# Some comments on wormholes and factorization

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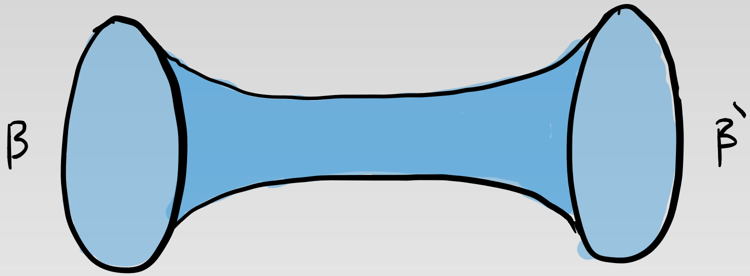
- In conventional AdS/CFT, spacetime wormholes lead to the "Factorization Problem" [Maldacena, Maoz].
- For example, the "cylinder" (in two dimensions) appears to prevent products of partition functions from factorizing.



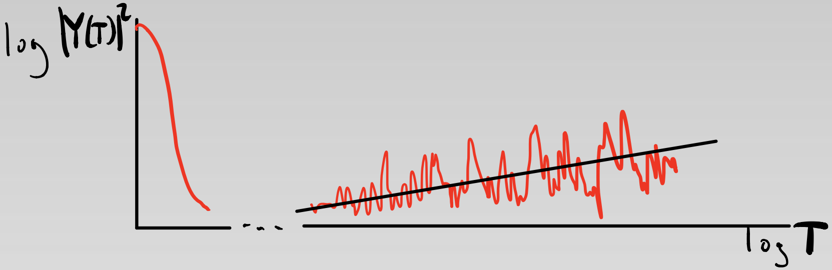
- Perhaps spacetime wormholes should not be included in the bulk path integral, or somehow sum to zero?

- However, there are indications that spacetime wormholes should play a nontrivial role. For other quantities, for which there is no "factorization problem", they seem to give sensible answers even in theories with a fixed Hamiltonian.

- The page curve of Hawking radiation [AHMST, PSSY], and the ramp in the *time-averaged* spectral form factor [P.S., Shenker, Stanford] are both quantities for which there are indications spacetime wormholes give a sensible answer, but those same spacetime wormholes lead to factorization problems in closely related quantities. [P.S., Shenker, Stanford; Stanford]

# Wormholes and the spectral form factor

- In this talk we will often use the analytically continued partition function $Z(\beta + iT)$, or its fixed-energy (with a smooth window) counterpart $Y_{E,\Delta E}(T)$, as an example.
- The mod squared of this quantity (spectral form factor) is noisy for sufficiently long times.



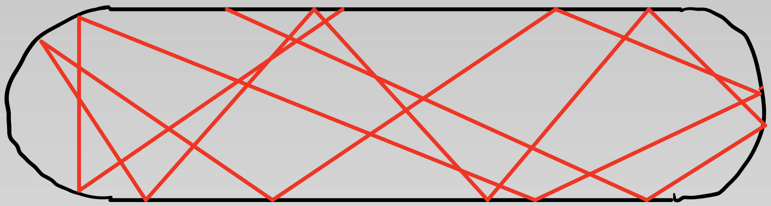

- The time-averaged $\langle |Y(T)|^2 \rangle_{\delta T}$ is smooth, and the answer given by wormholes matches our expectations for this quantity (as well as higher moments) based on random matrix universality [P.S., Shenker, Stanford].
- In a bulk description, how do we describe the noise?

- We seek an “effective description” of gravity with “microstate noise”, which captures some aspects of the noise and includes wormholes.

- Roughly analogous to “hydrodynamics with quantum noise” [Stanford], or hydrodynamics souped up to include Brownian motion.

# Periodic orbits and Berry's diagonal approximation

- An analogy that may be useful is the computation of the spectral form factor in a non-gravitational system - chaotic billiards.



- The path integral for $Y(T)$ can be done by saddle point; sum over orbits of period $T$

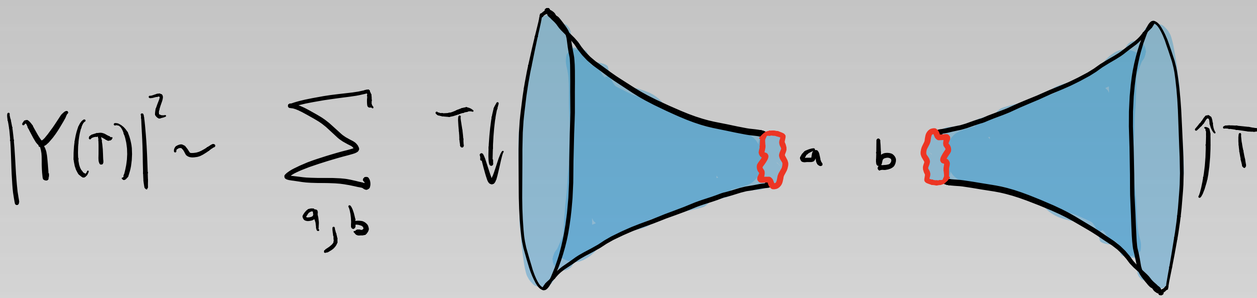
$$Y(T) \sim \sum_a e^{iS_a}$$

- The mod squared is a double-sum over orbits. Time averaging leaves only the diagonal part of the sum [Berry].

$$\langle |Y(T)|^2 \rangle_{\delta T} \sim \sum_{a,b} \langle e^{i(S_a - S_b)} \rangle_{\delta T} \longrightarrow \sum_a 1 \sim T$$

- We will think of this diagonal piece as analogous to the wormhole.

