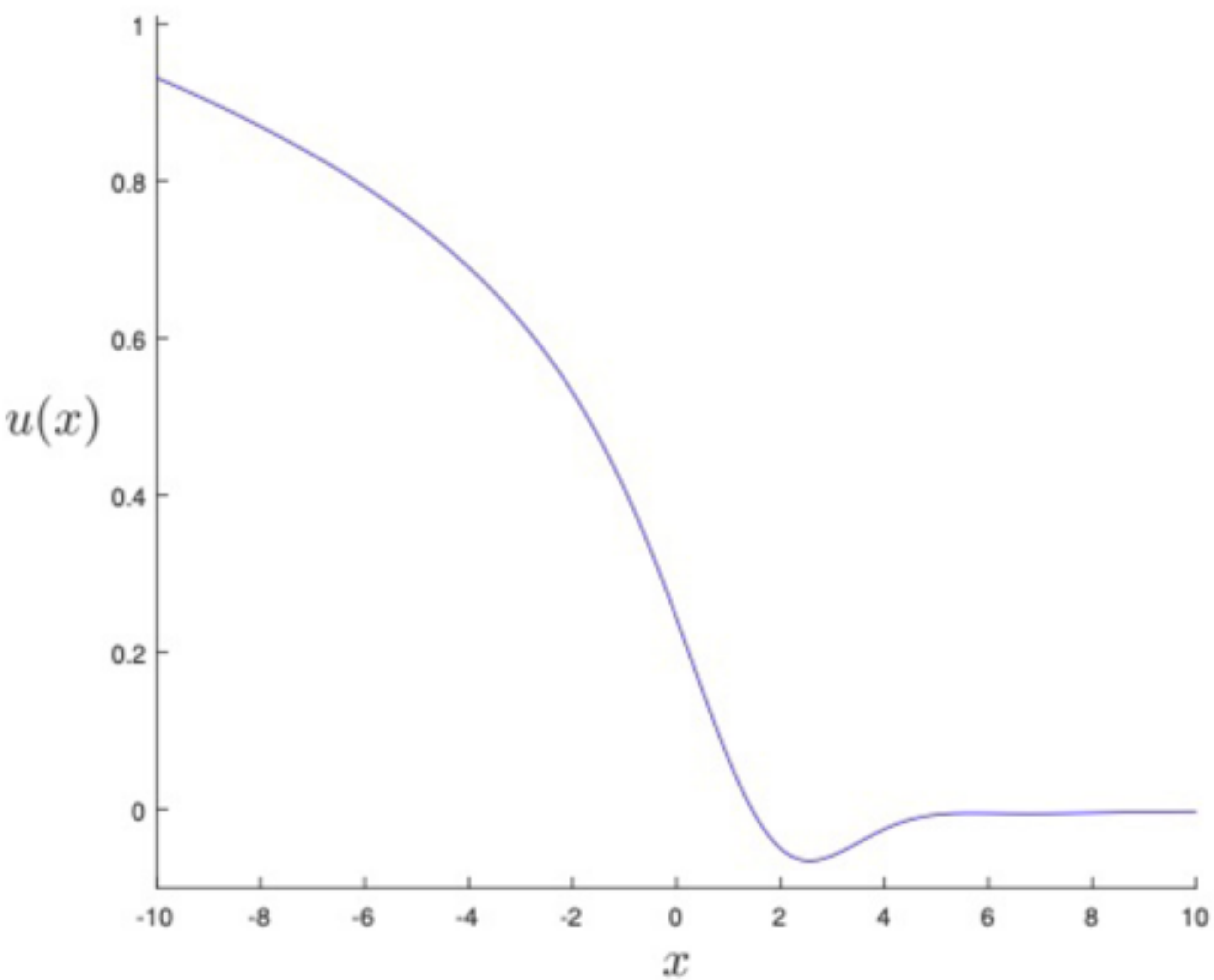
$$\boxed{\text{CMM:} \quad u\mathcal{R}^2 - \frac{\hbar^2}{2}\mathcal{R}\mathcal{R}'' + \frac{\hbar^2}{4}(\mathcal{R}')^2 = 0}$$

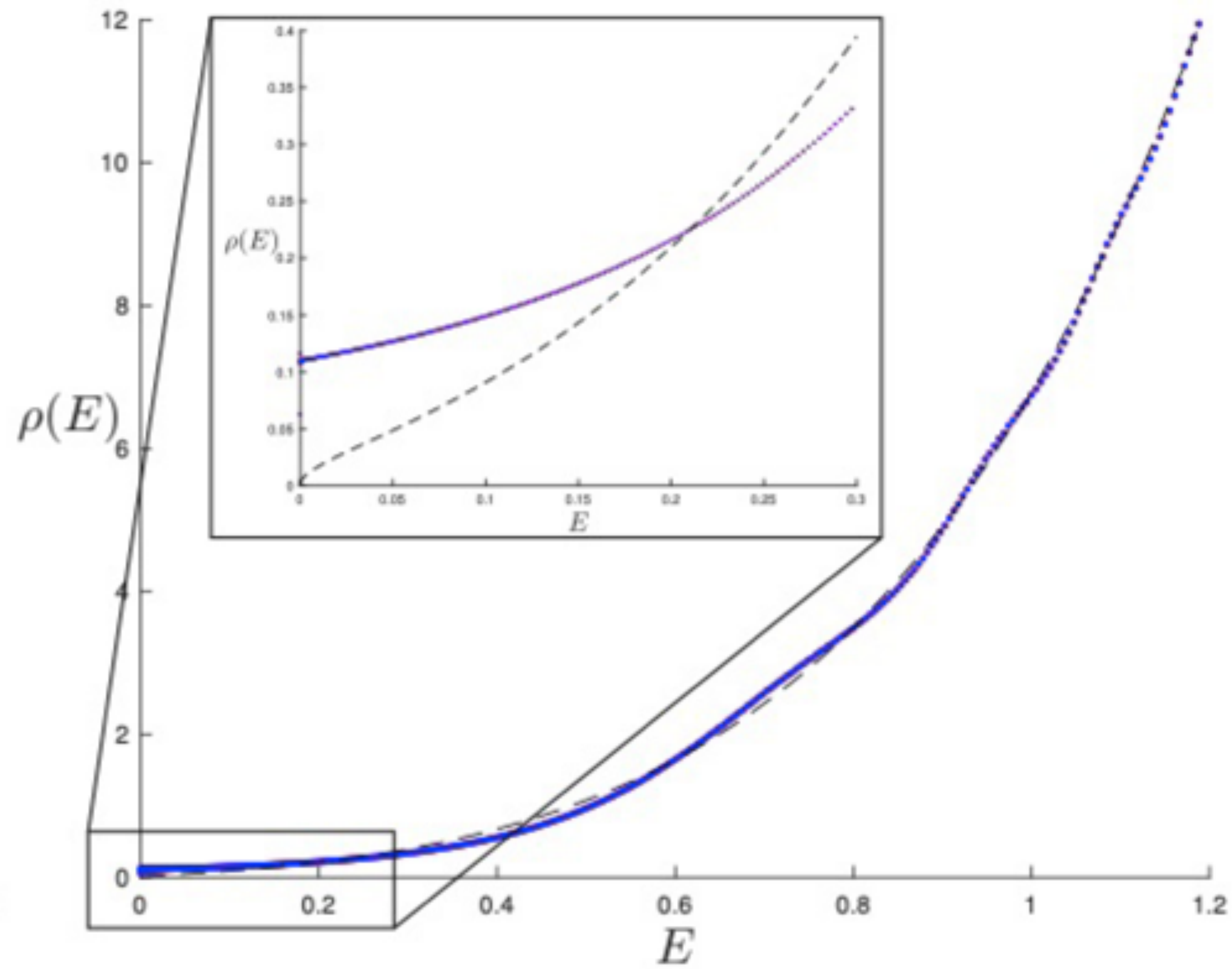
Dalley, CVJ, Morris '92

If t_k picked such that $\rho_0(E)$ is JT, this gives a non-perturbative definition

