Holographic duality for averaged free CFTs

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Island Hopping ✦ CERN ✦ November 17, 2020

Based on

[arXiv 2006.04839] with Nima Afkhami-Jeddi, Henry Cohn, and Amir Tajdini

and coordinated with Alex Maloney's talk, coming up next

[arXiv 2006.04855] by Maloney and Witten

The proposal

[Afkhami-Jeddi, Cohn, TH, Tajdini ’20]
and [Maloney, Witten ’20]

Consider N free bosons in two dimensions.

This is a CFT with N^2 moduli.

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N free bosons averaged over moduli	$=$	$U(1)^N \times U(1)^N$ 3d Chern-Simons theory summed over topologies

“U(1) gravity”

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I will discuss the torus partition function

$$\int dM\, Z_{\mathrm{CFT}}(\tau, \bar{\tau}; M) = Z_{\mathrm{bulk}}(\tau, \bar{\tau})$$

Background:
Narain CFTs

N free bosons

$$S = \int d^2x \left(G_{\mu\nu}\delta^{ab}\partial_a X^\mu \partial_b X^\nu + iB_{\mu\nu}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu\right)$$

The is a CFT with central charge $c = N$ and partition function

$$Z(\tau, \bar{\tau}; \Lambda) = \frac{1}{|\eta(\tau)|^{2N}} \sum_{(p,\bar{p})\in\Lambda} q^{p^2/2} \bar{q}^{\bar{p}^2/2}$$

$$\Lambda = \text{Narain lattice in } \mathbb{R}^{N,N}$$

(even, self-dual)

example: single compact boson, $\Lambda = (\frac{m}{R} - nR,\ \frac{m}{R} + nR)$

All Narain CFTs have (at least) a current algebra

$$U(1)^N_{\text{Left}} \times U(1)^N_{\text{Right}}$$

The Dedekind eta functions account for descendants under this algebra

$$\frac{1}{|\eta|^{2N}} = \frac{(q\bar{q})^{-N/24}}{|\prod_m (1-q^m)|^{2N}}$$

So points on the lattice Λ correspond to primaries under $U(1)^N \times U(1)^N$

$$(p,\bar{p}) \in \Lambda \quad \Rightarrow \quad \Delta = \frac{1}{2}(p^2+\bar{p}^2), \quad \ell = \frac{1}{2}(p^2-\bar{p}^2)$$

Moduli space

All Narain lattices of dimension *N* are related by *O(N,N)* rotations

Moduli space of Narain CFTs

$$\mathcal{M} \cong \frac{O(N,N)}{O(N)\times O(N)\times O(N,N,\mathbb{Z})}$$

example: single compact boson $R \geq 1$

Averaging over lattices

Why average?

In hindsight: because it gives an interesting answer with a holographic interpretation.

Initially: studying modular bootstrap bounds on the spectral gap

$$\text{max. } \Delta_1$$

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In the theory of lattices, averaging is a standard trick to derive bounds on the spectrum at large *N*.

Recall the moduli space of *N* free bosons is

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Zamolodchikov metric = Haar measure for $O(N,N)$

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Average partition function:

$$\langle\langle Z(\tau,\bar{\tau})\rangle\rangle = \frac{1}{\mathrm{vol}(\mathcal{M})} \int dM \, Z_{\mathrm{Narain}}(\tau,\bar{\tau};M)$$

Haar O(N,N)

This converges for $N > 2$

The average was calculated by C. Siegel in 1951!