- This motivates a cartoon of factorization in AdS/CFT; we view the wormhole as a diagonal part of a sum over "broken cylinders", and add in the off-diagonal terms.




- Is there evidence for this picture in a gravitational theory? We will find that this is related to an effective description for some theories.
- In this cartoon, the wormhole is *emergent*, not a *fundamental* ingredient in the calculation. Factorization is manifest.

# Background about spacetime wormholes and ensemble averaging

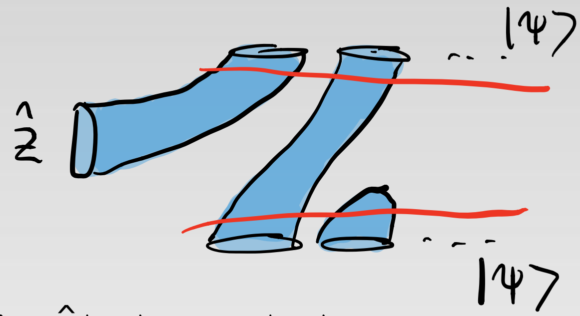
- Now we will move on to discussing the factorization problem in some simple gravitational theories which are dual to *ensemble averages* of boundary theories - different than conventional, non-averaged AdS/CFT.
- We can study factorization by focusing on a single member of the ensemble.
- In these theories, wormholes are *fundamental*.
- We first illustrate this using a simple toy model of gravity introduced by Marolf and Maxfield- a topological theory of two-dimensional surfaces.
- This model builds on old ideas of Coleman, Giddings, and Strominger.

- Instead of products of partition functions $Z(\beta)$, we compute *correlation functions* of a boundary operator $\hat{Z}$ in states $|\psi\rangle$ of closed universes.

$$\langle NB|\hat{Z}^3|NB\rangle \supset$$

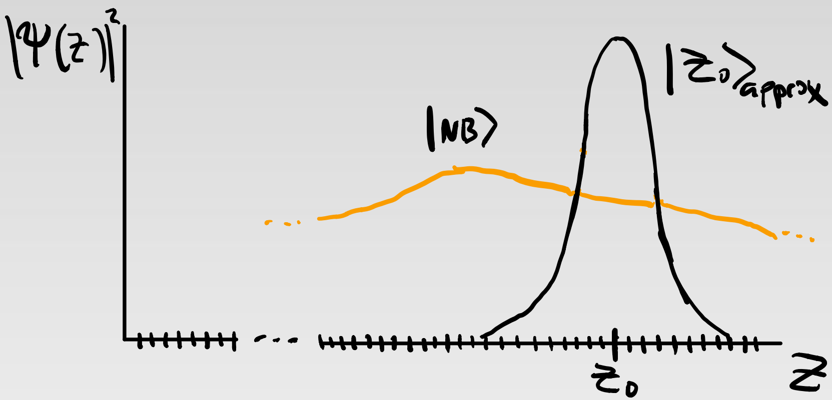

- In the No-Boundary state $|NB\rangle$ (like the vacuum of the closed universes), correlation functions do not factorize.
- In other states, $\hat{Z}|NB\rangle$, $\hat{Z}^2|NB\rangle$, or generally $|\psi\rangle \equiv \psi(\hat{Z})|NB\rangle$, we include "state boundaries" in the path integral.

$$\langle\psi|\hat{Z}|\psi\rangle \supset$$




- In eigenstates ("$\alpha$ states"), $\hat{Z}|z_0\rangle = z_0|z_0\rangle$, correlation functions factorize. The eigenstate corresponds to a single member of an ensemble of boundary theories.

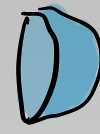
- How do correlation functions factorize in an eigenstate?
- Surfaces with many boundaries and high genus complicate the story; in an exact eigenstate, these surfaces are not suppressed order by order in the topological expansion.
- In *approximate* eigenstates, these surfaces are not important, and we can ignore them.
- Approximate Gaussian eigenstates ignore the effects from the discreteness of the spectrum of $\hat{Z}$, and correspond to only keeping disks and cylinders. The "Coleman-Giddings-Strominger" model.




- Study an approximate version of the factorization problem.