CVJ 1912.03637 2006.10959

Full result: (Stable Non-perturbative definition of JT gravity)
(solution of truncated eqn. CVJ: 2006.10959)





Application 2: Deformations of JT gravity

Witten
- 2006.03494
- 2006.13414

Maxfield & Turiaci
- 2006.11317

Turiaci Usatyuk Weng
- 2011.06038

$$I = -S_0 \chi - \frac{1}{2}\int d^2x \sqrt{g}\left[\phi(R+2) + U(\phi)\right]$$

with $$U(\phi) = 2\sum_{i=1}^{r} \lambda_i e^{-2\pi(1-\alpha_i)\phi}$$

$\alpha_i \leq 1$

$$W(\phi) = 2\phi + U(\phi)$$

Black hole solutions:

$$\boxed{ds^2 = \frac{1}{4} f(r)\, dt^2 + f(r)^{-1} dr^2}$$

$$f(r) = \int_{\phi_h}^{r} d\phi\, W(\phi)$$

$\phi(r) = r$

$$T = \frac{W(\phi_h)}{2\pi} \qquad S = S_0 + 2\pi\phi_h$$

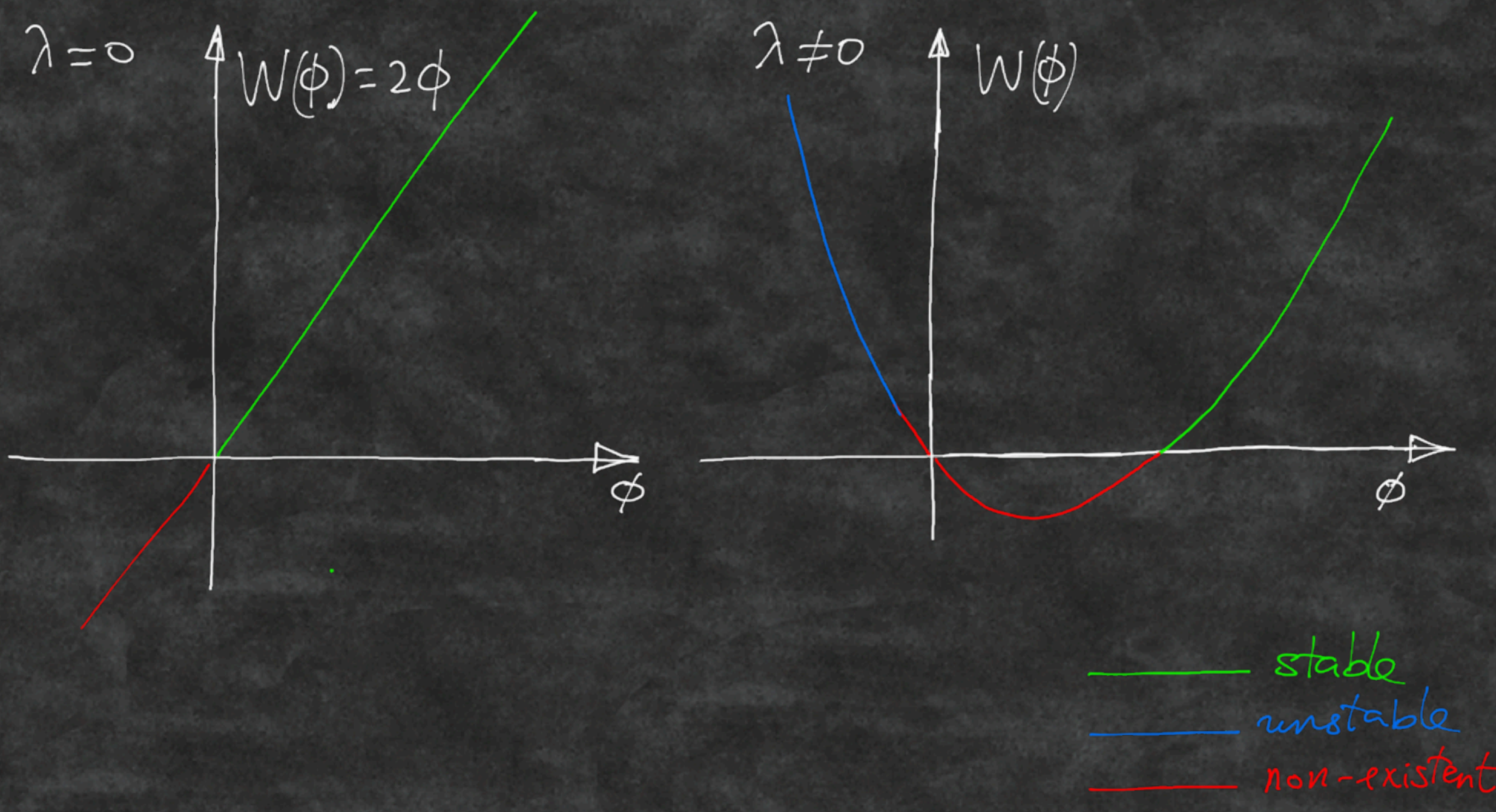
$$\int^{\phi} d\phi' W(\phi') \geq 0$$

+ve signature

$$E = \phi_h^2 - \int_{\phi_h}^{\infty} U(\phi)\, d\phi$$

(Witten 2006.13414)

Richer phases of black hole solution possible:



W and MT showed an equivalence to matrix models analogous to SSS for $\lambda=0$.

<u>But</u>: There are parts of parameter space where $\rho_0(E)$ goes negative!

- What is going on?
- How to resolve this?

Perturbatively?
Non-Perturbatively?
Phase transition?

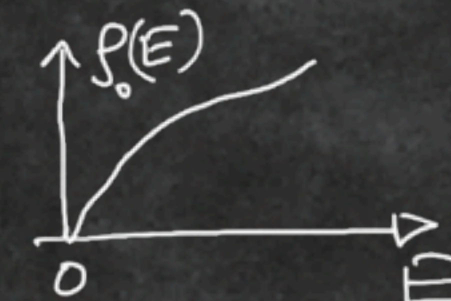
We have the tools! ⟹

Example A:

$$U(\phi) = 2\lambda\left(e^{-2\pi(1-\alpha_1)\phi} - e^{-2\pi(1-\alpha_2)\phi}\right)$$

$\alpha_1 < \alpha_2$ $\qquad \boxed{U(0)=0}$

$$\rho_0(E) = \frac{\sinh(2\pi\sqrt{E})}{4\pi^2\hbar} + \lambda\left[\frac{\cosh 2\pi\alpha_1\sqrt{E} - \cosh 2\pi\alpha_2\sqrt{E}}{2\pi\hbar\sqrt{E}}\right]$$



See problem by e.g. expanding

$$\rho_0(E) = \left(\frac{\lambda_c - \lambda}{2\pi\hbar\lambda_c}\right)\sqrt{E} + O\left(E^{3/2}\right)$$

