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QCD Coffee/QCD Lunch Meeting — CERN (October 9, 2020)

Factorization at Subleading Power and Endpoint Divergences in Soft-Collinear Effective Theory

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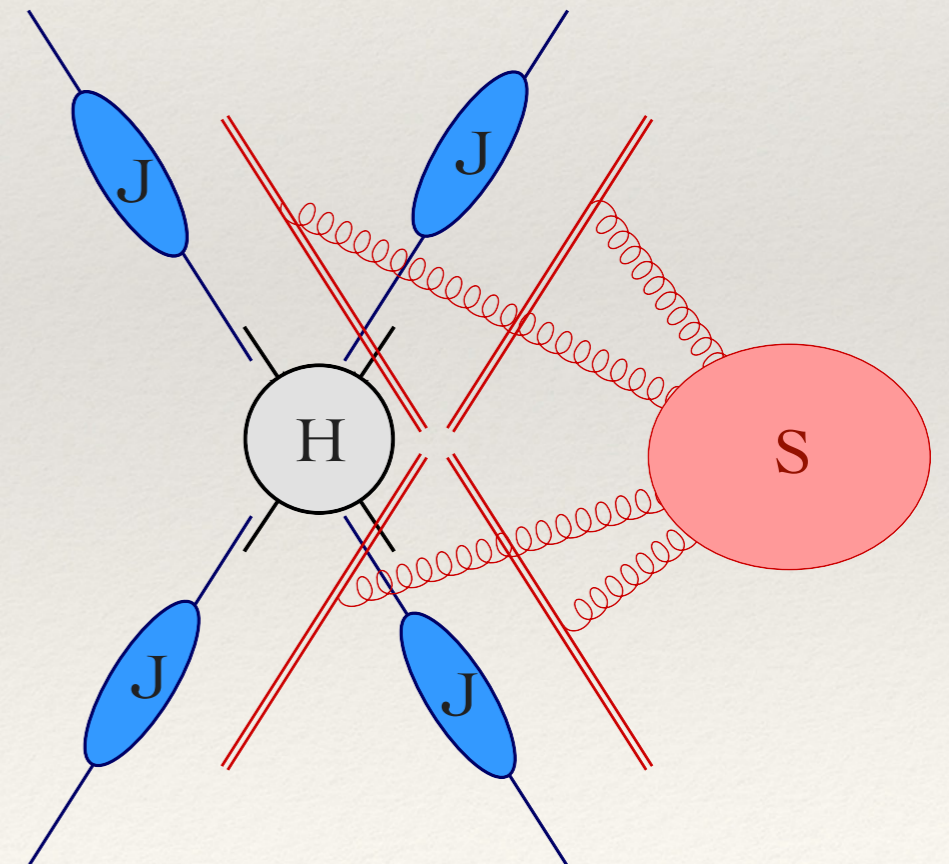


Introduction

- ❖ Factorization of scales is a fundamental concept in HEP:
 - ▶ LHC cross section $\sim \sigma_{\text{parton}} \otimes \text{PDFs}$
 - ▶ basis for the resummation of large logarithmic corrections