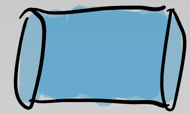
- In the Gaussian disk-and-cylinder approximation, $\hat{Z}$ is like the coordinate of a (shifted) harmonic oscillator. It is useful to introduce creation and annihilation operators, $a, a^\dagger$, $[a, a^\dagger] = \text{Cyl}$, such that

$$\hat{Z} = \text{Disk} + a + a^\dagger$$

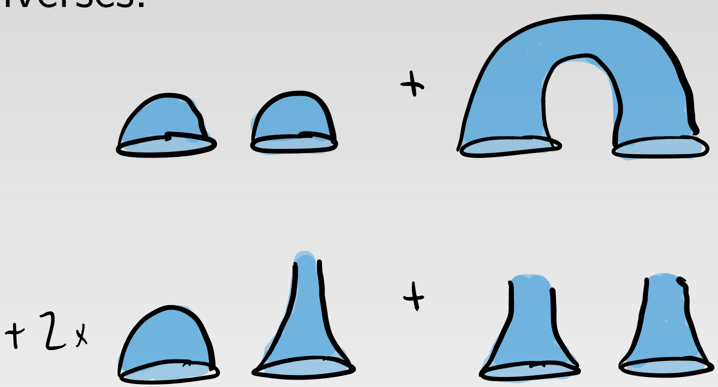
$\langle \hat{Z} \rangle_{NB} =$ 

and

$$a|NB\rangle = 0$$

$\langle \hat{Z}^2 \rangle_{NB,\,conn.} =$ 

- We think of $|n\rangle \propto (a^\dagger)^n |NB\rangle$ as an $n$-universe state.
- The state $|Z^k\rangle \equiv \hat{Z}^k |NB\rangle$ is a superposition of states $|n\rangle$ with at most $n = k$ universes.

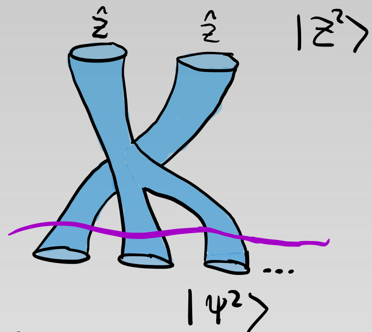
$|Z^2\rangle =$ 

- It is also useful to think about these correlation functions as overlaps of states.

$$\langle \hat{Z}^k \rangle_\psi = \langle \psi(\hat{Z})\ \hat{Z}^k\ \psi(\hat{Z}) \rangle_{NB} = \langle Z^k | \psi^2 \rangle$$

  We've defined the (not normalized!) state $|\psi^2\rangle \equiv \psi(\hat{Z})^2 |NB\rangle$ (combining $\psi(\hat{Z})$ from the bra and ket).

$\langle Z^2 | \psi^2 \rangle \supset$

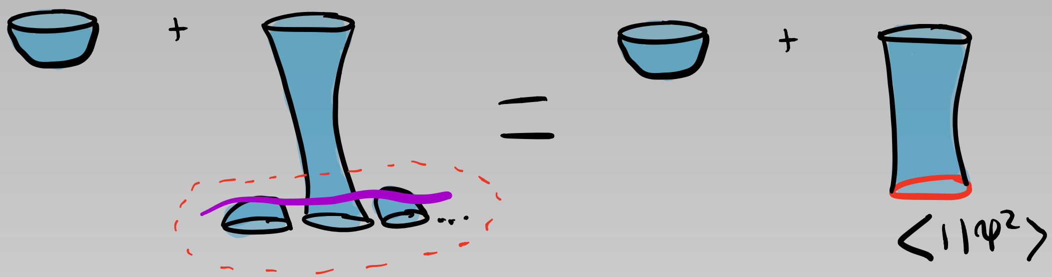



- Then correlation functions $\langle \hat{Z}^k \rangle_\psi$ *only* involve the components of the state $|\psi^2\rangle$ with up to $k$ universes

$$\langle \hat{Z}^k \rangle_\psi = \langle Z^k | \psi^2 \rangle = \sum_{n=0}^{k} \langle Z^k | n \rangle \langle n | \psi^2 \rangle$$

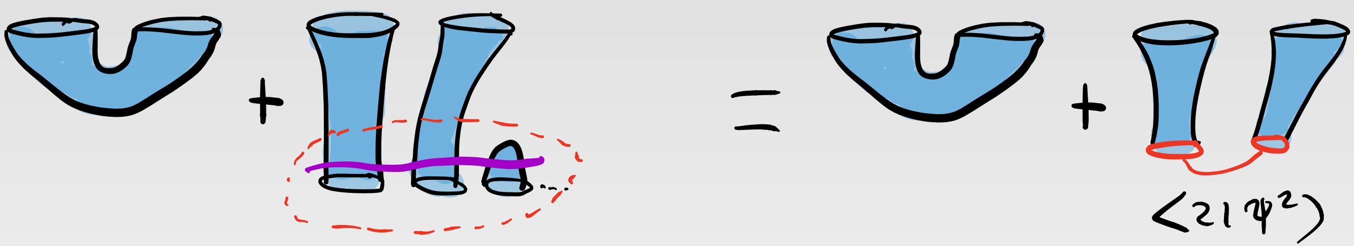
- For example,

$$\langle \hat{Z} \rangle_\psi = \text{Disk} + \langle 1|\psi^2\rangle$$

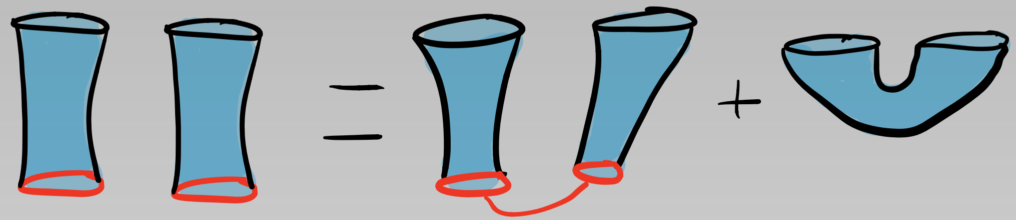



- The red boundary represents the wavefunction $\langle 1|\psi^2\rangle$, this wavefunction is prepared by summing over surfaces with many boundaries.
- The two-point function with the disk subtracted *only* involves the two-universe component

$$\langle (\hat{Z} - \text{Disk})^2 \rangle_\psi = \text{Cyl} + \langle 2|\psi^2\rangle$$



