# A general approximation scheme for addressing the black hole information loss paradox

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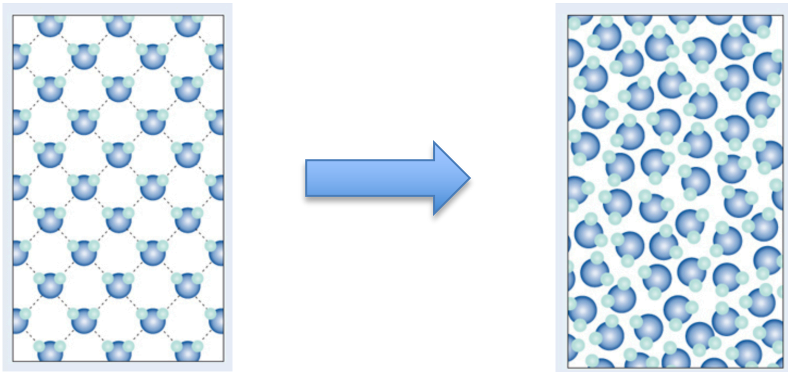
Island Hopping 2020

Based on work with Shreya Vardhan

arXiv: 2008.01089

# Equilibrated pure states



$|\Psi\rangle$

"an equilibrated pure state"

An initial pure state $|\Psi_0\rangle$

$$|\Psi\rangle = U|\Psi_0\rangle$$

Can use thermodynamic quantities to describe the state, which obey laws of thermodynamics.

Measurements on any small fraction of the system can be well approximated by those in an equilibrium density operator,

for example
$$\rho_e = \frac{1}{Z(\beta)} e^{-\beta H}$$

But if we use quantum informational properties of $\rho_e$
to approximate those of $|\Psi\rangle$

Leads to violation of unitarity.

Renyi and von Neumann entropies for some subsystem A
and its complement $\bar{A}$

$$|\Psi\rangle \qquad S_n^{(A)} = S_n^{(\bar{A})}, \quad n = 1, 2, \cdots$$

$$\rho_e \qquad S_n^{(A)} \neq S_n^{(\bar{A})}$$

This is precisely the situation we face in the evaporation of a black hole.

An evaporating black hole formed from collapse of a star in a pure state is in an equilibrated pure state (quasi-static).

apparent violation of unitarity if we use Renyi or von Neuman entropies of $\rho_e$ to approximate those of the black hole.

This is a version of the information paradox.

Unitarity: Page curve

Key question: what level of knowledge of $|\Psi\rangle$ is needed to obtain $S_n^{(A)}$ that is compatible with unitarity

(What inputs are needed to derive the Page curve?)

Is knowledge of $\rho_e$ enough?

Or do we need more detailed knowledge of $|\Psi_0\rangle$?

Recent derivation of the Page curve from gravity used only knowledge of $\rho_e$

Penington; Almheiri, Engelhardt, Marolf and Maxfield; Almheiri, Mahajan, Maldacena and Zhao; Penington, Shenker, Stanford and Yang; Almheiri, Hartman, Maldacena, Shaghoulian and Tajdini, …….

This has often been referred to as the "magic" or "miracle'' of Euclidean gravity path integrals.

Main messages to convey in this talk:

1. Introduce a general approximation scheme to find $S_n^{(A)}$ that is compatible with unitarity using equilibrium properties (i.e. $\rho_e$ ) alone.

Applies to general quantum many-body systems that equilibrate as well as gravity (can be used to address the information paradox) in the limit of large number of d. o. f.

The magic of computing the Page curve is more of magic in observables such entanglement entropies rather than the magic of Euclidean gravity path integrals.

2. Generalization of Page's results to finite temperature and infinite dimensional Hilbert space

Page's results: based on average over pure states for a finite dimensional Hilbert space, corresponding to infinite temperature

The approximation can in principle be systematically improved.

3. The approximation scheme provides a precise prescription for deriving replica wormholes and explains why they give results compatible with unitarity.

Replica wormholes can arise in a theory with a fixed Hamiltonian, no ensemble average is needed.

# Setup of the problem

$$|\Psi\rangle = U|\Psi_0\rangle$$ "an equilibrated pure state"

Macroscopically well described by an equilibrium density operator

$$\rho_e = \frac{1}{Z(\alpha)}\mathcal{I}_\alpha, \qquad Z(\alpha) = \mathrm{Tr}\mathcal{I}_\alpha$$ ("effective dimension")