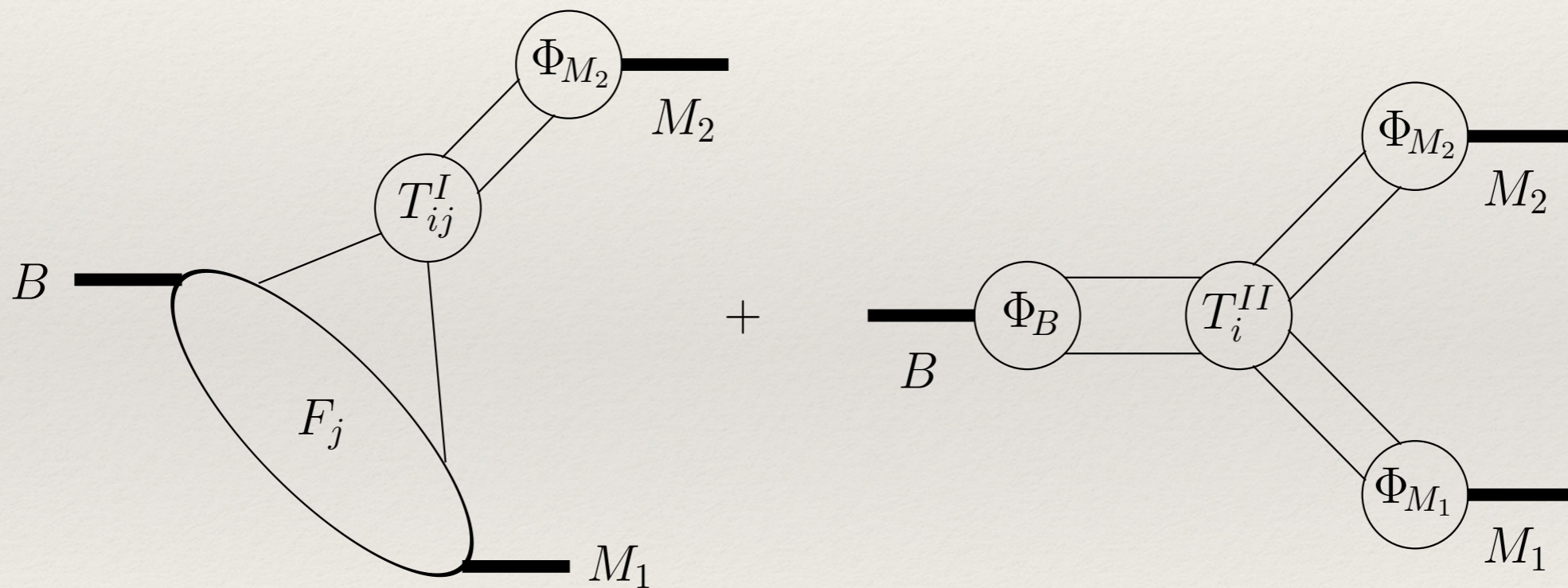
- ❖ **Soft-collinear effective theory (SCET)** provides a framework for studying factorization and resummation for processes involving light energetic particles using tools of effective field theory (EFT)

[Bauer et al. 2000, 2001; Beneke et al. 2002]



Introduction

- ❖ SCET was constructed as the EFT for **QCD factorization**, as applied to hadronic decays of B mesons:



[Beneke, Buchalla, MN, Sachrajda 1999, 2000]

Introduction

- ❖ **Puzzling observation:**
 - ▶ some power-suppressed corrections (e.g. weak annihilation) involve divergent convolution integrals $\sim \int_0^1 \frac{dx}{x}$, which were interpreted as a **breakdown of factorization** [\[Beneke, Buchalla, MN, Sachrajda 2000\]](#)
 - ▶ proof of factorization requires showing that certain soft-collinear modes parameterizing these endpoint contributions cancel out to all orders [\[Hill, MN 2003; Becher, Hill, MN 2005\]](#)
- ❖ It was believed that SCET may overcome the problem of endpoint divergences (“everything factorizes”) or at least shed new light on it

Introduction

- ❖ Conventional EFTs provide a systematic expansion in inverse powers of a large scale Q :

$$\mathcal{L}_{\text{eff}} = \sum_i C_i(Q, \mu) O_i(\mu) + \frac{1}{Q} \sum_j C_j^{(1)}(Q, \mu) O_j^{(1)}(\mu) + \frac{1}{Q^2} \sum_k C_k^{(2)}(Q, \mu) O_k^{(2)}(\mu) + \dots$$

- ❖ Examples: $\mathcal{H}_{\text{eff}}^{\text{weak}}$, χPT , HQET, NRQCD, SMEFT, ...

- ❖ Extension to higher orders “straightforward if tedious”

- ▶ χPT : 2, 12, 117, 1959, 45171, 1170086, ... [Graf et al. 2020]

- ▶ SMEFT: 12, 3045, 1542, 44807, 90456, 2092441, ...