- For $|\psi\rangle$ an eigenstate, factorization requires that the two-point function is equal to the square of the one-point function.



- In this model, the two-point function has a *fundamental* wormhole, but the square of the one-point function does not.
- So the square of the one-universe component $\langle 1|\psi^2\rangle$ must "contain" the cylinder.

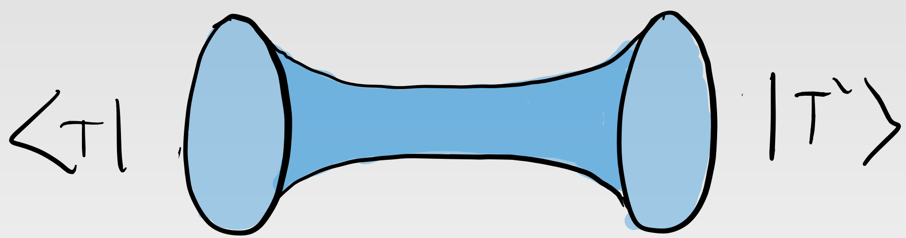
$$\langle 1|\psi^2\rangle^2 = \langle 2|\psi^2\rangle + \text{Cyl}$$

- Making an analogy with the periodic orbits, the cylinder plays the role of the diagonal sum, while the component $\langle 2|\psi^2\rangle$ is tuned so that it behaves like the "off-diagonal" sum.
- Analogy is sharper in JT gravity, where there is a natural double-sum.

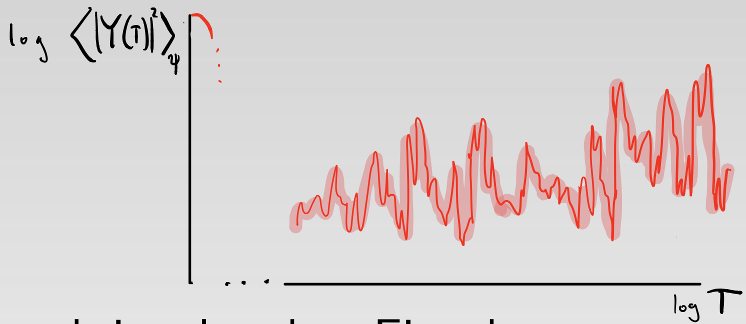
# JT gravity on the ramp

- In JT gravity, instead of a single operator $\hat{Z}$ we have many operators $\hat{Z}_i$, such as $\hat{Z}(\beta + iT)$ or $\hat{Y}(T)$ for different values of $T$.
- On the ramp, the $\hat{Y}(T)$ for (sufficiently) different $T$ behave like independent Gaussian variables in the No-Boundary state. Many copies of the Marolf-Maxfield model.
- After subtracting the disk, $\hat{Y}(T)$ creates and annihilates approximately orthogonal one-universe states $|T\rangle$. More general $\hat{Z}_i$ create one-universe states $|i\rangle$, and the Hilbert space is spanned by $n$-universe states $|i_1 \dots i_n\rangle$.
- The overlaps between one-universe states $|T\rangle$ are roughly

$$\langle T|T'\rangle \equiv \mathrm{Cyl}(T, T') \sim T\delta(T - T')$$



- The disk-and-cylinder approximation works for many observables in approximate eigenstates. Here this corresponds to ignoring the discreteness of the spectrum of the random matrix dual- $\rho(E) = \rho_{smooth}(E) + \delta\rho(E)$, with $\delta\rho(E)$ continuous and Gaussian-distributed. After a Fourier transform, the $\delta\rho(E)$ behave like the independent Gaussian variables $Y(T)$.
- In this approximation we can calculate the noisy part of the spectral form factor on the ramp (up to a time which depends on the allowed error).



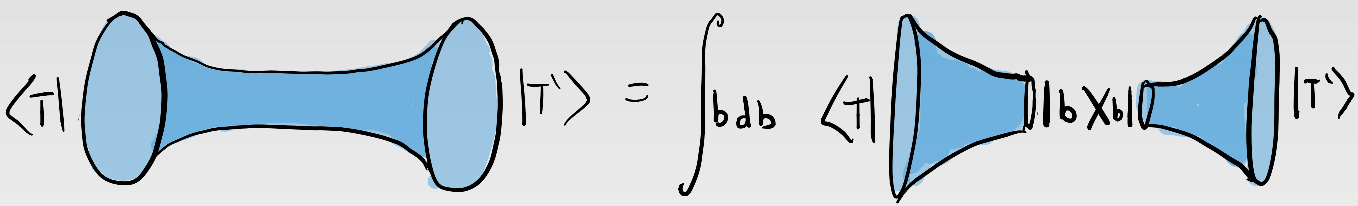

- Our approach is related to Eigenbranes [Blommaert, Mertens, Verschelde], a method for constructing exact eigenstates, including the discreteness. However, the gravity computations are difficult to control in these states.