$\alpha$ : equilibrium parameters such as temperature, chemical potential

1. Finite d, infinite temperature $\mathcal{I} = \mathbf{1}, \quad Z = d$

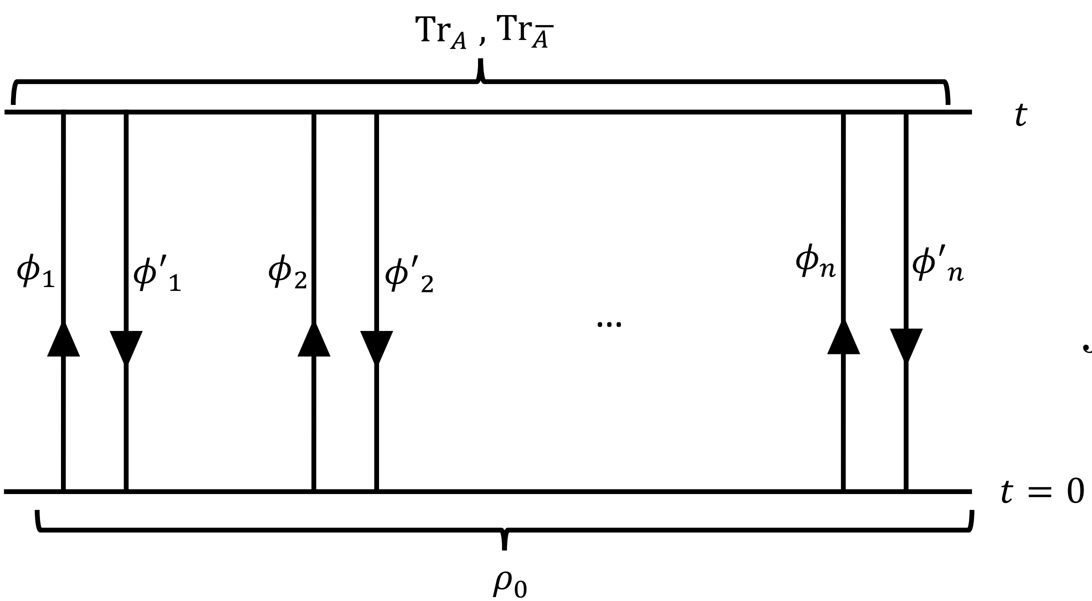
2. Microcanonical ensemble $\mathcal{I}_E = \sum_{E_n \in I} |n\rangle\langle n|, \quad Z(E) = \mathrm{Tr}\mathcal{I}_E = N_I$

3. Canonical ensemble $\mathcal{I}_\beta = e^{-\beta H}, \quad Z(\beta) = \mathrm{Tr}e^{-\beta H}$

# Renyi entropies

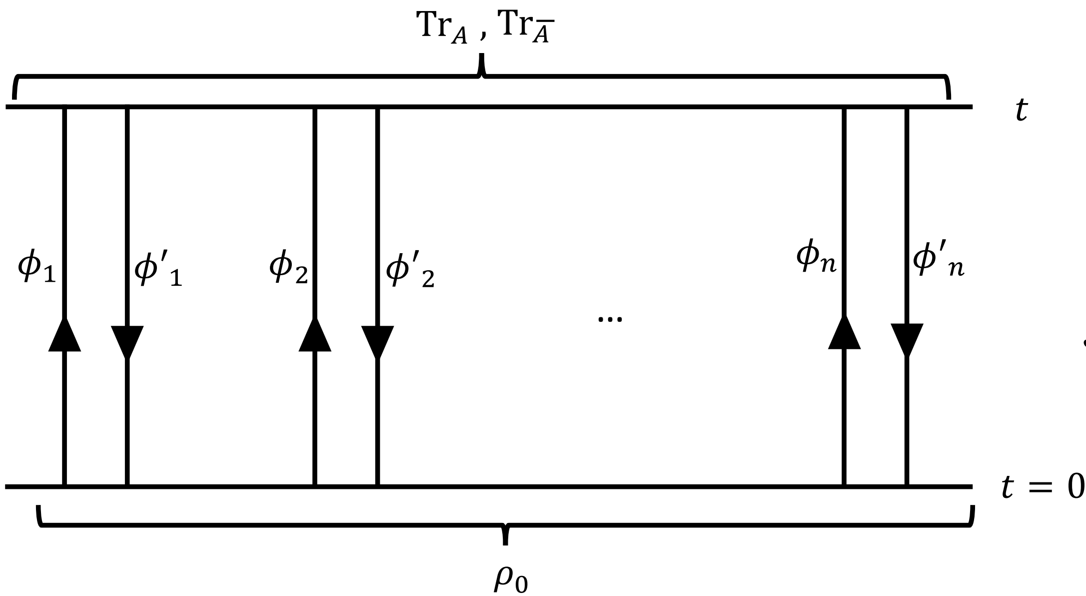
$$\mathcal{Z}_n^{(A)} = e^{-(n-1)S_n^{(A)}} = \mathrm{Tr}_A \left( \mathrm{Tr}_{\bar{A}} U \rho_0 U^\dagger \right)^n$$

$$\rho_0 = |\Psi_0\rangle\langle\Psi_0|$$


$$\int \exp\left( i \sum_{i=1}^{n} (S[\phi_i] - S[\phi'_i]) \right)$$

# Relevance of permutations




$$\int \exp\left( i \sum_{i=1}^{n} (S[\phi_i] - S[\phi'_i]) \right)$$

Zhou and Nahum, 2019

A special subset of configurations

$$\phi_i(t) = \phi'_{\sigma(i)}(t), \quad i = 1, \cdots, n, \quad \sigma \in \mathcal{S}_n$$

permutation group

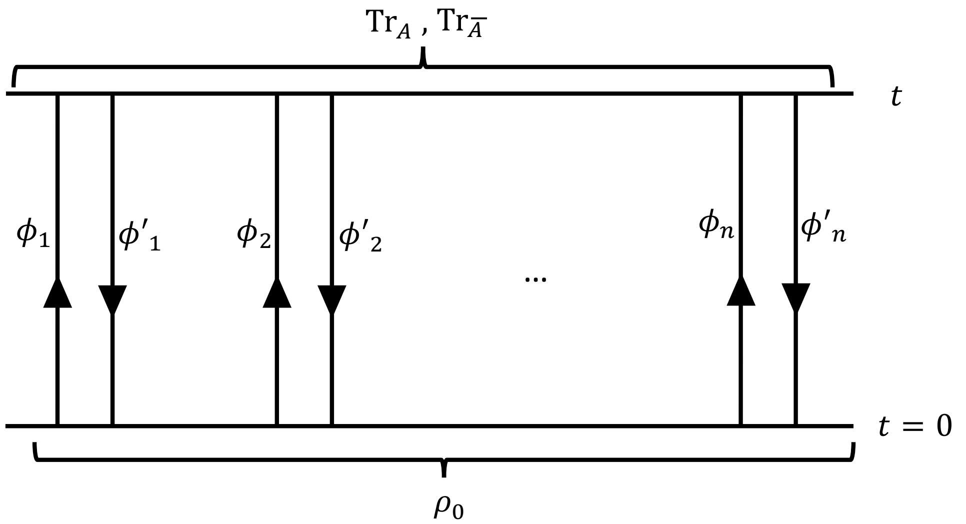
Intuitively, they could dominate the path integrals.

