More on the Average of Free Theories

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A. M. & E. Witten

N. Afkhami-Jeddi, H. Cohn, T. Hartman, & A. Tajdini

Also: work in progress with S. Collier, S. Datta, S. Duary, P. Kraus, P. Maity, . . .

Recap

Gravity in 3 Dimensions

AdS_3 gravity should be dual to a CFT_2 with $c = \frac{3\ell}{2G} \sim \frac{1}{\hbar}$.

One might think it is dual to a single compact, unitary CFT_2.

- Is it actually an ensemble average of many such CFTs?

Two pieces of evidence support this. . .

The Spectrum of 3D gravity

First, AdS_3 gravity should have a discrete spectrum.

But the torus partition function

$$Z_{GR}(T^2) \equiv \int_{\partial M = T^2(\tau)} Dg\ e^{-S} = \sum_{g_o} e^{-cS(g_o) + S^{(1)}(g_0) + \ldots}$$

$$\approx \sum_J \int d\Delta\ \rho_J(\Delta)\ q^{\Delta+J} \bar{q}^{\Delta-J}$$

appears to have a continuous spectrum $\rho_J(\Delta)$.

Perhaps this is because

$$Z_{GR}(T^2) = \langle Z_{\mathcal{C}}(T^2) \rangle \equiv \sum_{\text{CFTs } \mathcal{C}} Z_{\mathcal{C}}(T^2) p(\mathcal{C})?$$

A.M. & Witten
many others!

Euclidean Wormholes

Second, a CFT partition function on $\Sigma_1 \cup \Sigma_2$ will factorize.

But AdS_3 gravity has Euclidean wormholes

- Connected solutions M with $\partial M = \Sigma_1 \cup \Sigma_2$.

So it appears that

$$Z_{GR}(\Sigma_1 \cup \Sigma_2) = \int_{\partial M = \Sigma_1 \cup \Sigma_2} Dg \; e^{-S} \neq Z_{GR}(\Sigma_1) Z_{GR}(\Sigma_2)$$

Perhaps this is because

$$Z_{GR}(\Sigma_1 \cup \Sigma_2) = \langle Z_{\mathcal{C}}(\Sigma_1) Z_{\mathcal{C}}(\Sigma_2) \rangle \neq \langle Z_{\mathcal{C}}(\Sigma_1) \rangle \langle Z_{\mathcal{C}}(\Sigma_2) \rangle ?$$

Maldacena & Maoz, ...

Random CFT

Goal: Develop a theory of random CFTs

The data that define a CFT are

- A list of operator dimensions Δ_i and spins J_i
- A list of OPE coefficients C_{ijk}

for primary operators.

To average over this data we must

1. Determine the allowed values of $\{C_{ijk}, \Delta_i, J_i\}$.
2. Fix a probability distribution over these allowed values.
3. Compute ensemble averages and compare to gravity in AdS_3.

This is hard!

A Simpler Case

Consider a simple class of CFTs,

- $c = N$ and $U(1)^N \times U(1)^N$ current algebra.

There is a natural distribution on the space $\mathcal{M}_N$ of such CFTs.

We can compute the average of *any* observable $\langle \cdot \rangle$ over $\mathcal{M}_N$.

The result looks like the path integral

$$\langle Z(\Sigma) \rangle = \sum_{g_o} e^{-cS(g_o)+S^{(1)}(g_o)}$$

of an exotic theory of 3D gravity, "$U(1)$ Gravity."

With certain assumptions, we can enumerate all saddles and compute all loop corrections in this theory of gravity.

Plan for Today

- The Siegel-Weil Formula and CFT Averaging
- Bulk Gravity Interpretation
- Speculations & Conclusions

Averaging over Moduli Space

N Frozons

The most general action of N free bosons is

$$I = \int d^2z \ \left(G_{pq}\,\partial X^p\,\bar{\partial} X^q + B_{pq}\,dX^p \wedge dX^q\right)$$

The $\{G_{pq}, B_{pq}\}$ are coordinates on Narain's moduli space

$$\mathcal{M}_N = O(N,N,\mathbb{Z})\backslash O(N,N)/O(N)\times O(N)$$

The Zamolodchikov metric is the homogeneous one:

$$ds^2 = G^{mn}G^{pq}\left(dG_{mp}dG_{nq} + dB_{mp}dB_{nq}\right)$$

We can average over $m \in \mathcal{M}_N$ with this measure:

$$\langle\cdot\rangle = \frac{1}{Vol(\mathcal{M}_N)}\int_{\mathcal{M}_N} d\mu\,(\cdot)$$

Partition Function

For a general surface Σ of genus g, the frozon partition function is

$$Z(m, \Omega) = \frac{1}{(\det \Delta_0)^{N/2}} \Theta(m, \Omega)$$

where Δ_0 is the scalar Laplacian on Σ and $\Omega_{ij} = \int_{B_j} \omega_i$ is the period matrix of Σ.