- It is useful to define another (approximately) orthogonal basis. In the one-universe sector, these are states $|b\rangle$ corresponding to circular spatial slices with length $b$ and zero extrinsic curvature.

$$\langle b|b'\rangle = \frac{1}{b}\delta(b-b')$$

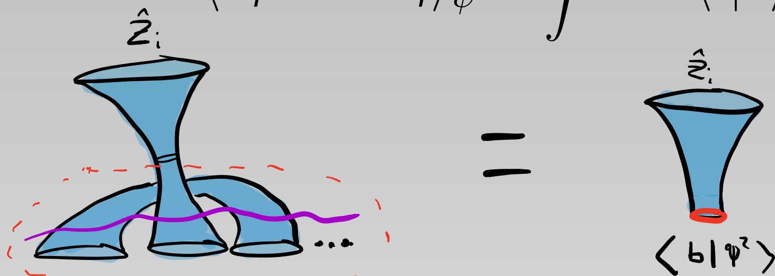
- In the n-universe sector, we have states $|b_1 \ldots b_n\rangle$, corresponding to $n$ universes with lengths $b_1 \ldots b_n$.
- This basis allows us to describe the cylinder as two "trumpets" glued together

$$\mathrm{Cyl}(T, T') \equiv \langle T|T'\rangle = \int_0^\infty bdb \; \langle T|b\rangle\langle b|T'\rangle$$



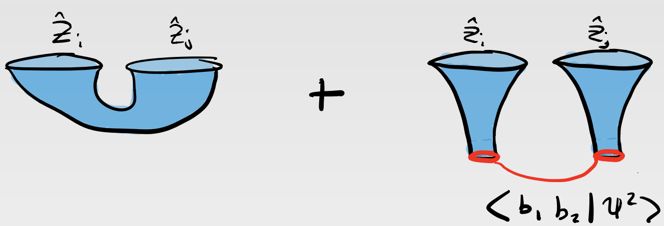
- In this basis, what is the condition on $|\psi^2\rangle$ such that correlation functions factorize?
- For a general operator $\hat{Z}_i$, the one-point function is a "sum" over $b$

$$\langle \hat{Z}_i - \text{Disk}_i \rangle_\psi = \int b db \, \langle i|b\rangle\langle b|\psi^2\rangle$$



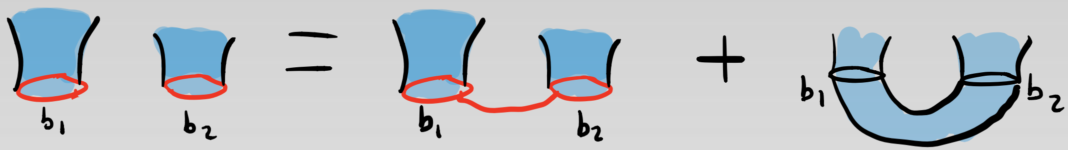

- The two-point function with the disks subtracted is the cylinder plus a "double sum" over $b_1$ and $b_2$

$$\langle (\hat{Z}_i - \text{Disk}_i)(\hat{Z}_j - \text{Disk}_j) \rangle_\psi$$
$$= \text{Cyl}_{ij} + \int b_1 db_1 b_2 db_2 \, \langle i|b_1\rangle\langle j|b_2\rangle \, \langle b_1 b_2|\psi^2\rangle$$



- $\mathrm{Cyl}_{ij} \equiv \langle i|j\rangle = \int b_1 db_1 \, \langle i|b_1\rangle\langle b_1|j\rangle$. This allows us to write both the two-point function and the square of the one-point function as "double sums" over $b_1$ and $b_2$, with the cylinder contributing a "diagonal "part of the sum.
- The condition on $|\psi^2\rangle$ so that they are equal is

$$\langle b_1|\psi^2\rangle\langle b_2|\psi^2\rangle = \langle b_1 b_2|\psi^2\rangle + \frac{1}{b_1}\delta(b_1 - b_2)$$

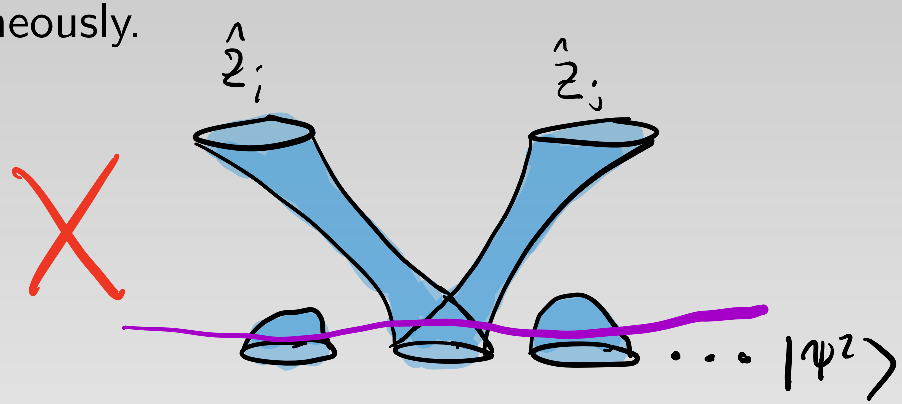



- The "diagonal" is being singled out with the delta function; this gives the cylinder in the two-point function.
- The two-universe component $\langle b_1 b_2|\psi^2\rangle$ is analogous to the off-diagonal terms in the periodic orbit sum; adding it to the "diagonal" cylinder gives a factorized answer.