Lead to time-independent expressions

Suggests relevance of permutation group

# Transition amplitude in replica space

$$\mathcal{Z}_n^{(A)} = e^{-(n-1)S_n^{(A)}} =$$




a transition amplitude in $(\mathcal{H} \otimes \mathcal{H})^n$ with $(U \otimes U^\dagger)^n$

$$\mathcal{Z}_n^{(A)} = \langle \eta_A \otimes e_{\bar{A}} | (U \otimes U^\dagger)^n | \rho_0, e \rangle$$

$$|\rho_0, e\rangle, |\eta_A \otimes e_{\bar{A}}\rangle \in (\mathcal{H} \otimes \mathcal{H})^n$$

# Replica space

For any operator O acting on $\mathcal{H}$, we can define a set of n! states

$$|O,\sigma\rangle \in (\mathcal{H}\otimes\mathcal{H})^n \qquad \sigma\in\mathcal{S}_n$$

$$\langle i_1 \bar{i}'_1 i_2 \bar{i}'_2 \cdots i_n \bar{i}'_n | O,\sigma\rangle = O_{i_1 i'_{\sigma(1)}} O_{i_2 i'_{\sigma(2)}} \cdots O_{i_n i'_{\sigma(n)}}$$

$$O_{ij} = \langle i|O|j\rangle$$

For the identity operator

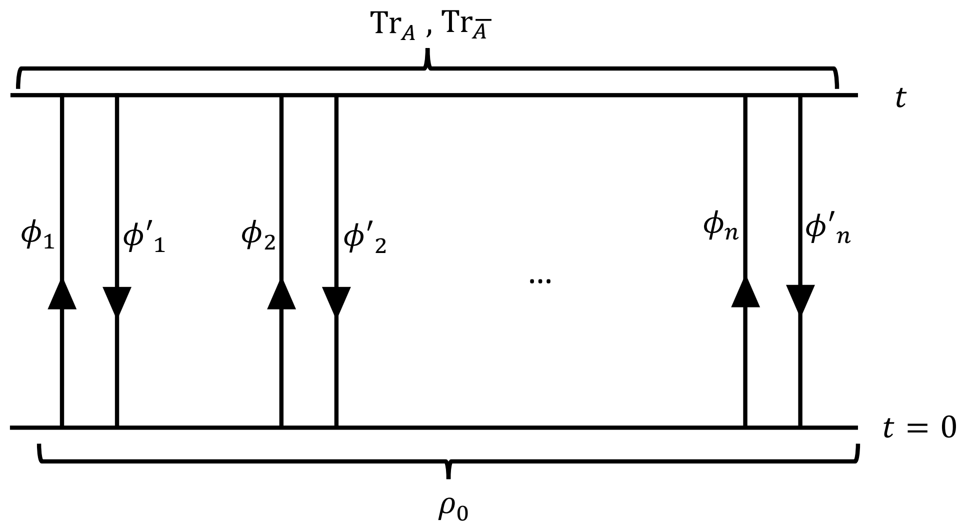
$$\langle i_1 \bar{i}'_1 i_2 \bar{i}'_2 \cdots i_n \bar{i}'_n | \sigma\rangle = \delta_{i_1 i'_{\sigma(1)}} \delta_{i_2 i'_{\sigma(2)}} \cdots \delta_{i_n i'_{\sigma(n)}}$$

e: identity permutation $\quad \eta = (n, n-1, \cdots, 1)$

$$\langle e|O,e\rangle = (\mathrm{Tr}O)^n, \quad \langle \eta|O,e\rangle = \mathrm{Tr}O^n$$

# Transition amplitude in replica space

$$\mathcal{Z}_n^{(A)} = e^{-(n-1)S_n^{(A)}} =$$




$$\mathcal{Z}_n^{(A)} = \mathrm{Tr}_A \left(\mathrm{Tr}_{\bar{A}} U \rho_0 U^\dagger\right)^n = \langle \eta_A \otimes e_{\bar{A}} | (U \otimes U^\dagger)^n | \rho_0, e \rangle$$

e: identity permutation $\qquad \eta = (n, n-1, \cdots, 1)$

# Equilibrium approximation

Consider: $|\mathcal{I}_\alpha, \sigma\rangle \in (\mathcal{H} \otimes \mathcal{H})^n$

$$P_\alpha = \frac{1}{Z_2^n} \sum_{\sigma,\tau} g^{\sigma\tau} |\mathcal{I}_\alpha, \sigma\rangle\langle\mathcal{I}_\alpha, \tau| \qquad g_{\tau\sigma} \equiv \frac{1}{Z_2^n} \langle\mathcal{I}_\alpha, \tau|\mathcal{I}_\alpha, \sigma\rangle$$

$$(U \otimes U^\dagger)^n |\mathcal{I}_\alpha, \sigma\rangle = |\mathcal{I}_\alpha, \sigma\rangle, \qquad (U^\dagger \otimes U)^n |\mathcal{I}_\alpha, \sigma\rangle = |\mathcal{I}_\alpha, \sigma\rangle$$

$$\mathbf{1} = P_\alpha + Q$$

$$\mathcal{Z}_n^{(A)} = \langle \eta_A \otimes e_{\bar{A}} | (U \otimes U^\dagger)^n | \rho_0, e\rangle$$

$$= \mathcal{Z}_{n,P}^{(A)} + \cancel{\mathcal{Z}_{n,Q}^{(A)}}$$ (cross terms vanish)

$$\approx \mathcal{Z}_{n,P}^{(A)}$$ (in the limit of large number of d. o. f. )