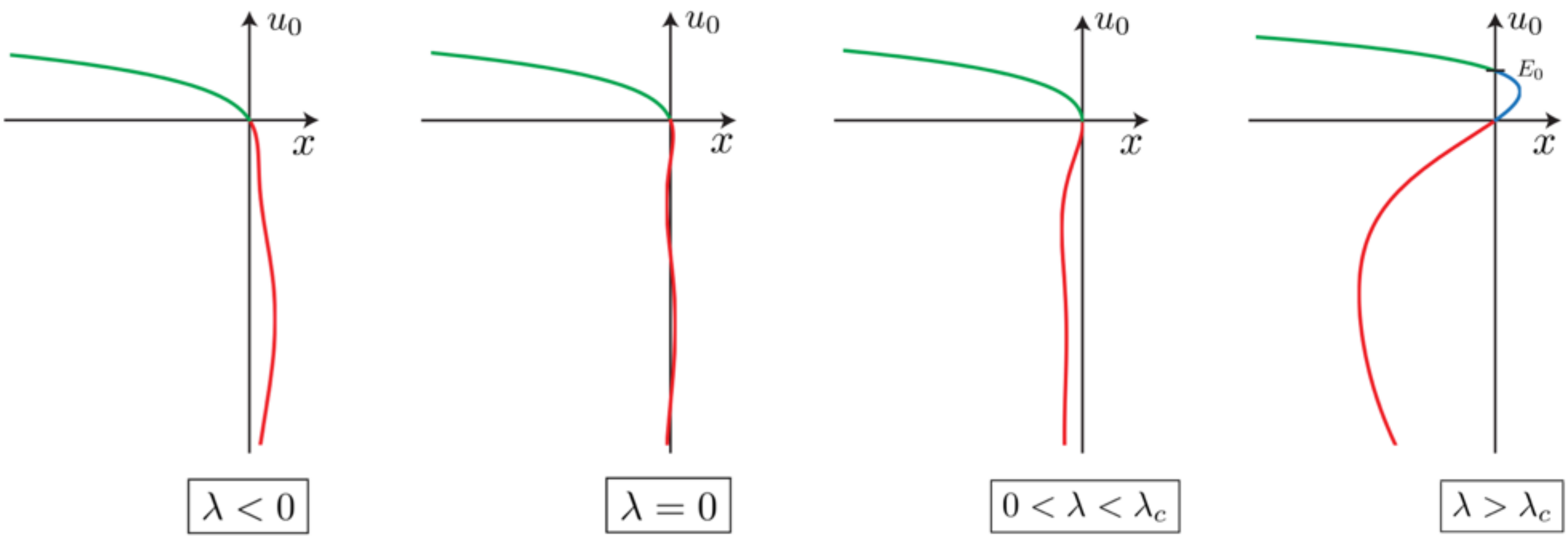
$$\lambda_c = \frac{-1}{2\pi^2(\alpha_1^2 - \alpha_2^2)}$$

$\rho_0(E) < 0$ for $\lambda > \lambda_c$ at small E

Rewrite, using integral transform, as disc string equation:

$$R_0 = \frac{\sqrt{u_0}}{2\pi} I_1\left(2\pi\sqrt{u_0}\right) + \lambda\left(I_0\left(2\pi\alpha_1\sqrt{u_0}\right) - I_0\left(2\pi\alpha_2\sqrt{u_0}\right)\right) + x = 0$$

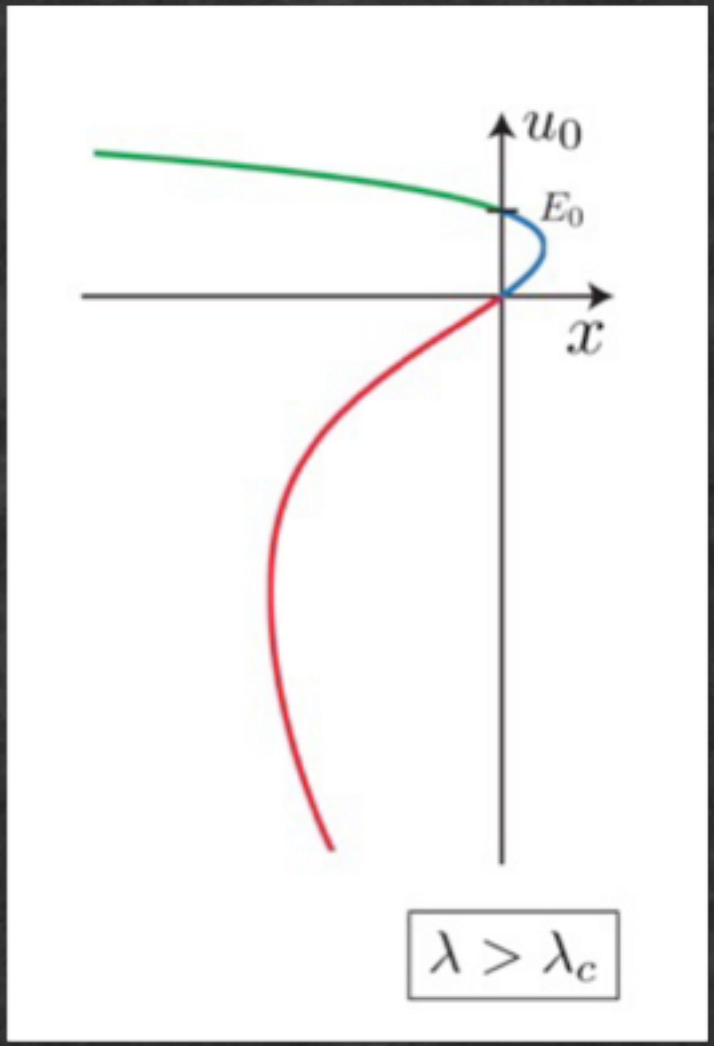
The issue becomes clear when visualized:



Key: Above λ_c, $u_0(x)$ develops a mult-valuedness at low E...

Resolution:

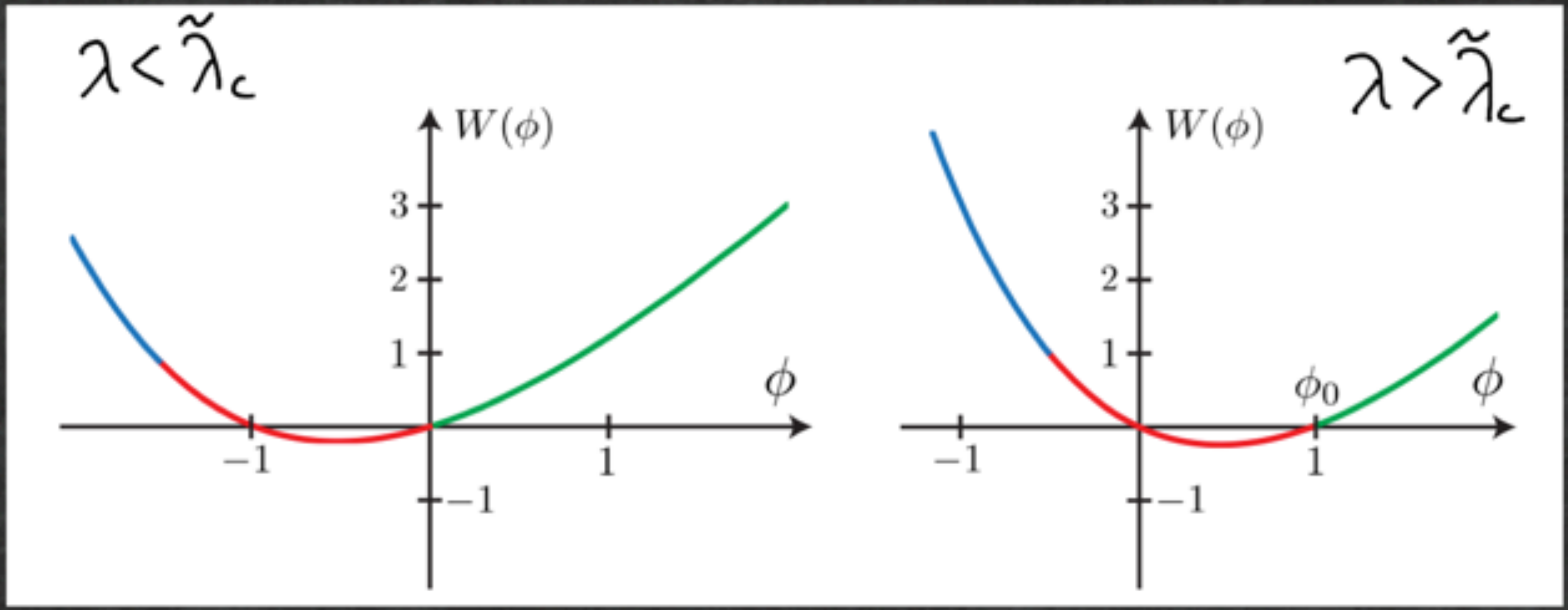
- Phase transition to new $\rho_0(E)$ with threshold E_0.