The "Siegel-Narain" theta function is the momentum-winding sum

$$\Theta(m, \Omega) = \sum_{\mathbf{n}, \mathbf{w} \in \mathbb{Z}^{g \times N}} e^{-\pi y_{ij} \left(G^{pq} v^i{}_p v^j{}_q + G_{pq} w^{ip} w^{jq} \right) + 2\pi i x_{ij} n^i{}_p w^{jp}}$$

where $v^i{}_p = n^i{}_p + B_{pq} w^{iq}$ and $\Omega = x + iy$.

It is a function on $\mathcal{M}_N \times \mathcal{H}_g$, where $\mathcal{H}_g = \{\Omega_{g \times g} | \mathrm{Im}(\Omega_{g \times g}) > 0\}$ is Siegel upper half-space.

Siegel-Weil at Higher Genus

The Siegel-Weil formula for the average

$$\langle Z(m,\Omega)\rangle = \frac{1}{(\det \Delta_0)^{N/2}} E_{N/2}(\Omega)$$

gives a degree g Siegel-Eisenstein series:

$$E_{N/2}(\Omega) = \sum_{\gamma \in Sp(2g,\mathbb{Z})/P} (\det \mathrm{Im}\, \gamma\Omega)^{N/2} = \sum_{(C,D)=1} \left(\frac{\det \mathrm{Im}\, \Omega}{|C\Omega + D|^2} \right)^{N/2}$$

which is a sum over the modular group $Sp(2g,\mathbb{Z})$

$$\gamma\Omega = (A\Omega + B)(C\Omega + D)^{-1}, \quad \text{with} \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g,\mathbb{Z})$$

Siegel-Weil Formula

One can prove this using (since $T \sim J^2$)

$$\left(\Delta_{\mathcal{H}_g} - \Delta_{\mathcal{M}_N} + \frac{gN(N-g-1)}{4}\right)\left((\det \operatorname{Im}\Omega)^{N/2}\,\Theta(m,\Omega)\right) = 0.$$

Integrating over $\mathcal{M}_N$ and discarding boundary terms gives

$$\left(\Delta_{\mathcal{H}_g} + \frac{gN(N-g-1)}{4}\right)\left((\det \operatorname{Im}\Omega)^{N/2}\,\langle\Theta(m,\Omega)\rangle\right) = 0$$

Our Eisenstein series is the only modular invariant solution with the right boundary conditions.

Bulk Gravity Interpretation

$U(1)$ Gravity

Our bulk theory has $2N$ perturbative $U(1)$ Chern-Simons fields.

Since $N \sim 1/\hbar$ there is no distinction between "classical" and "loop" effects.

- But $U(1)$ Chern-Simons is simple enough to study on any 3-manifold.

The simplest bulk geometry Y with $\partial Y = \Sigma$ is a handlebody:

For any Σ there is a unique hyperbolic metric on $Y = \mathbb{H}_3/\Gamma$.

Bulk Computation

The one loop determinant of $U(1) \times U(1)$ Chern-Simons is

$$Z^{CS}(Y) = \frac{(\det D_0)^{3/2}}{(\det D_1)^{1/2}} = e^{-\frac{\mathrm{Vol}(Y)}{6\pi}} \prod_{\gamma \in \pi_1(Y)} \left(\prod_{n=1}^{\infty} \frac{1}{|1 - {q_\gamma}^n|^2} \right)$$
$$= \sqrt{\frac{\det \mathrm{Im}\,\Omega}{\det \Delta_0}}$$

This is one of the terms in $\langle Z(m, \Omega) \rangle$!

It is a $U(1) \times U(1)$ current algebra block on Σ.

Our theory does not include a separate Einstein-Hilbert term:

- In a sense, it is induced one at one-loop.

Giombi, A.M. & Yin
Zograf; McIntyre & Takhtajan

Non-Perturbative Corrections

The sum over geometries comes from the Eisenstein series:

$$E_{N/2}(\Omega) = \sum_{\gamma \in Sp(2g,\mathbb{Z})/P} (\det \operatorname{Im} \gamma\Omega)^{N/2} \sim \sum_{g_o} e^{-\frac{1}{\hbar} S(g_o)}$$

The coset $Sp(2g, \mathbb{Z})/P$ labels the handlebodies Y which fill in Σ:

The choice of γ labels which cycles are contractible in the bulk.