When the system is large and for a pure initial state:

$$\mathcal{Z}_n^{(A)} \approx \frac{1}{Z_1^n} \sum_{\tau \in \mathcal{S}_n} \langle \eta_A \otimes e_{\bar{A}} | \mathcal{I}_\alpha, \tau \rangle$$

$$Z_1 = \mathrm{Tr}\mathcal{I}_\alpha >> 1$$

Expressed solely in terms equilibrium density operator, independent of the initial state

Sum over all permutations

naïve equilibrium approximation corresponds to taking $\tau = e$ which gives answer which violates unitarity.

The approximation can also be applied to other observables in a pure state.

# General features (I)

$$\mathcal{Z}_n^{(A)} \approx \frac{1}{Z_1^n} \sum_{\tau \in \mathcal{S}_n} \langle \eta_A \otimes e_{\bar{A}} | \mathcal{I}_\alpha, \tau \rangle \equiv \sum_{\tau \in \mathcal{S}_n} \mathcal{Z}_n^{(A)}(\tau)$$

- Satisfy the unitarity constraint $\mathcal{Z}_n^{(A)} = \mathcal{Z}_n^{(\bar{A})}$

- Two particularly simple contributions

$$\mathcal{Z}_n^{(A)}(e) = \mathrm{Tr}_A \left( \mathrm{Tr}_{\bar{A}} \rho_{\mathrm{eq}} \right)^n = \mathcal{Z}_n^{(A,\mathrm{eq})} = e^{-(n-1) S_n^{(A,\mathrm{eq})}}$$

$$\mathcal{Z}_n^{(A)}(\eta) = \mathrm{Tr}_{\bar{A}} \left( \mathrm{Tr}_A \rho_{\mathrm{eq}} \right)^n = \mathcal{Z}_n^{(\bar{A},\mathrm{eq})} = e^{-(n-1) S_n^{(\bar{A},\mathrm{eq})}}$$

- When one of A or $\bar{A}$ is much smaller than the other,

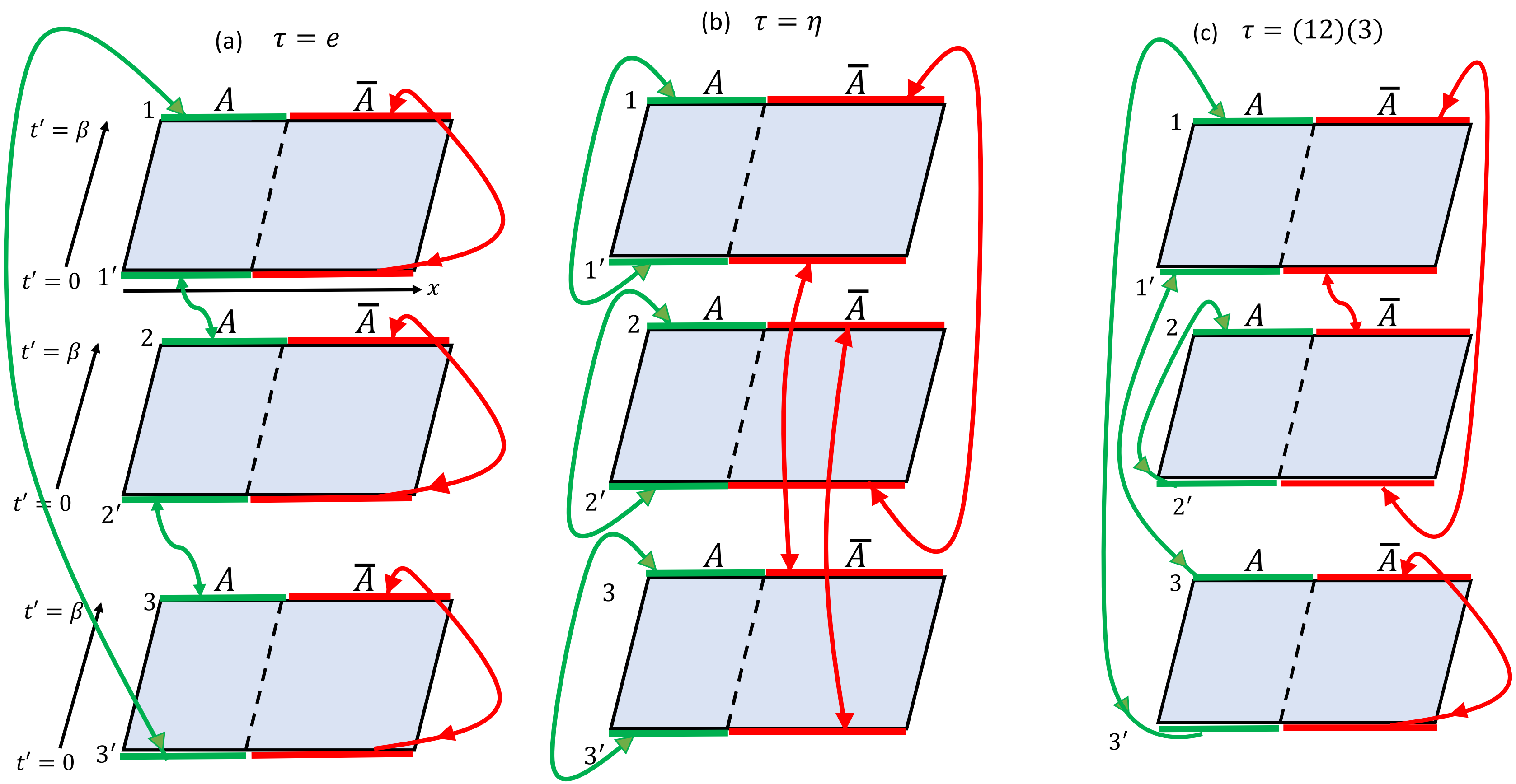
$$S_n^{(A)} = \min \left( S_n^{(A,\mathrm{eq})}, S_n^{(\bar{A},\mathrm{eq})} \right),$$