# An effective description with dynamical boundaries

- What is special about this diagonal component in the many-universe description?

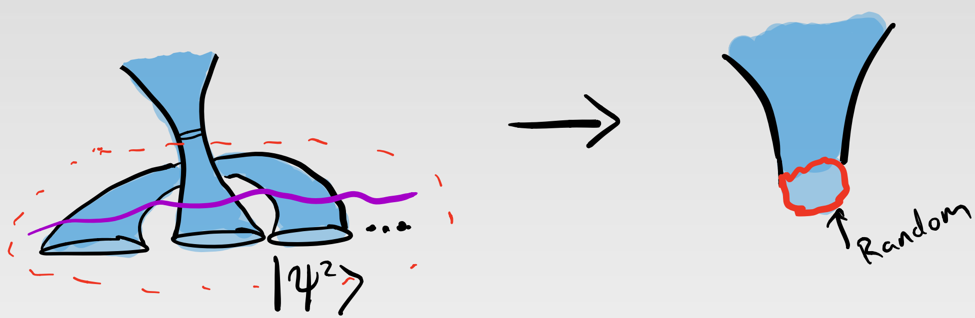
- The many-universe state $|\psi^2\rangle$ is *shared* by the $\hat{Z}_i$s in the correlation function. A consequence of this is that they cannot connect to the same closed universe with a cylinder simultaneously.




- For $|\psi\rangle$ an eigenstate, $|\psi^2\rangle$ is tuned so that the two-universe component almost factorizes, except for a correction from subtracting the cylinder due to this geometric "exclusion" effect.

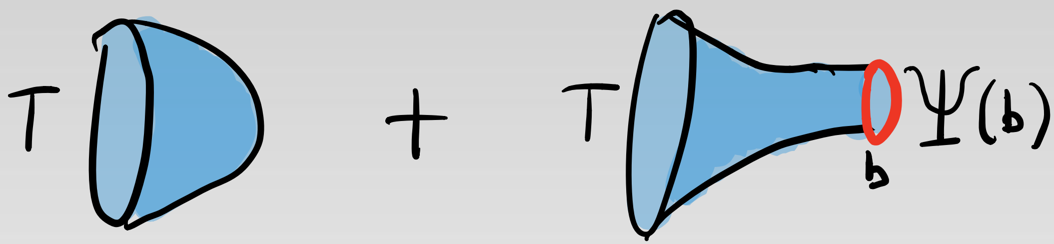
# An effective description with dynamical boundaries

- We now introduce an effective model which takes advantage of the simple description in terms of the single-universe wavefunction $\langle b|\psi^2\rangle$.
- In the full model, the gravity computation involves many "closed universe state" boundaries which represent the full $|\psi\rangle$ and $|\psi^2\rangle$. However, for simple correlators we only need the components of $|\psi^2\rangle$ on a few boundaries, which are simple functions of the single-universe components $\langle b|\psi^2\rangle$.
- In the effective model we trade the component $\langle b|\psi^2\rangle$, created by summing over many "state" boundaries, for a single "dynamical boundary" with a random boundary condition.



- We can describe this random boundary condition with a random Gaussian function $\Psi(b)$, with variance $\langle \Psi(b)\Psi(b')\rangle = \frac{1}{b}\delta(b-b')$. This replaces the single-universe component $\langle b|\psi^2\rangle$. For example, the equation for $Y(T)$ becomes

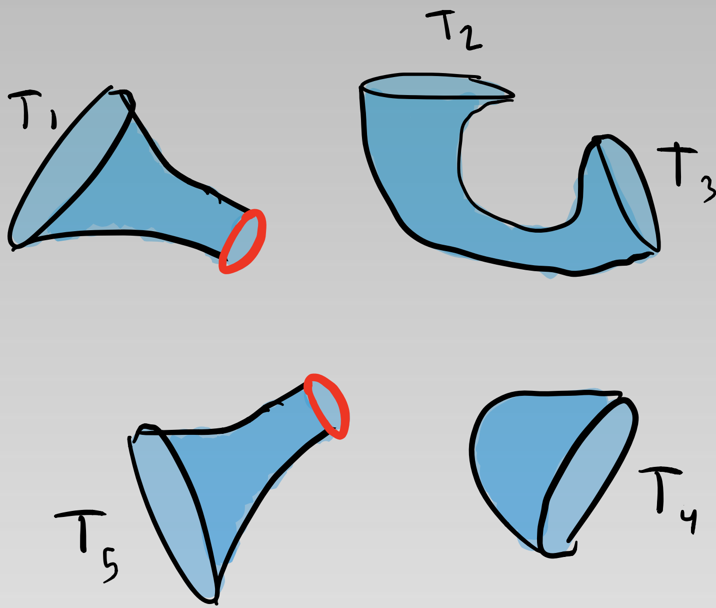
$$Y(T) = \text{Disk}(T) + \int bdb\, \langle T|b\rangle\, \Psi(b)$$




- For a single draw of $\Psi(b)$, this reproduces the noisy signal on the ramp.