In CFT language, Siegel's result for the average density of states is

$$\rho_\ell(\Delta) = \frac{2\pi^N \sigma_{1-N}(\ell)}{\Gamma(N/2)^2 \zeta(N)} (\Delta^2 - \ell^2)^{N/2-1} \quad + \delta_{\ell 0}\delta(\Delta)$$

$$\ell = \text{spin}$$

$$\Delta = \text{dimension}$$

Comments

- Continuous
- Extends down to the unitarity bound $\quad \Delta \geq |\ell|$
- Vacuum state

Warm-up:
Averaging over Euclidean lattices of unit determinant

moduli space = $SL(N,\mathbb{R})/SL(N,\mathbb{Z})$

Claim: The average density of lattice vectors is

$$\rho(\vec{x}) = \delta(\vec{x}) + 1$$

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Proof [Siegel]:

$\mathbb{R}^N$ has two orbits under *SL(N,R)*

1. the origin {0}
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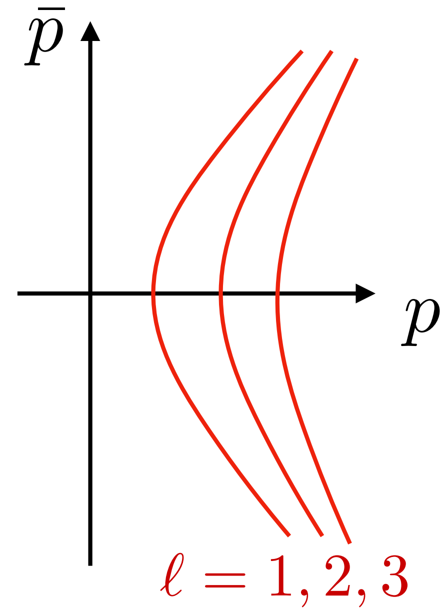
The action of *O(N,N)* preserves ℓ

So now we have an infinite set of orbits labeled by spin.

The orbits are the hyperboloids

$$p^2 - \bar{p}^2 = 2\ell$$

with "radial coordinate" Δ



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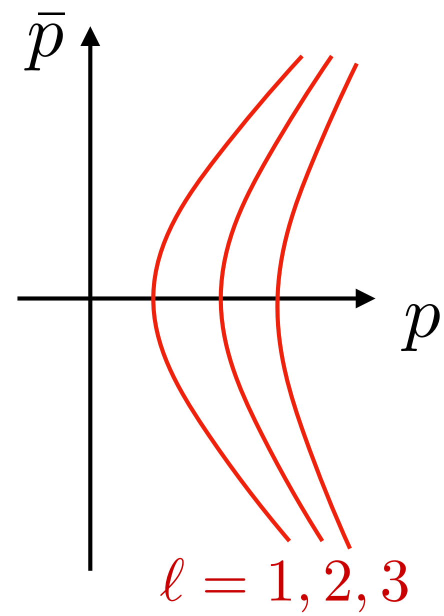

On each orbit, symmetry fixes

$$\rho_\ell(\Delta) \propto \sqrt{|g|} \propto (\Delta^2 - \ell^2)^{N/2-1}$$

metric on the hyperboloid

All that's left is to fix the coefficients; use the asymptotics.
This can be done by an explicit counting [Siegel] or by modular invariance.

Hardy-Littlewood circle method

$$Z \sim \sum_{\ell=-\infty}^{\infty} \int_{|\ell|}^{\infty} d\Delta\, \rho_\ell(\Delta) e^{-\beta\Delta + 2\pi i x \ell} \qquad \tau = x + \frac{i\beta}{2\pi}$$

$$\rho_\ell(\Delta) \sim \mathcal{L}^{-1}\left[\int_0^1 dx e^{-2\pi i \ell x} Z\right]$$

For $\beta \to 0$ this integral is dominated near the cusps of SL(2,Z)

$$\tau \sim \frac{a}{b} + i0^+$$

Use modular invariance to evaluate Z near cusps and sum over coprime *(a,b)*

(*cf.* the usual Cardy formula comes from a single cusp.)

$$\rho_\ell(\Delta) = \frac{2\pi^N \sigma_{1-N}(\ell)}{\Gamma(N/2)^2 \zeta(N)} (\Delta^2 - \ell^2)^{N/2-1} \quad + \delta_{\ell 0} \delta(\Delta)$$

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Spectral gap

$\Delta_1 =$ scaling dimension of the lightest nontrivial primary ("spectral gap")

Strictly speaking the gap of the averaged theory is zero. But for large *N,* the low-lying density of states is $\ll 1$ so most Narain CFTs have no light states!