(generalization of Page's results to finite temperature and infinite dimensional Hilbert space.)

# General features (II)

$$\mathcal{Z}_n^{(A)} \approx \frac{1}{Z_1^n} \sum_{\tau \in \mathcal{S}_n} \langle \eta_A \otimes e_{\bar{A}} | \mathcal{I}_\alpha, \tau \rangle$$

can be represented in terms of Euclidean path integrals for n replicas of $\mathcal{I}_\alpha$



# Validity of approximation

Consider

$$\Delta^2 \equiv \left[\left(\mathcal{Z}^{(A)}_{n,Q}\right)^2\right]_{\text{eq app}} = \left[\left(\mathcal{Z}^{(A)}_{n} - \mathcal{Z}^{(A)}_{n,P}\right)^2\right]_{\text{eq app}} = \left[\left(\mathcal{Z}^{(A)}_{n}\right)^2\right]_{\text{eq app}} - \left(\mathcal{Z}^{(A)}_{n,P}\right)^2$$

The approximation is self-consistent if $\Delta \ll \mathcal{Z}^{(A)}_{n,P}$

Equivalently, this means that:

$$\left[\left(\mathcal{Z}^{(A)}_{n}\right)^2\right]_{\text{eq app}} \approx \left((\mathcal{Z}^{(A)}_{n})_{\text{eq app}}\right)^2$$

One finds that:

$$\frac{\Delta}{\mathcal{Z}^{(A)}_{n,P}} \sim \frac{1}{Z_1^{\frac{1}{2}}} \ll 1$$

Larger than subleading terms in $\mathcal{Z}^{(A)}_{n,P}$

It is also possible to derive the equilibrium approximation by making some minimal assumptions on spectral properties of the Hamiltonian and initial states.

in progress with Darius Zhengyan Shi and Shreya Vardhan

# Gravity systems

The equilibrium approximation can be applied to gravity systems in equilibrated pure states

$$\mathcal{Z}_n^{(A)} \approx \frac{1}{Z_1^n} \sum_{\tau \in \mathcal{S}_n} \langle \eta_A \otimes e_{\bar{A}} | \mathcal{I}_\alpha, \tau \rangle$$

Amplitudes in gravity

- If gravity respects the usual rules of quantum mechanics, the prescription leads to answers consistent with unitarity.
- May be implemented at semi-classical level using Euclidean path integrals
- For holographic systems, the boundary version of the prescription provides the boundary conditions for formulating the prescription for bulk gravity.
- Intuitively, "couplings" among different replicas could lead to replica wormholes, i.e. geometries connecting different replica "universes".
- Still an approximation even if one can calculate these amplitudes in exact QG

# A model of black hole evaporation

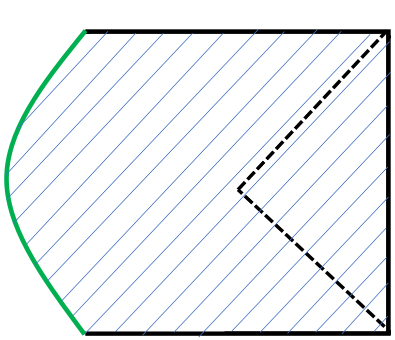
For the black hole mode considered by Penington, Shenker, Stanford and Yang

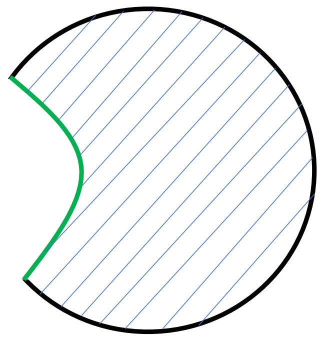
$$\mathcal{I}_\alpha = \mathbf{1}_R \otimes \mathcal{I}_\beta^{(B)}, \quad \mathrm{Tr}_R \mathbf{1}_R = N \quad Z_1^{(B)} = \mathrm{Tr}_B \mathcal{I}_\beta^{(B)} = e^{S_0} z_1(\beta)$$

Consider the regime: $e^{S_0}, N \to \infty, \qquad Ne^{-S_0} = \text{finite}$

$$\mathcal{Z}_n^{(R)} = \frac{1}{(Z_1^{(B)})^n} \sum_{k=1}^{n} N^{1-k} \sideset{}{'}\sum_{\sum_i m_i = k} N(\{m_i\}) \prod_{j=1}^{n} (Z_j^{(B)})^{m_j}$$