[Henning, Lu, Melia, Murayama 2015]

Introduction

- ❖ SCET is more complicated in several aspects:
 - ▶ operators contain **non-local products of fields** (unavoidable consequence of $E \sim Q$ but $p^2 \ll Q^2$), need to introduce **Wilson lines** for gauge invariance
 - ▶ Wilson coefficients depend on large momentum components in addition to heavy masses of particles integrated out
 - ▶ fields are split up in **momentum modes** (method of regions):
[\[Beneke, Smirnov 1997\]](#)

$$\phi(x) \rightarrow \underbrace{\phi_{n_1}(x) + \phi_{n_2}(x) + \dots}_{\text{collinear}} + \underbrace{\phi_s(x) + \phi_{us}(x) + \dots}_{\text{soft}}$$

collinear
(regions of large momentum flow)

soft

Introduction

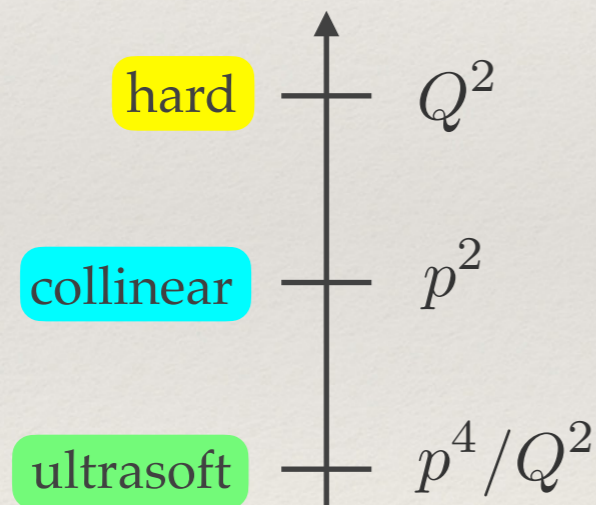
- ❖ SCET is more complicated in several aspects:
 - ▶ hard modes are integrated out
(Wilson coefficients = hard matching coefficients)
 - ▶ different collinear sectors appear decoupled in the effective Lagrangian except for soft interactions
 - ▶ soft interactions can be decoupled by means of field redefinitions
→ **factorization theorems**
 - ▶ large logarithms can be resummed systematically by solving RGEs

Introduction

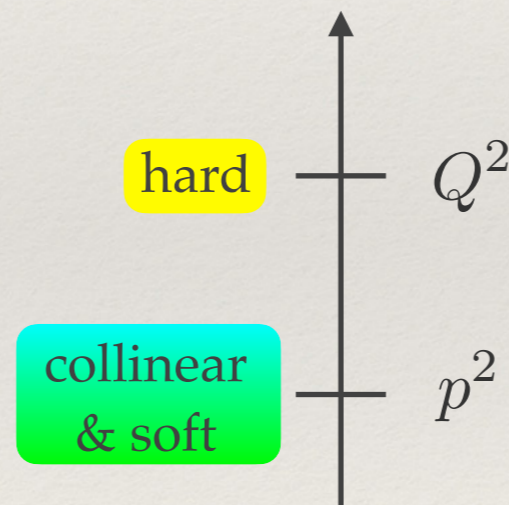
❖ Typical SCET factorization theorem:

$$\sigma \sim \underset{\text{hard}}{H} \int \underset{\text{collinear}}{J \otimes J} \otimes \underset{\text{soft}}{S}$$

❖ Two common scale hierarchies:



SCET-1



SCET-2

In SCET-2 the product $J \otimes J \otimes S$ contains an extra dependence on Q^2 due to the **collinear anomaly**.

[Becher, MN 2010]

Introduction

❖ Examples:

- ▶ threshold resummation for DIS, DY, Higgs $t\bar{t}$ production, ...
- ▶ p_T resummation, jet vetoes, event shapes, jet substructure, ...
- ▶ non-global logarithms, super-leading logarithms (ongoing work)
- ▶ high-order structure of IR divergences of scattering amplitudes, subtraction methods for N^n LO fixed-order calculations (e.g. based on N-jettiness)

[Becher, MN et al. 2006-2016; ...]

Introduction

- ❖ Extension to next-to-leading power?
 - ▶ generically (all known examples), find **endpoint-divergent convolution integrals!** [[Beneke et al.](#), [Moult et al.](#), [Stewart et al.](#), [MN et al. 2018-2020](#); ...]
 - ▶ upsets scale separation and breaks factorization
 - ▶ prevents systematic resummation of large logarithms
 - ▶ failure of standard OPE based on dimensional regularization and $\overline{\text{MS}}$ subtractions
- ❖ Questions usefulness of entire SCET framework!
 - ▶ a hard problem; many groups world-wide work on this...