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$$\boxed{\Delta_1 \approx \frac{N}{2\pi e}}$$

This is surprisingly large, and suggests looking for a holographic dual.

In fact this is the Cardy threshold for a theory with *U(1)^N* symmetry

$$S_{\text{Cardy}} = N \log\left(\frac{\Delta}{\Delta_1}\right)$$

$$\Delta_1 = \frac{N}{2\pi e}$$

This is analogous to the BTZ threshold in a theory with only Virasoro

$$S_{\text{Cardy}} = 2\pi\sqrt{\frac{c}{3}(\Delta - \Delta_1)}$$

$$\Delta_1 \approx \frac{c}{12}$$

Status report:
3d Pure Gravity

3d gravity is dual to a CFT with

$$c = \frac{3\ell}{2G_N} \gg 1$$

Assuming the graviton is the only massless field with spin, the chiral algebra in the CFT is Virasoro (nothing more).

A tentative definition of "pure" gravity is a theory with gap

$$\Delta_1 = O(c) \qquad \text{or} \qquad \Delta_1 \approx \frac{c}{12}$$

i.e., the only perturbative excitations are gravitons.

No such theory is known.

In 2007 Maloney and Witten studied the sum over saddles

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- The JT/Random matrix duality suggests an ensemble interpretation. There is growing evidence for this, but so far, we do not know how to define an "average CFT".

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Status: Unresolved

Let's replace

$$\text{Virasoro} \quad \rightarrow \quad U(1)^N$$

and try again.

First, we need a 3d theory for which the 1-loop partition function is the correct vacuum character.

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$U(1)^N \times U(1)^N$ Chern-Simons

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Each U(1) gauge field gives a U(1) current algebra at the boundary.

Therefore perturbatively,

$$Z_{\rm pert}(\tau, \bar{\tau}) = \chi_0^{U(1)^N} = \frac{1}{|\eta(\tau)|^{2N}}$$

e.g. [Porrati, Yu '19]

The tentative definition of bulk theory is this theory summed over topologies.

Poincare series:

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This Poincaré series is proportional to a *non-holomorphic Eisenstein series:*

$$Z_{\rm bulk}(\tau,\bar\tau) = ({\rm Im}\ \tau)^{-N/2}|\eta|^{-2N} E(\tau,\frac{N}{2})$$

$$E(\tau,s) = \sum_{\gamma\in SL(2,Z)/\Gamma_\infty} ({\rm Im}\ \gamma\tau)^s$$

The sum can be done explicitly to read off the density of states:

$$Z_{\text{bulk}}(\tau,\bar{\tau}) = \sum_{SL(2,Z)/\Gamma_\infty} Z_{\text{pert}}(\tau,\bar{\tau})$$

$$= \frac{1}{|\eta|^2} \sum_{\ell} \int_{|\ell|}^{\infty} d\Delta\, \rho_\ell(\Delta) q^{(\Delta-\ell)/2} \bar{q}^{(\Delta+\ell)/2}$$

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Siegel's measure on random Narain lattices!

"Siegel-Weil formula"

$$(\text{Eisenstein}) = \int \Theta$$

Recap

We have shown that on the torus,

N free bosons averaged over moduli	$=$	$U(1)^N \times U(1)^N$ 3d Chern-Simons theory summed over topologies

"U(1) gravity"

for $N > 2$

Remarks

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It has a composite boundary gravity,

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It has a large spectral gap,

$$\Delta_1 \approx \frac{N}{2\pi e}$$

compare: $\Delta_1^{\mathrm{BTZ}} \approx \dfrac{c}{12}$

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Compare: ordinary 3d gravity agrees perturbatively with SL(2,C) Chern-Simons but is not equivalent.

Note that perturbatively, compact U(1) = non-compact U(1).

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Thank you