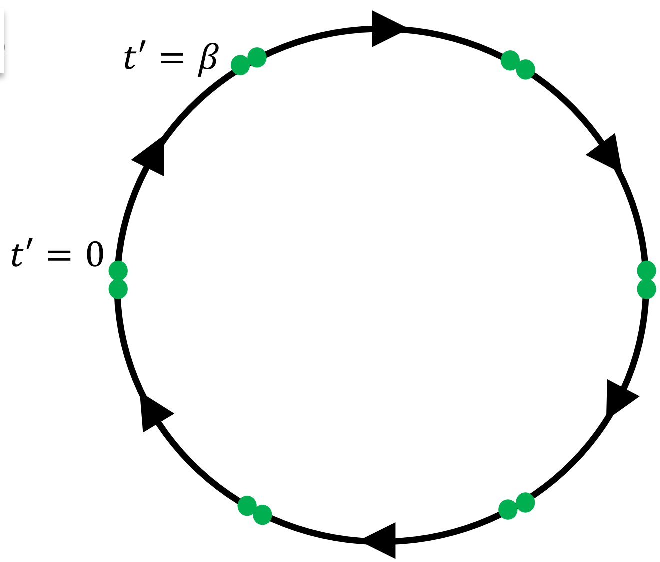
$$Z_j^{(B)} = \mathrm{Tr}_B \left(\mathcal{I}_\beta^{(B)}\right)^j$$

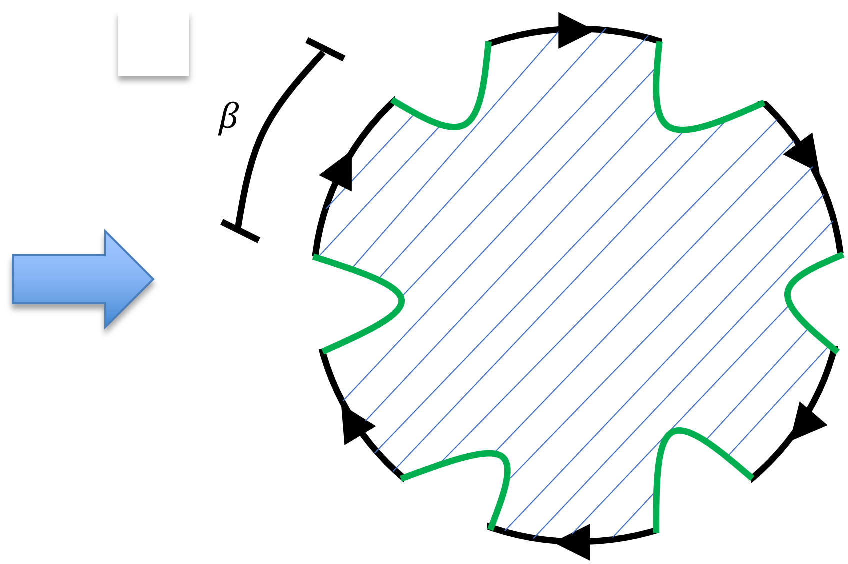




$$\mathcal{I}_\beta^{(B)} = f(H_B)e^{-\beta H_B}$$

$$Z_j^{(B)} =$$



- This leads to a derivation of the prescription of replica wormholes introduced by Penington, Shenker, Stanford and Yang.

  Provides precise prescription when and how to introduce replica wormholes.

- Explains the emergence of Euclidean gravity in describing entanglement entropies of a pure state, and why it is compatible with unitarity.

- the underlying mathematical structure: permutations, and for planar diagrams: non-crossing permutations. Closed form for Renyi entropies.

- Physically, replica wormholes can arise in systems with a fixed Hamiltonian. No averages needed.

Incompatibility between $\left(\mathrm{Tr}(\rho^{(R)})^2\right)_{\mathrm{wormhole}}$ and $(\rho_{ij}^{(R)})_{\mathrm{wormhole}}$

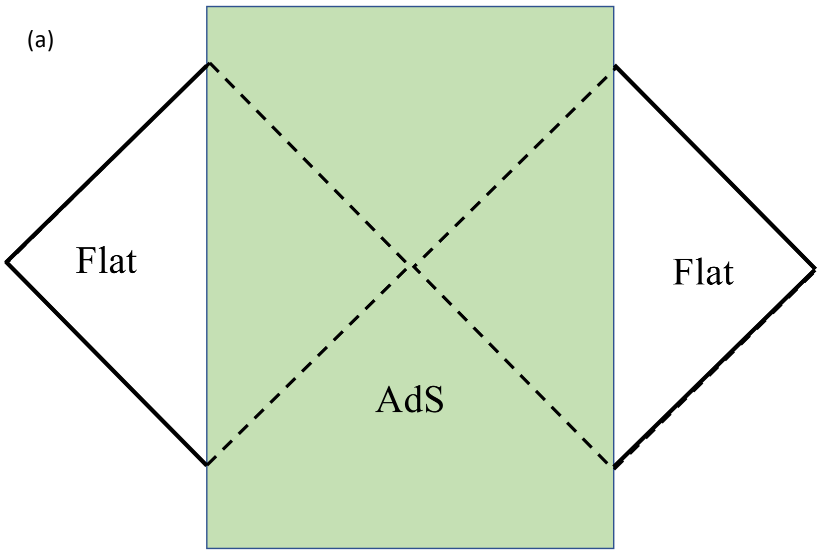
can be naturally explained in terms of

$$\left(\mathrm{Tr}(\rho^{(R)})^2\right)_{\mathrm{eq.approx.}} \qquad (\rho_{ij}^{(R)})_{\mathrm{eq.apporx.}}$$