- Previous $\rho_0(E)$ incorrect for $\lambda > \lambda_c$.

- Instead, $\rho_0(E) = \frac{1}{2\pi\hbar} \int_{E_0}^{E} \frac{\partial_{u_0} R_0[u_0]\, du_0}{\sqrt{E - u_0}}$

$\boxed{\text{Notice: semiclassical phase structure mirrors this!}}$



$T=0$ black hole develops $\phi_h \neq 0$.

Example B:

$$\mathcal{U}(\phi) = 2\lambda e^{-2\pi(1-\alpha)\phi}$$

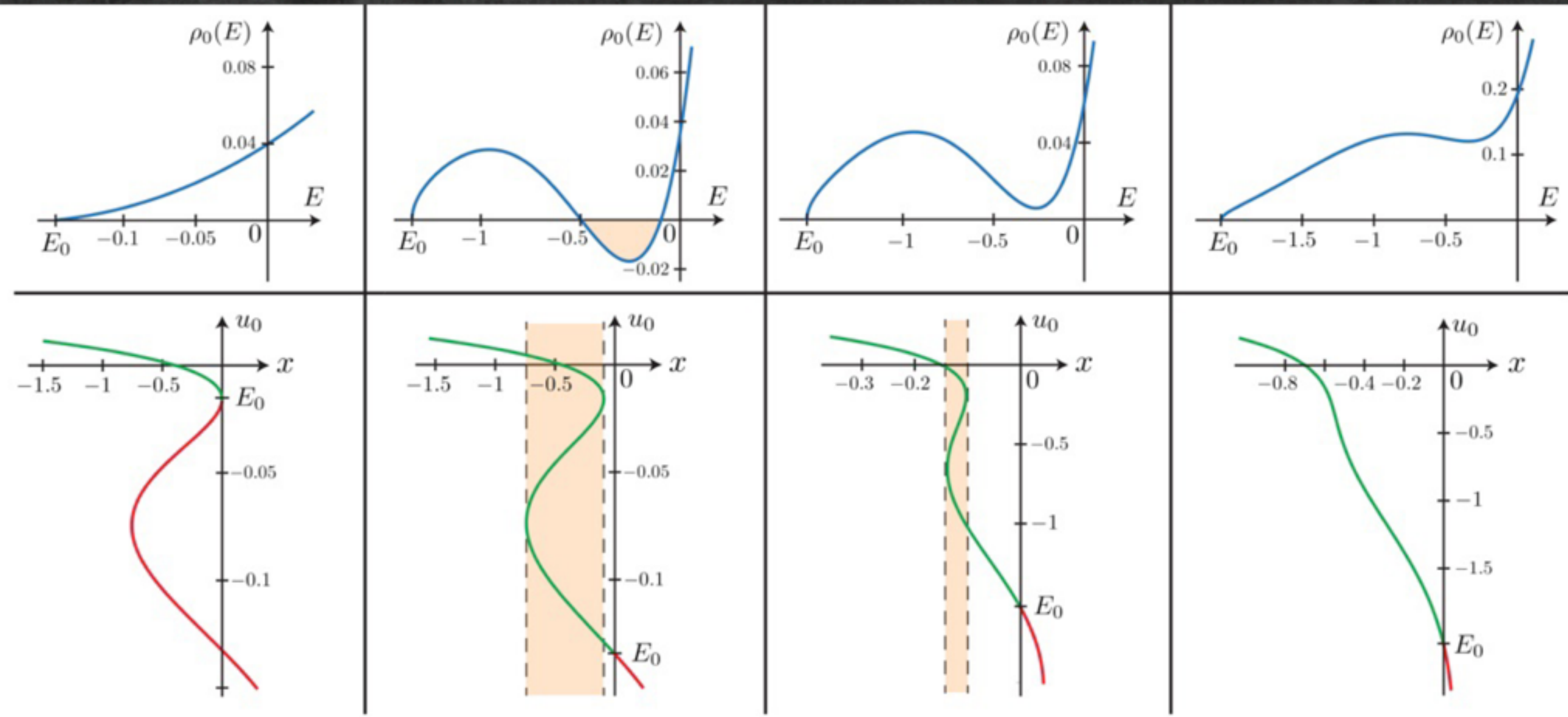
$$\boxed{U(0) \neq 0}$$

The disc string equation is:

$$R_0(u_0, x) = \frac{\sqrt{u_0}}{2\pi} I_1\left(2\pi\sqrt{u_0}\right) + \lambda I_0\left(2\pi\sqrt{u_0}\right) + x = 0$$

Resuting $\rho_0(E)$ goes $-$ve in a finite range but there's more:

$\lambda < \lambda_c^*$ $\qquad$ $\lambda > \lambda_c^*$



- E_0 jumps
- phase transition?
- $\rho_0(E) < 0$ & multi-valued in $x < 0$

- multivaluedness in $x < 0$ still present even if $\rho_0(E)$ +ve!

- New, fine, perturbative phase.

Example B:

$$\mathcal{U}(\phi) = 2\lambda e^{-2\pi(1-\alpha)\phi}$$

$\boxed{\mathcal{U}(0) \neq 0}$

The disc string equation is:

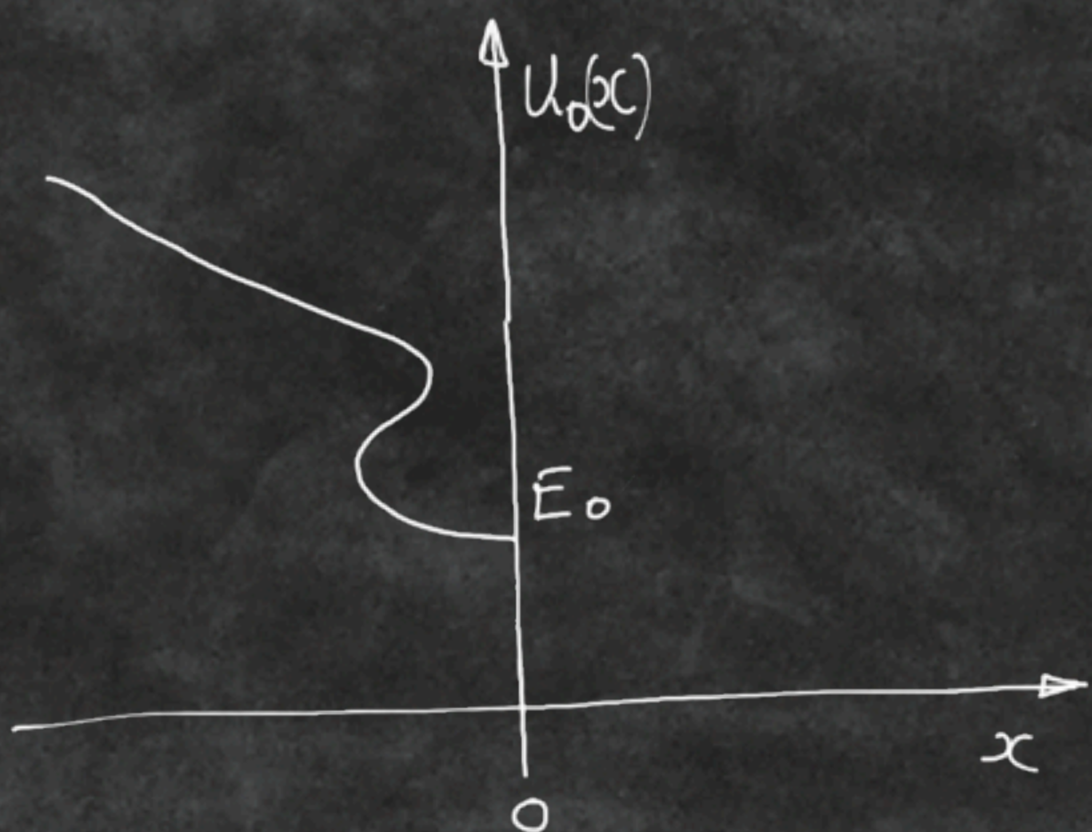
$$\mathcal{R}_0(u_0, x) = \frac{\sqrt{u_0}}{2\pi} I_1(2\pi\sqrt{u_0}) + \lambda I_0(2\pi\sqrt{u_0}) + x = 0$$

Resulting $\rho_0(E)$ goes −ve in a finite range but there's more:

> −ve $\rho_0(E)$ only a symptom of true pathogen!

> Multivalued $u_0(x)$ for $x < 0$ dooms the perturbative physics.

Multivalued $U_0(x)$ for $x<0$ dooms the perturbative physics.



- $\mathcal{H} = -\hbar^2 \dfrac{\partial^2}{\partial x^2} + U_0(x)$ is ill-defined as a mechanics problem.

- Recall: $U_0(x) \longleftrightarrow A_n$

recursion coeffs for Orthog polys

$\rightarrow$ Information loss/ ambiguity.

Hilbert space ill-defined...

Non-perturbative physics

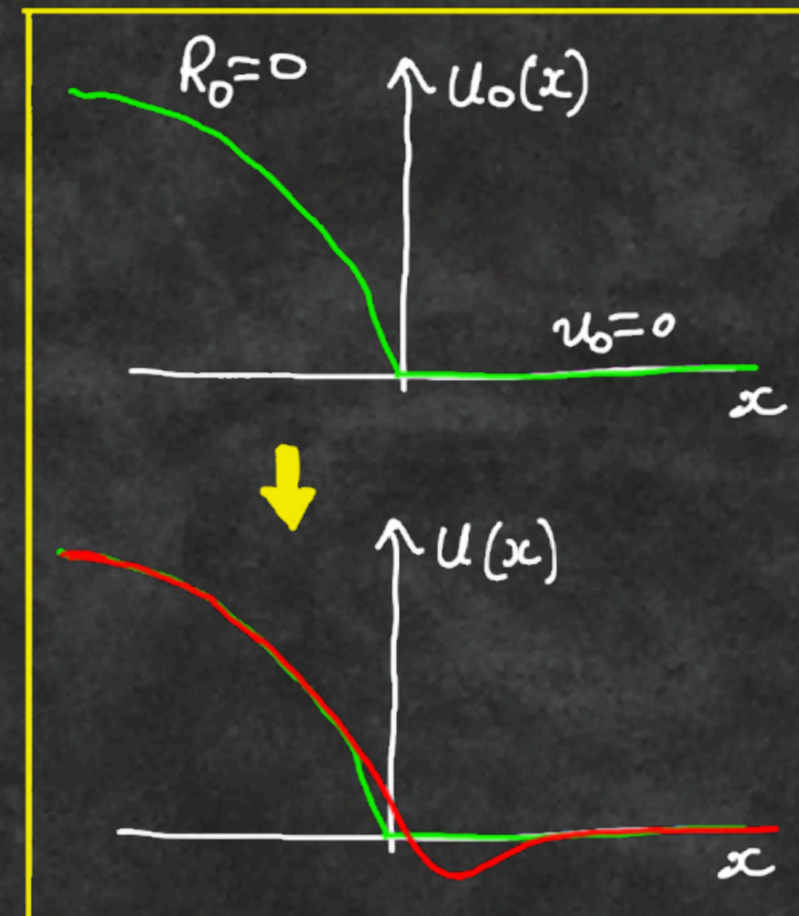
Successful formulation can be extended!

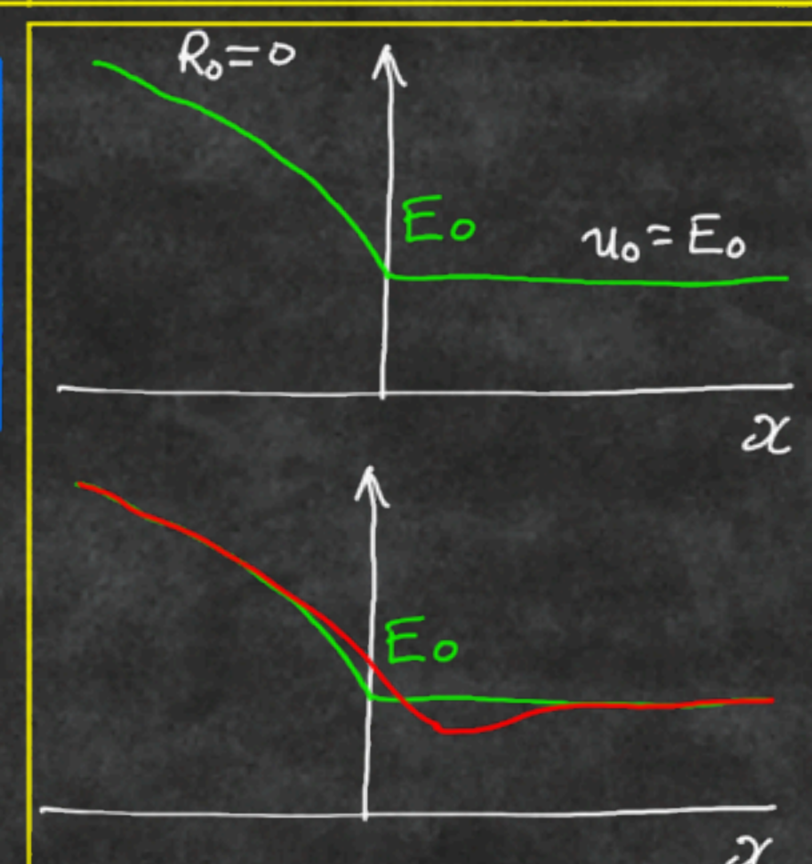
$$u R^2 - \frac{\hbar^2}{2} R R'' + \frac{\hbar^2}{4} (R')^2 = 0$$

$$\boxed{(u - E_0) R^2 - \frac{\hbar^2}{2} R R'' + \frac{\hbar^2}{4} (R')^2 = 0}$$

So: What physics emerges non-perturbatively?