SCET 2020

Bern, Switzerland
June 8 – 11, 2020

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UNIVERSITÄT
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Cancelled due to the COVID-19 pandemic

XVIIth annual workshop on Soft-Collinear Effective Theory

Organizers:

Thomas Becher, Christoph Greub, Thomas Rauh, Xiaofeng Xu, Marcel Balsiger, Samuel Favrod, Francesco Saturnino

<http://scet.itp.unibe.ch/>

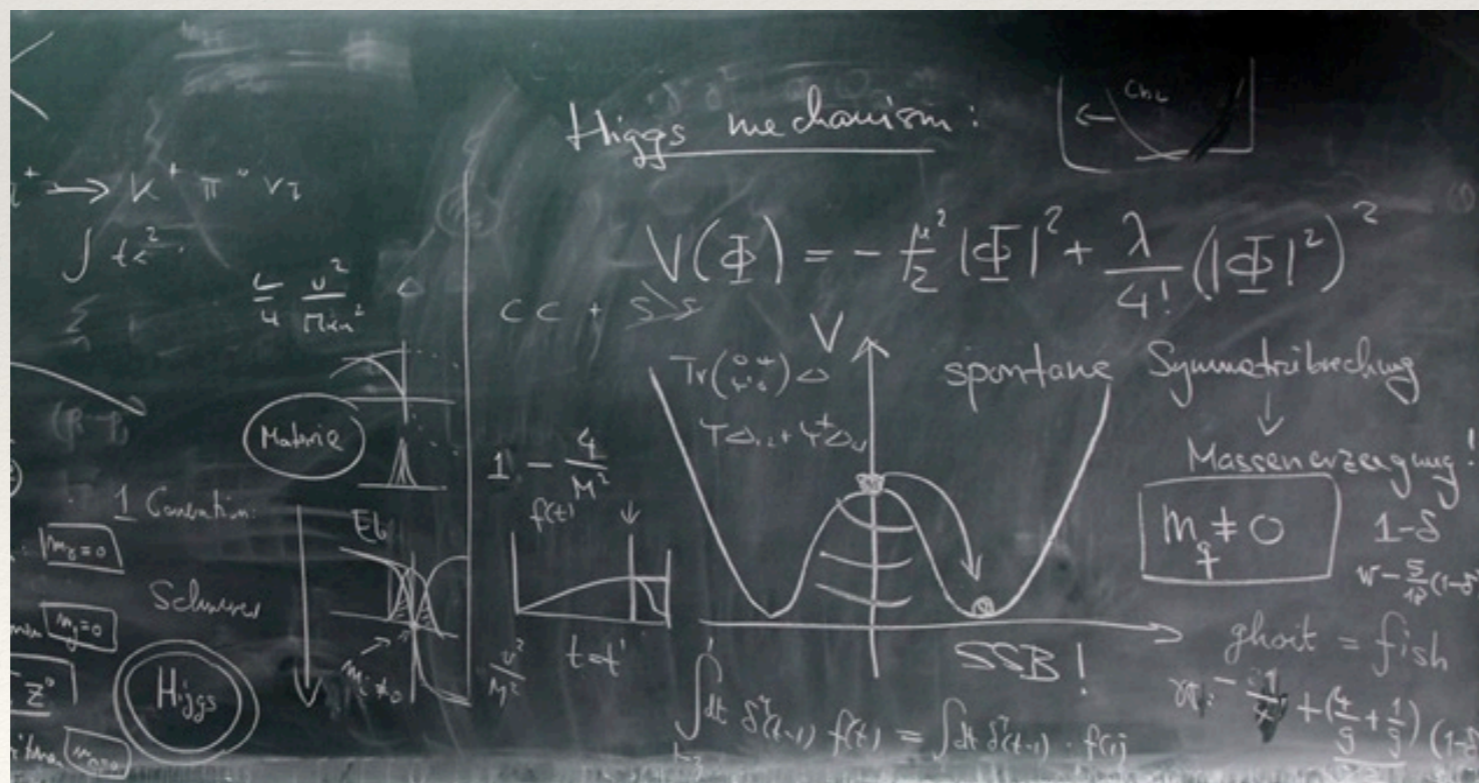
First SCET factorization theorem at subleading power