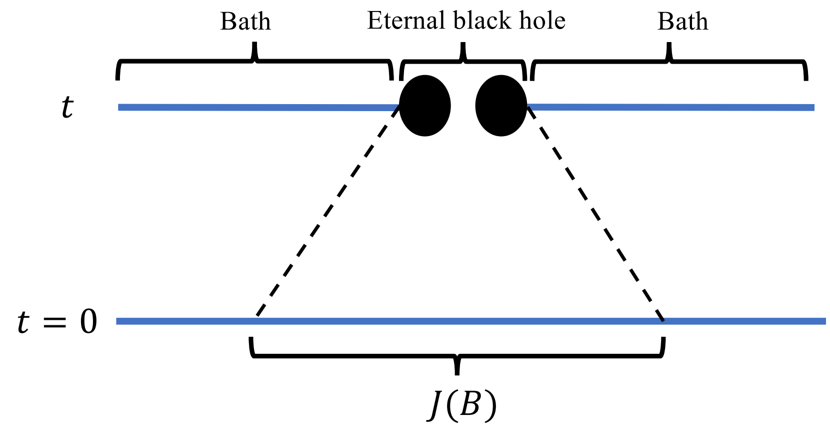
# Eternal BH coupled to bath

Almheiri, Mahajan Maldacena;
Rozali, Sully, Van Raamsdonk
Waddell, Wakeham

(a)



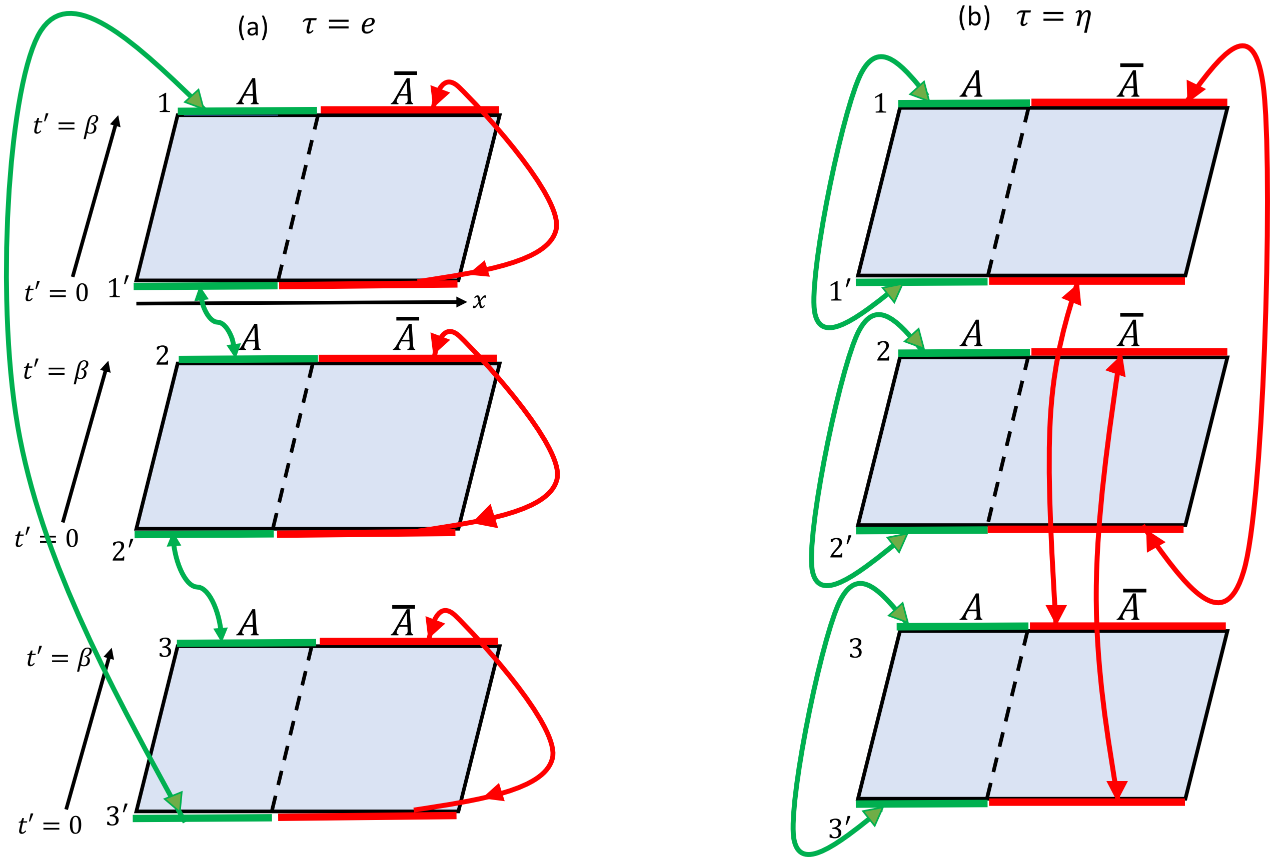

(b)




$$S_n^{(\mathrm{BH})} = S_n^{(\mathrm{bath})} = \min(2S_n^{(\mathrm{BH,th})}, S_n^{(\mathrm{bath,eq})}(t)\,)$$

Confirm the replica wormhole prescription of Almheiri, Hartman, Maldacena, Shaghoulian and Tajdini.