Recall:





Non-perturbative physics

Minimal model decomposition is:

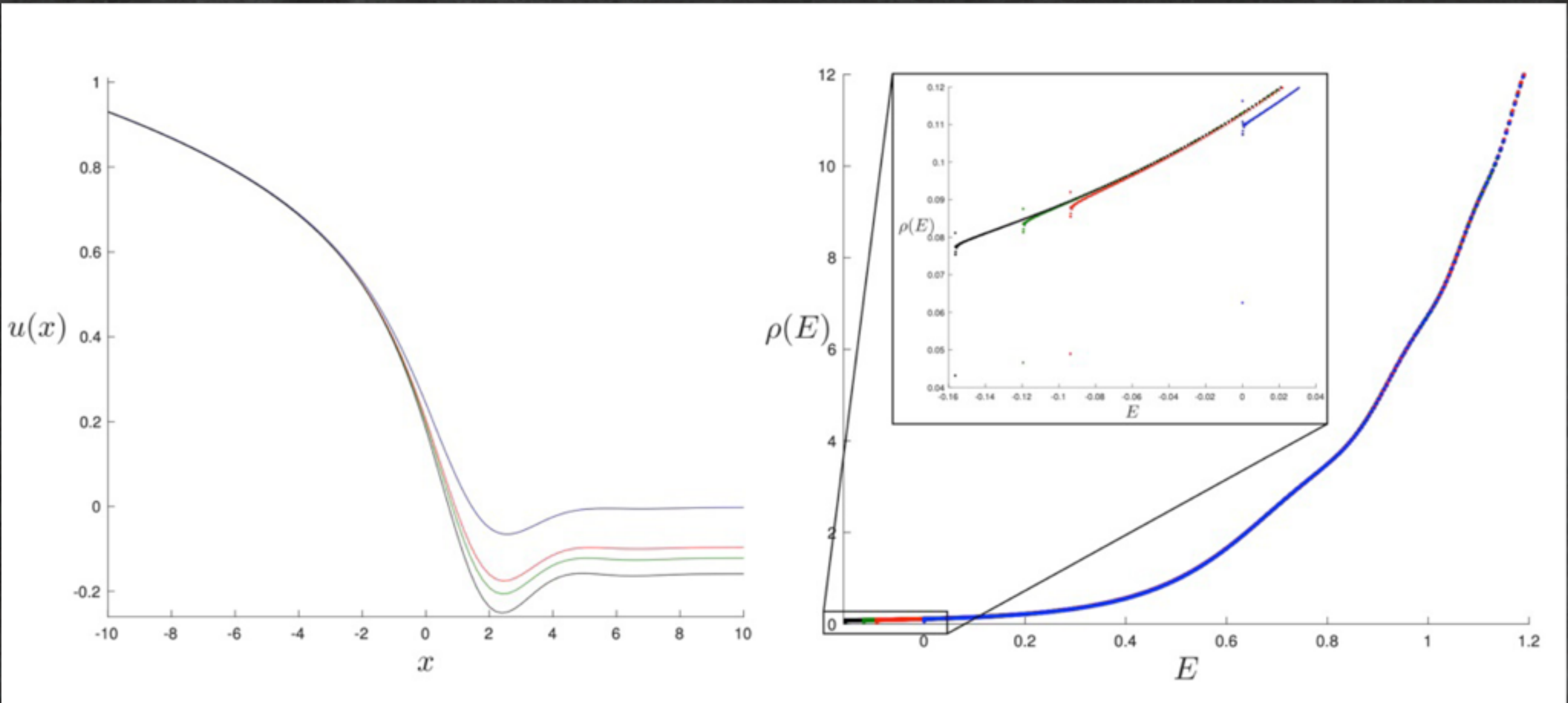
- Example A:

$$\boxed{t_k(\lambda) = \frac{\pi^{2(k-1)}}{2(k!)^2}\left(k + 2\pi^2\lambda\left(\alpha_1^{2k} - \alpha_2^{2k}\right)\right)}$$

- Example B:

$$\boxed{t_k(\lambda) = \frac{\pi^{2(k-1)}}{2(k!)^2}\left(k + 2\pi^2\lambda\alpha^{2k}\right)}$$

For Example B $\lambda < \lambda_c^*$



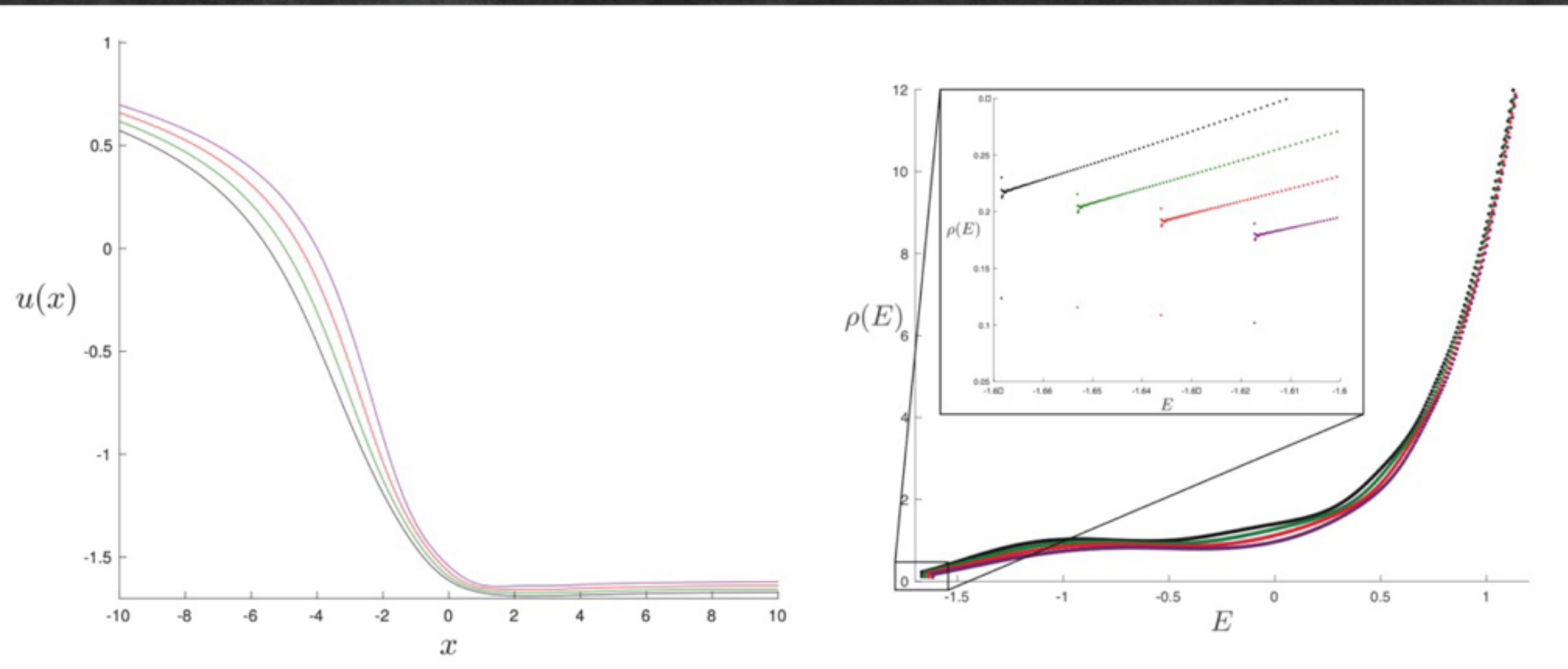
Full non-perturbative solutions found and spectrum solved.