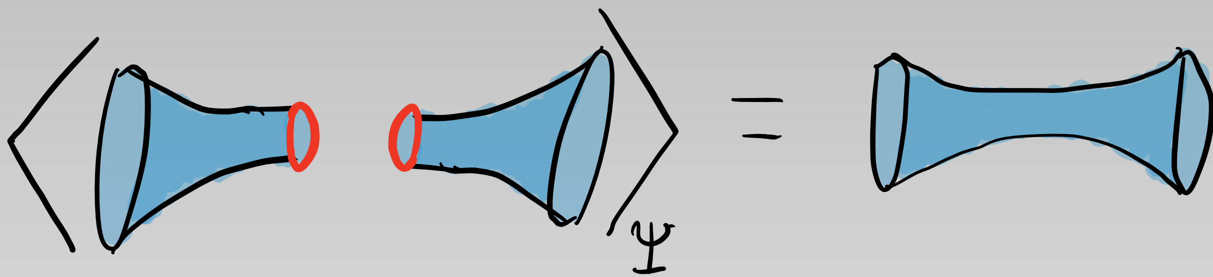
- Now, instead of summing over many "state" boundaries, we only need a few. For $k$ partition functions, we need up to $k$ dynamical boundaries.

$$Y(T_1)\cdots Y(T_5) \supset$$



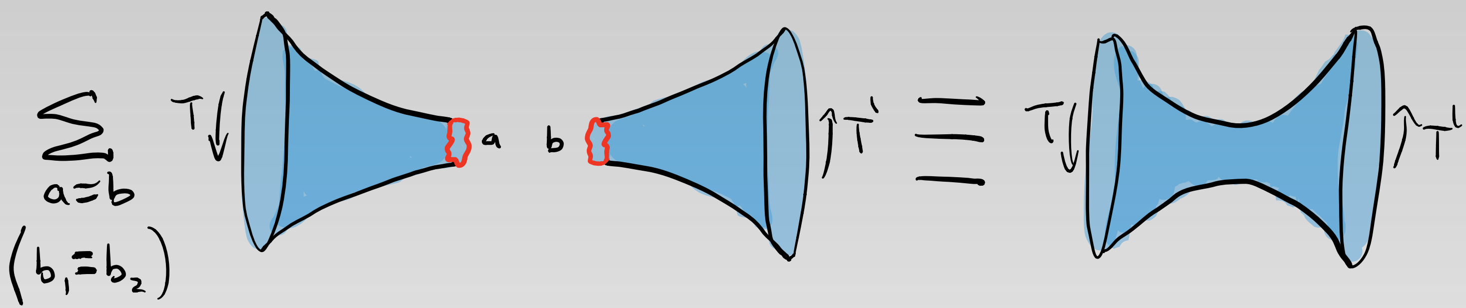

- We could also choose a different boundary condition for the dynamical boundaries, instead of the zero extrinsic curvature boundary condition related to the $\langle b|\psi^2\rangle$ wavefunctions. We focus on this choice for simplicity.

- Taking the answer for a single Y and squaring it, then averaging over the ensemble (the random $\Psi(b)$ with $\langle\Psi(b)\Psi(b')\rangle = \frac{1}{b}\delta(b-b')$), we recover the wormhole by gluing together the random boundaries.




- However, the gravitational path integral for two $Y$s includes an explicit wormhole **and** the pairs of random boundaries- we appear to have overcounted. By giving up the many-universe description, we've given up the geometrical mechanism for "excluding" the cylinder.
- In order for this effective model to match with the full model, we must introduce an *ad hoc* modification to the sum over geometries- the "exclusion rule".

- The exclusion rule instructs us to view a “diagonal” part of the pairs of dynamical boundary contributions as *equivalent* to the cylinder. We should include one *or* the other, not *overcount* them.



- This “exclusion rule” was introduced by hand to match the full theory - we did not derive it directly from any bulk reasoning.
- The origin of the exclusion rule in the full many-universe theory is geometrical.
- However, it is tempting to take the “no overcounting” idea more seriously. Are there other theories described by this effective model in which the exclusion arises because the full theory views the cylinder and the corresponding diagonal piece of the dynamical boundary contributions as two descriptions of the same object- related by some sort of “quantum equivalence” or “gauge equivalence” (Marolf and Maxfield), or “duality”? Reminiscent of recent work by Eberhardt, Jafferis.