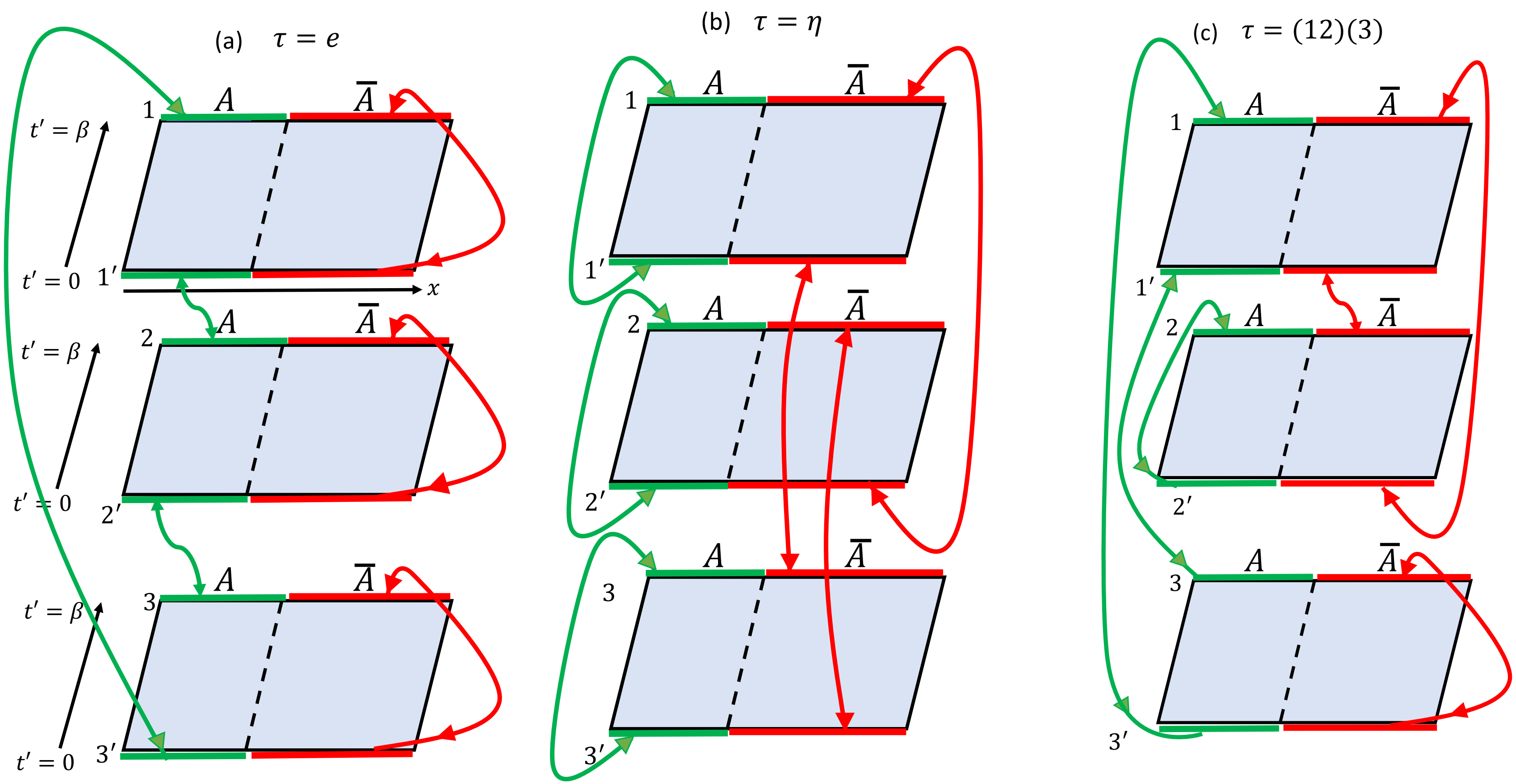
A= bath $\bar{A}$ = BH




(replica wormhole)

# Big black hole in AdS from collapse

One can also consider a big black hole in AdS resulted from gravitational collapse, e.g  Vaidya solution.



# Further applications

The equilibrium approximation provides powerful tools for studying quantum informational properties of general quantum many-body systems, both condensed matter and gravity systems.

For example:

Finite temperature effects on the Page time

quantum correlations among different parts of radiation and their Euclidean gravity descriptions

information transfer of from BH to radiation

Finite temperature generalization of Petz map

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in progress with Jonah Kudler-Flam, Shreya Vardhan

# Discussions

$$\mathcal{Z}_n^{(A)} = \langle \eta_A \otimes e_{\bar{A}} | (U \otimes U^\dagger)^n | \rho_0, e \rangle$$
$$= \mathcal{Z}_{n,P}^{(A)} + \mathcal{Z}_{n,Q}^{(A)}$$

To understand the corrections from the neglected part, we may need intrinsically Lorentzian path integrals in 2n copies of the original system.

Also interesting to study its behavior in some model systems.

In progress with Shreya Vardhan and Ziqi Zhou

# Caution: interpretations of wormholes

Euclidean wormholes arise in many contexts of semi-classical gravity discussions.

Even though they look alike on the gravity side, depending on contexts, they may be responsible for completely different physics, so may have completely different boundary theory interpretations.

For example, replica wormholes are relevant for physics at time scales much much shorter than those for wormholes arising in the computation of spectral form factors.

Our discussion of replica wormholes arising in systems with a fixed Hamiltonian may not say anything regarding wormholes in other physical contexts.

Thank You !