For Example B

$\lambda_c^* < \lambda < \lambda_{upper}^*$

No solutions whenever $u_0(x)$ multivalued for $x<0$

For Example B $\lambda^*_{upper} < \lambda$ i.e, on other side of window.



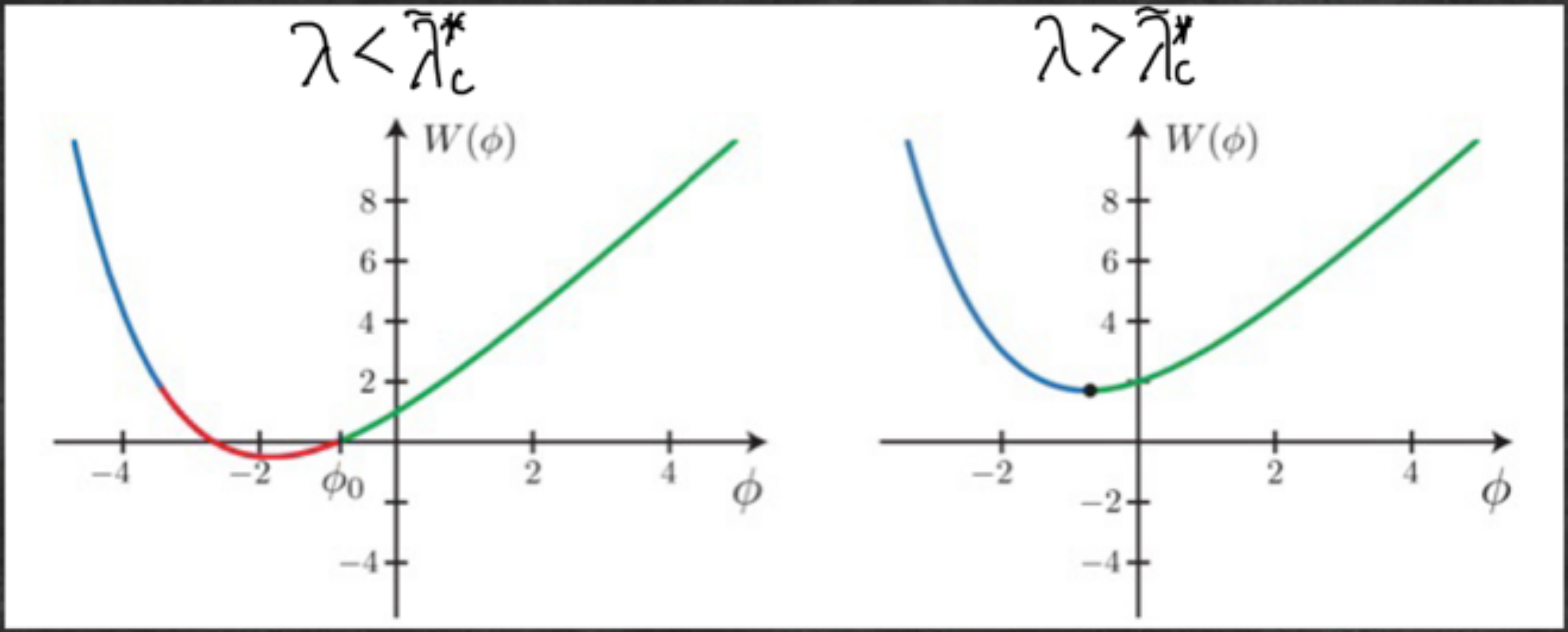
Stable non-perturbative solutions return when $u_0(x)$ multivaluedness disappears

Example B

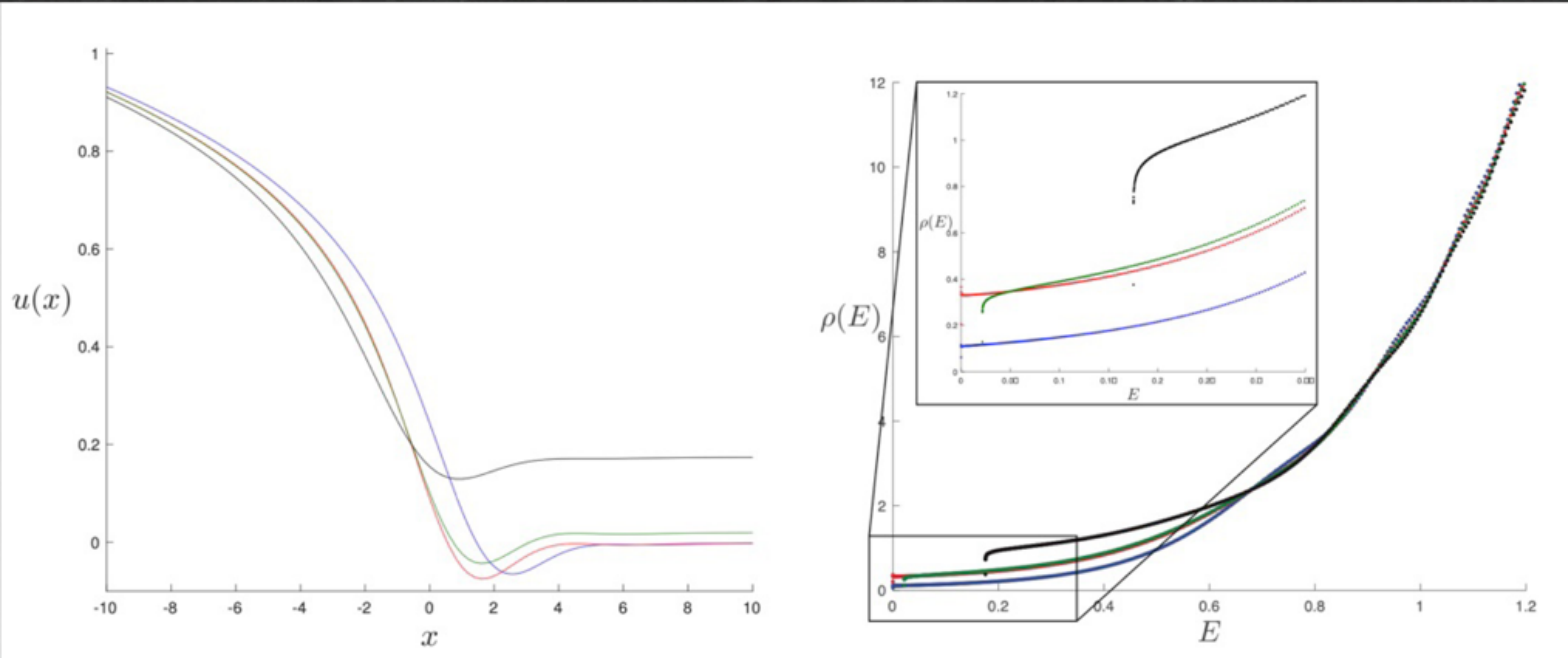
Remark:

Again, semi-classical regime sees something interesting at $\lambda > \tilde{\lambda}_c^*$