Sum over Topologies

This is an all-genus version of the black hole Farey tail.

Dijkgraaf, Maldacena, Moore & Verlinde

- This sum enforces $Sp(2g, \mathbb{Z})$ modular invariance.

When $g > 1$ there are non-handlebody hyperbolic manifolds (related to wormholes):

- Exponentially suppressed, but non-zero Einstein action.

But our result can be written as a sum over handlebodies:

- Perhaps $U(1)$ gravity cannot distinguish handlebodies from non-handlebodies.
- Or perhaps non-handlebodies simply aren't solutions.

Speculations & Confusions

Disconnected Geometries

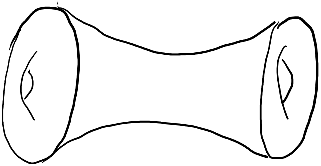
Siegel-Weil formulas work when the boundary is disconnected, by evaluating the Eisenstein series with

$$\Omega(\Sigma \cup \Sigma') = \begin{pmatrix} \Omega_\Sigma & 0 \\ 0 & \Omega_{\Sigma'} \end{pmatrix}$$

The result does not factorize:

$$\langle Z(\Sigma \cup \Sigma') \rangle \neq \langle Z(\Sigma) \rangle \langle Z(\Sigma') \rangle$$

This is interpreted as coming from wormholes:



These "quasi-Fuchsian" conformal blocks give (exponentially) small but non-zero contributions.

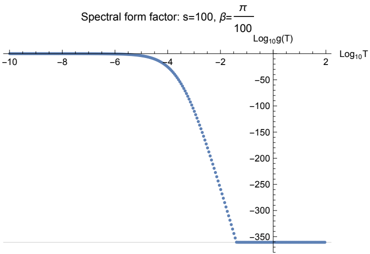
Spectral Form Factor

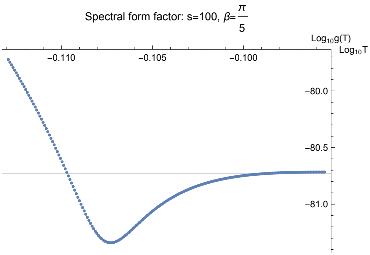
We can compute $\langle Z(\beta + it)Z(\beta - it)\rangle$.

This has the expected Plateau behaviour at $t \to \infty$:

$$\langle Z(\beta + it)Z(\beta - it)\rangle \to Z(2\beta)$$

But no linear ramp:





w/ S. Collier (in progress)

Cotler & Jensen

The Sum over Geometries Diverges

The "sum over geometries" for

$$\langle Z_{T^2}(\tau_1) \dots Z_{T^2}(\tau_n) \rangle$$

diverges when $n > N - 1$.

Perhaps gravity only computes "coarse" observables with n small:

- "Refined" statistics are not independent random variables at finite N?

Perhaps "doubly non-perturbative" effects are necessary to render the theory sensible at finite N?

Or perhaps we must define the sum via analytic continuation?

Other Ensembles of CFTs?

Other measures on Narain moduli space:

- Using the "theta correspondence"?
- Or slices of moduli space?

Other spaces of CFTs:

- Minimal models?
- Symmetric Products?
- Orbifolds?
- Enhanced chiral algebras?
- Classical/Quantum Codes?

Chiral CFTs and holomorphic Eisenstein series?

- Discrete average: $E_{N/2}(\Omega) = \sum_{\Lambda} \frac{1}{|Aut(\Lambda)|}\theta_{\Lambda}(\Omega)$

The Narain ensemble is perched at the edge of chaos: $c = c_{currents}$

- Can we find other, more chaotic examples?

Pure Gravity

Is pure 3D gravity an average over all CFTs?

$$Z_{GR}(\Sigma) \equiv \int_{\partial M=\Sigma} Dg\ e^{-cS} \stackrel{?}{=} \sum_{CFTs\ \mathcal{C}} \frac{1}{|Aut(\mathcal{C})|} Z_{\mathcal{C}}(\Sigma)$$

Does this imply that a typical large c CFT is sparse (or extremal)?

- i.e. that $\Delta_1 = \mathcal{O}(c)$?

Is pure gravity an "averaged" solution to the bootstrap?