# SYK

- The SYK model provides a possible example of this. [Harlow; Harlow, Jafferis]
- In the SYK model, partition functions are typically averaged over the couplings $J_{i_1 \dots i_q}$. These averaged partition functions can be represented with a path integral over $G - \Sigma$ collective fields.
- For example, for $\langle Y_L(T) Y_R(T')^* \rangle_J$ we have a $2 \times 2$ matrix of collective fields, with same-side (LL and RR) and off-diagonal (LR) fields.
- The $G - \Sigma$ collective fields are closely related to the JT gravity approximation of SYK. The "double-cone" saddle point of the LR field plays the role of a spacetime wormhole [P.S., Shenker, Stanford].
- What happens when we don't average over the couplings?

- If we don't average over the couplings, and instead focus on a single choice, partition functions factorize.

- We can still introduce the $G - \Sigma$ collective fields and integrate out the fermions (in principle). This adds a noisy term, depending on the couplings, to the action.

- For two copies of $Y(T)$, we may choose to introduce, or not introduce, the LR collective fields. The result of the calculation is the same - the fields integrate to one.

- Computing with the LR fields and without them is analogous to calculating $\langle|\hat{Y}|^2\rangle_\psi$ vs. $\left|\langle\hat{Y}\rangle_\psi\right|^2$ in JT gravity - The first computation has a *fundamental* wormhole plus "off-diagonal" corrections, the second is a "double-sum" which manifestly factorizes, with a potentially *emergent* wormhole.

- This gives an example of an "equivalence" between something like the wormhole, and matched "disconnected" configurations. How does this work?

- The noisy term in the action depends on whether or not we include the LR fields. If we *do* include the LR fields, do we see the "diagonal plus off-diagonal" structure, with configurations of the LR field playing the role of the "diagonal" sum, and the noisy terms behaving like the "off-diagonal" terms?

- Difficult to control the model with exactly fixed couplings. Can use small $J$ perturbation theory as a tool.

- Perturbative fluctuations of the LR fields sum up certain fermion diagrams with the flavor indices of the L and R fermions matched - $G^{LR} \sim \sum_a \chi_a^L \chi_a^R$. However, it is difficult to connect this to the double-cone saddle point [CGHPSSSST].

- To have better control, we approximately fix the couplings in a way which allows us to indirectly study these noisy terms using the $G - \Sigma$ saddle points.

- We can copy our procedure from JT gravity and work in an ensemble for which $Y(T)$ is approximately fixed for an interval on the ramp.

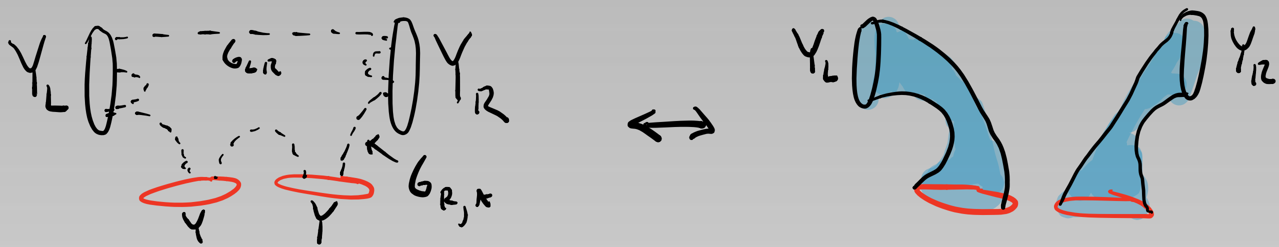
- We can enforce this by inserting approximate delta functions into the usual $J$-average. Representing these approximate delta functions as tight Gaussians,

$$\exp\left[-\frac{(Y(T)-y_0(T))^2}{2\Delta^2}\right], \qquad \Delta \ll 1,$$

  we can expand out the exponential and introduce $G-\Sigma$ fields for these "auxiliary" copies of $Y(T)$.

- For appropriate observables the many-replica $G-\Sigma$ integral can then be done by saddle point. The result mirrors the calculation in JT gravity, with the double-cone saddle point playing the role of the cylinder, and the auxiliary $Y(T)$'s playing the role of the state boundaries.

- Like in JT, we can find an effective description for $Y_L(T) Y_R(T')$ with just two auxiliary boundaries.




- If we integrate out the auxiliary boundaries we would find an approximate action for just the "physical" $G - \Sigma$ fields, with the anticipated noisy terms. The effective description allows us to study these terms using saddle points.
- If we do not introduce the LR fields, the resulting noisy terms contribute a manifestly factorizing double-sum.
- If we *do* introduce the LR fields, the noisy terms give a purely "off-diagonal" contribution, with the "diagonal" double cone excluded.
- There are two equivalent representations of the $G - \Sigma$ integral: with the LR fields and including the double-cone; or without the LR fields and including the corresponding "diagonal" noisy terms.

# Discussion and questions

- Can a variant of this effective description apply to conventional AdS/CFT, like SYM?
- What would provide a microscopic realization of the dyanamical boundaries? Perhaps periodic "fuzzball" configurations? We would also need a bulk explanation for the exclusion rule.
- The Disk (Euclidean black hole) also poses a similar problem- in a full bulk description, is the black hole "equivalent" to some contributions of "microstates"?
- For the tensionless string in $AdS_3$, Eberhardt found an equivalence between the black hole and "microstate" contributions, as well as indications of an equivalence between the wormhole and disconnected quantities. It would be nice to understand how this works in more detail.