$\boxed{T=0 \text{ black holes disappear.}}$

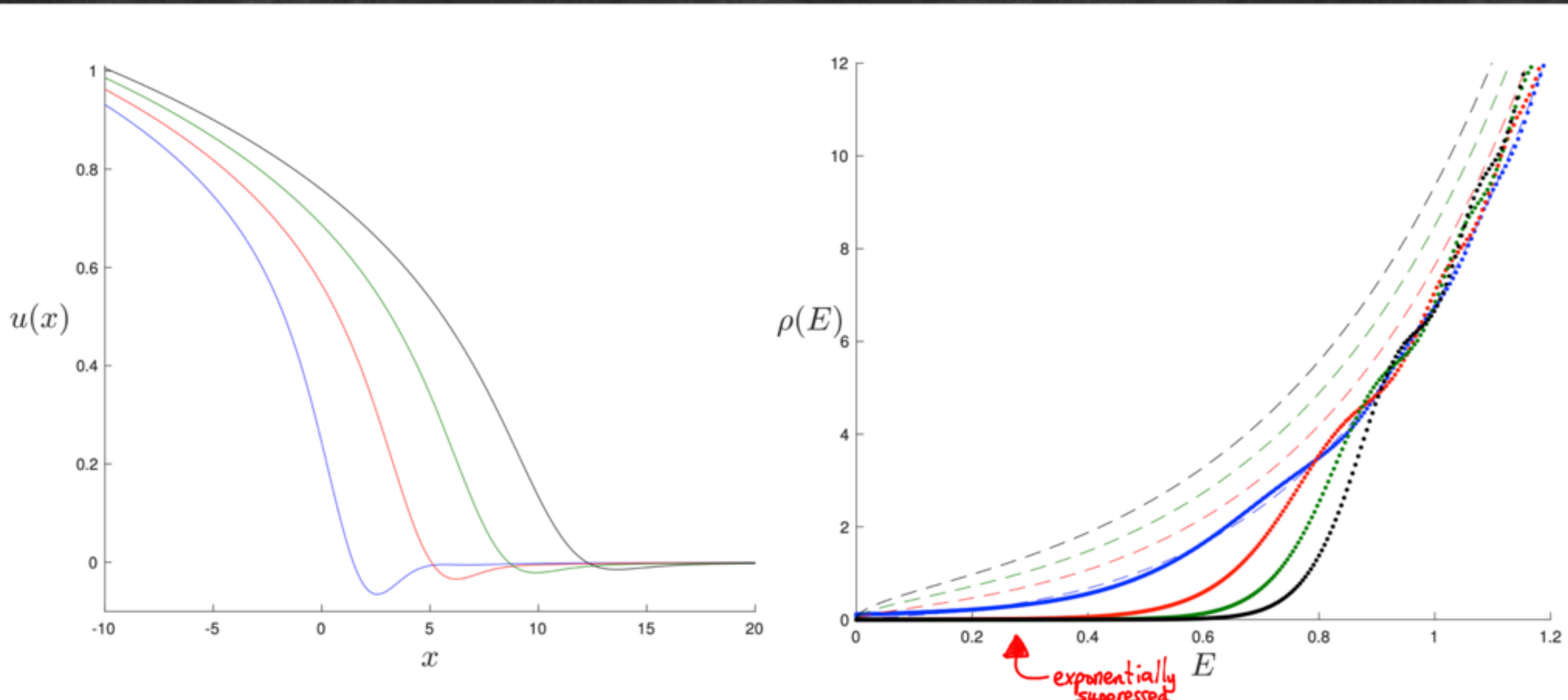


Example A, $\lambda > 0$.



Nice non-perturbative completions found for all λ. Above λ_c, an $E_0 \neq 0$ develops.

Example A, $\lambda < 0$.

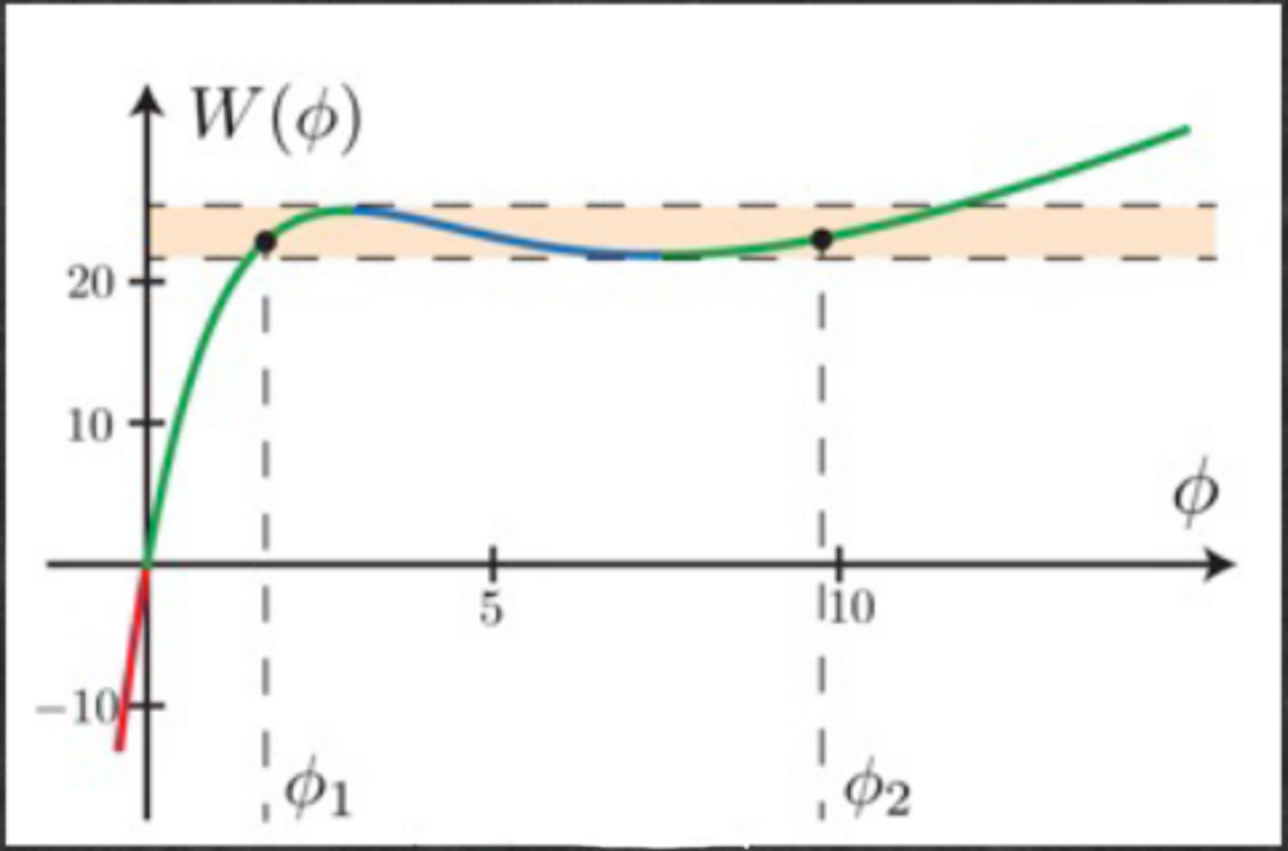


$\boxed{\text{An effective gap develops!}}$

(Non-perturbative effect)

Example A
remark:

Semi-classical analysis also reveals a gap for $\lambda < \tilde{\lambda}_c'$!



Summary:

Based on work with Felipe Rosso
CVJ + FR: 2011.06026

- Casting JT gravity + deformations into minimal string language is a powerful tool.
- It gives both perturbative and non-perturbative insights, by connecting to the underlying matrix model.
- Many fascinating non-perturbative phenomena to explore!
- Clearly, deformations of other members of the JT family can/should be explored with these tools.

Thank You!