Liu, MN: 1912.08818 (JHEP)

Liu, Mecaj, MN, Yang: 2009.04456 & 2009.06779

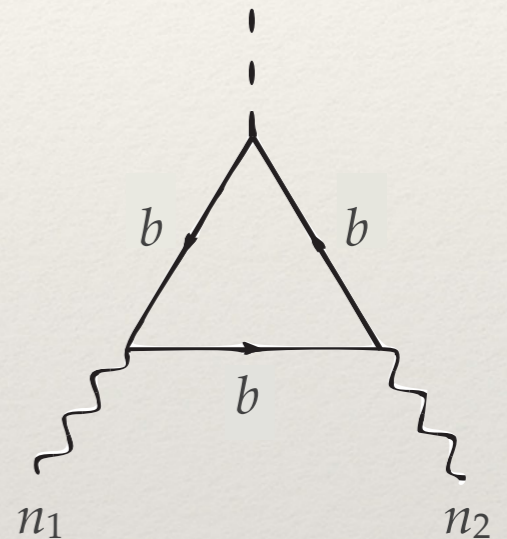
Liu, MN: 2003.03393 (JHEP)

Liu, Mecaj, MN, Yang, Fleming: 2005.03013 (JHEP)



A subleading-power observable

- ❖ Consider b -quark induced contribution to $h \rightarrow \gamma\gamma$ decay amplitude (pseudo observable)
 - ▶ this and related $gg \rightarrow h$ process may be relevant for high-precision Higgs studies, but here are considered for academic purposes mainly
 - ▶ “sufficiently complicated but simple enough”
- ❖ Relevant modes are hard, collinear (n_1 and n_2) and soft, with SCET-2 scaling
- ❖ Scale hierarchy: $m_b^2 \ll M_h^2$

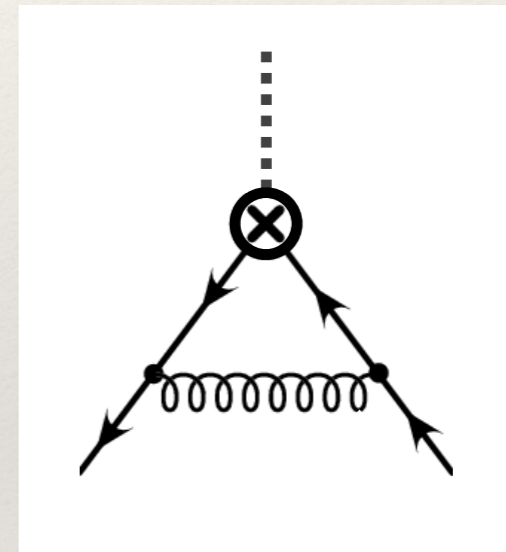


A subleading-power observable

- ❖ Same momentum regions appear in analysis of the Sudakov form factor (e.g. electroweak Sudakov resummation)

- ▶ standard factorization theorem without endpoint divergences:

$$\sigma \sim H \int J \otimes J \otimes S$$



- ▶ a single, leading-order SCET operator arises at $O(\lambda^2)$:

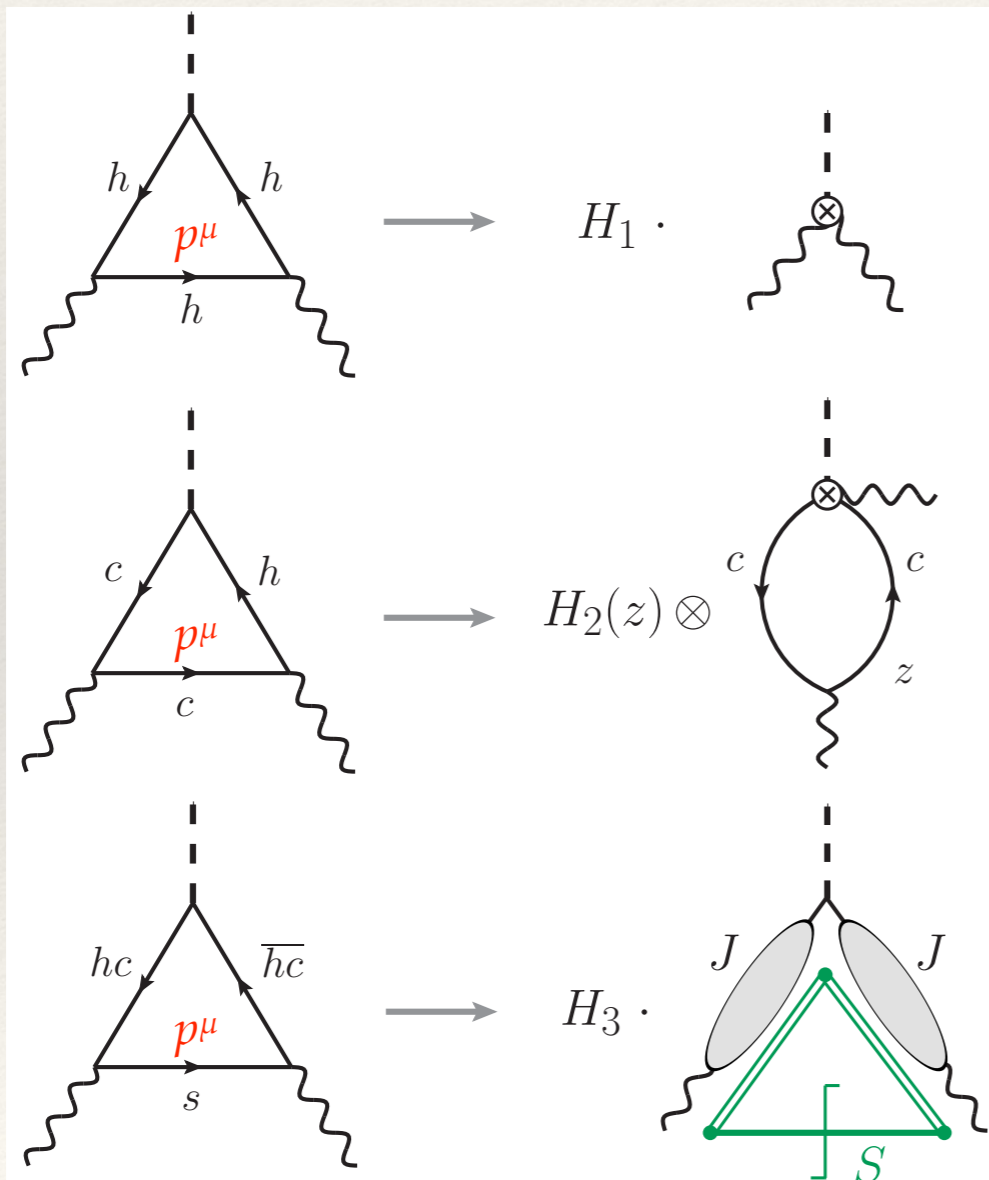
$$V_\mu \bar{\chi}_{n_1} \gamma_\perp^\mu \chi_{n_2}$$

λ λ

- ▶ crucial difference: **soft quark** can appear at subleading power

A subleading-power observable

- ❖ Relevant momentum regions at 1-loop order:



$$p^\mu \sim M_h$$

hard

$$p^\mu \parallel k_1, \quad p^2 \sim m_b^2$$

n_1 -collinear

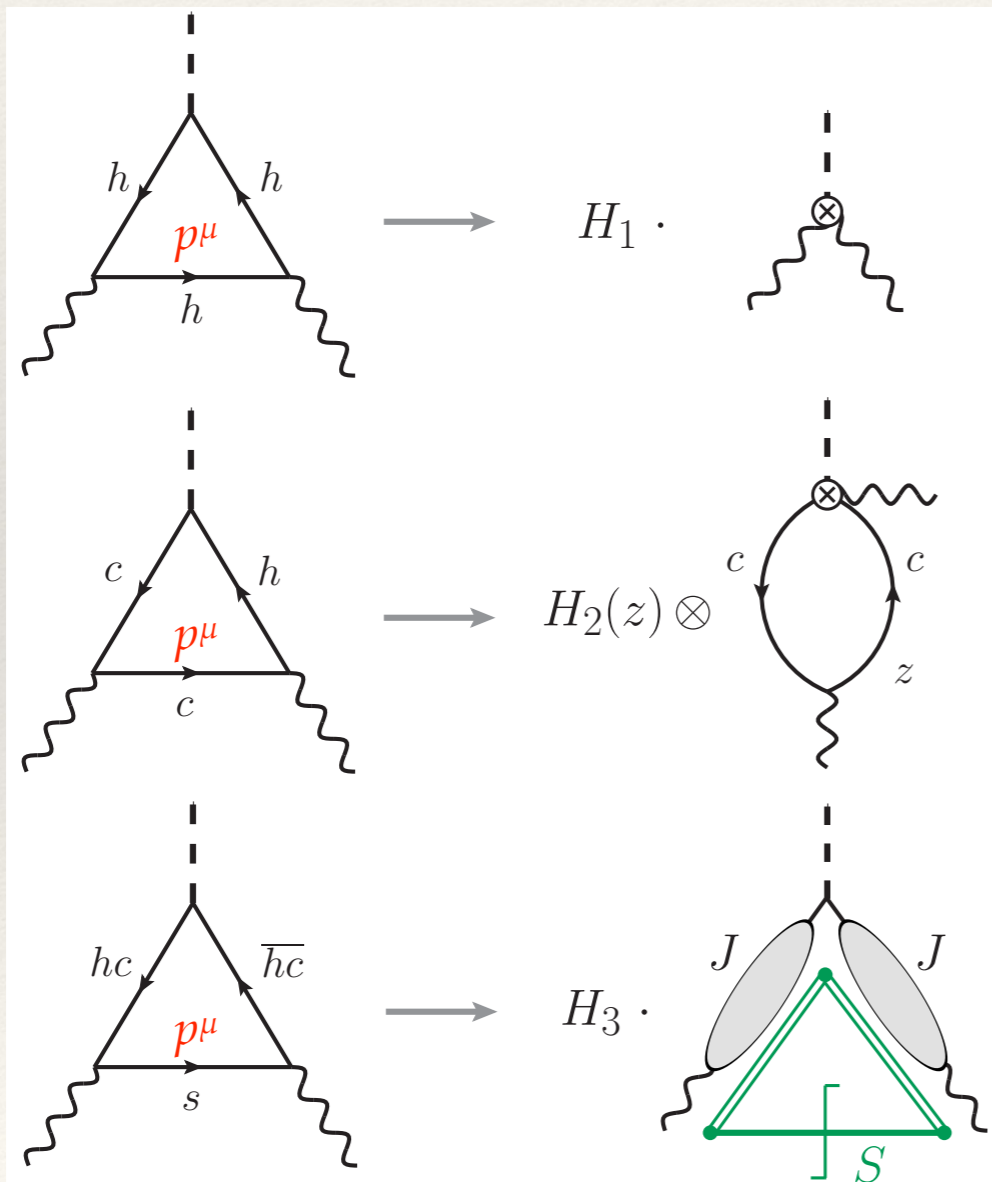
$$p^\mu \sim m_b$$

soft

A subleading-power observable

❖ Relevant momentum regions at 1-loop order:

$$\lambda \sim m_b/M_h$$



$$p^\mu \sim M_h (1, 1, 1)$$

$n_1 \ n_2 \perp$

hard

$$p^\mu \sim M_h (1, \lambda^2, \lambda)$$

n_1 -collinear

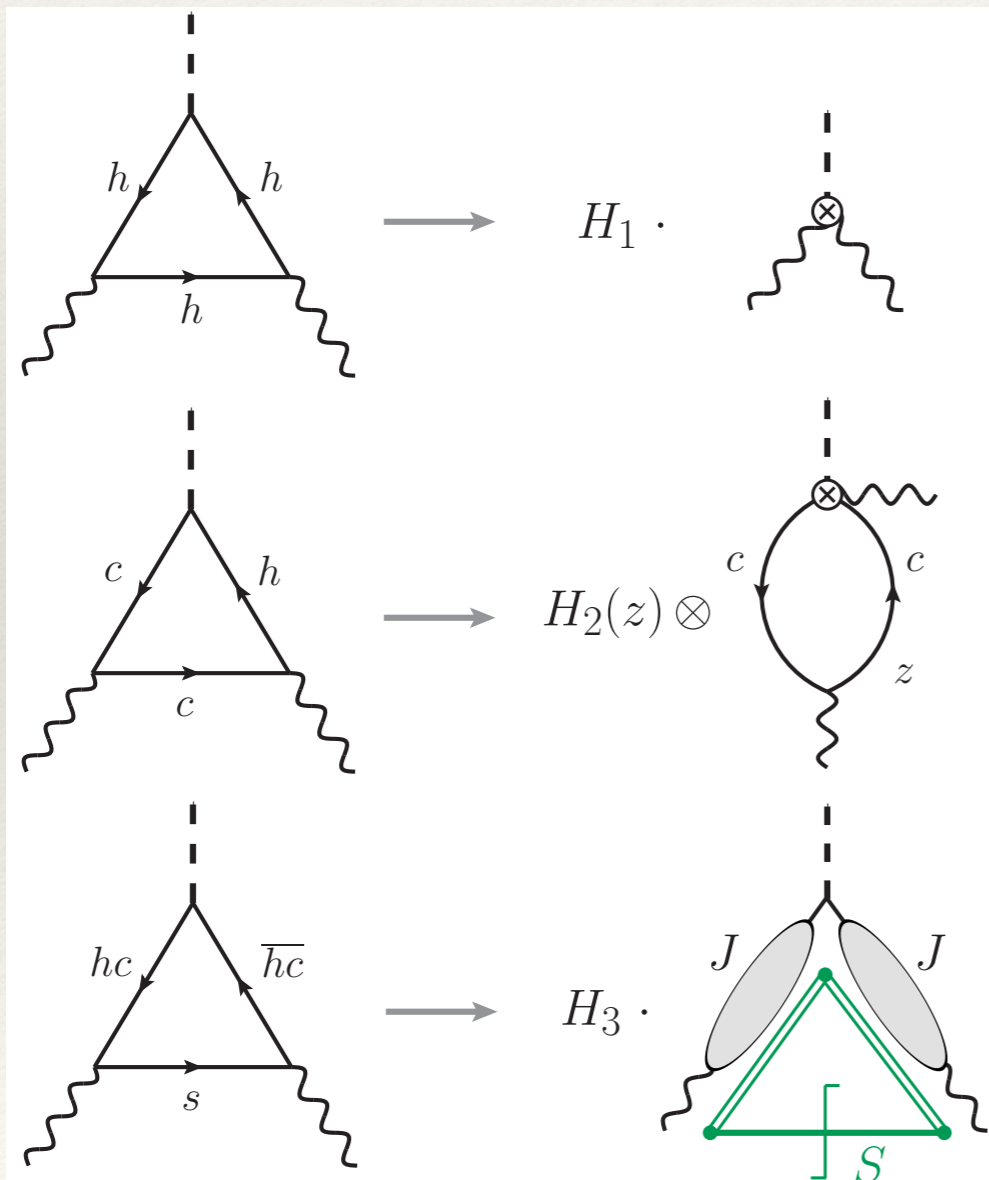
$$p^\mu \sim M_h (\lambda, \lambda, \lambda)$$

soft

A subleading-power observable

❖ Corresponding SCET operators at $O(\lambda^3)$:

$$\lambda \sim m_b/M_h$$



$$O_1^{(0)} = \frac{\lambda}{e_b^2} h \mathcal{A}_{n_1}^{\perp\mu} \mathcal{A}_{n_2,\mu}^{\perp}$$

dressed collinear photon fields

$$O_2^{(0)}(z) = h \left[\bar{\mathcal{X}}_{n_1} \gamma_{\perp}^{\mu} \frac{\not{n}_{\perp 1}}{2} \delta(z \bar{n}_1 \cdot k_1 + i \bar{n}_1 \cdot \partial) \mathcal{X}_{n_1} \right] \mathcal{A}_{n_2,\mu}^{\perp}$$

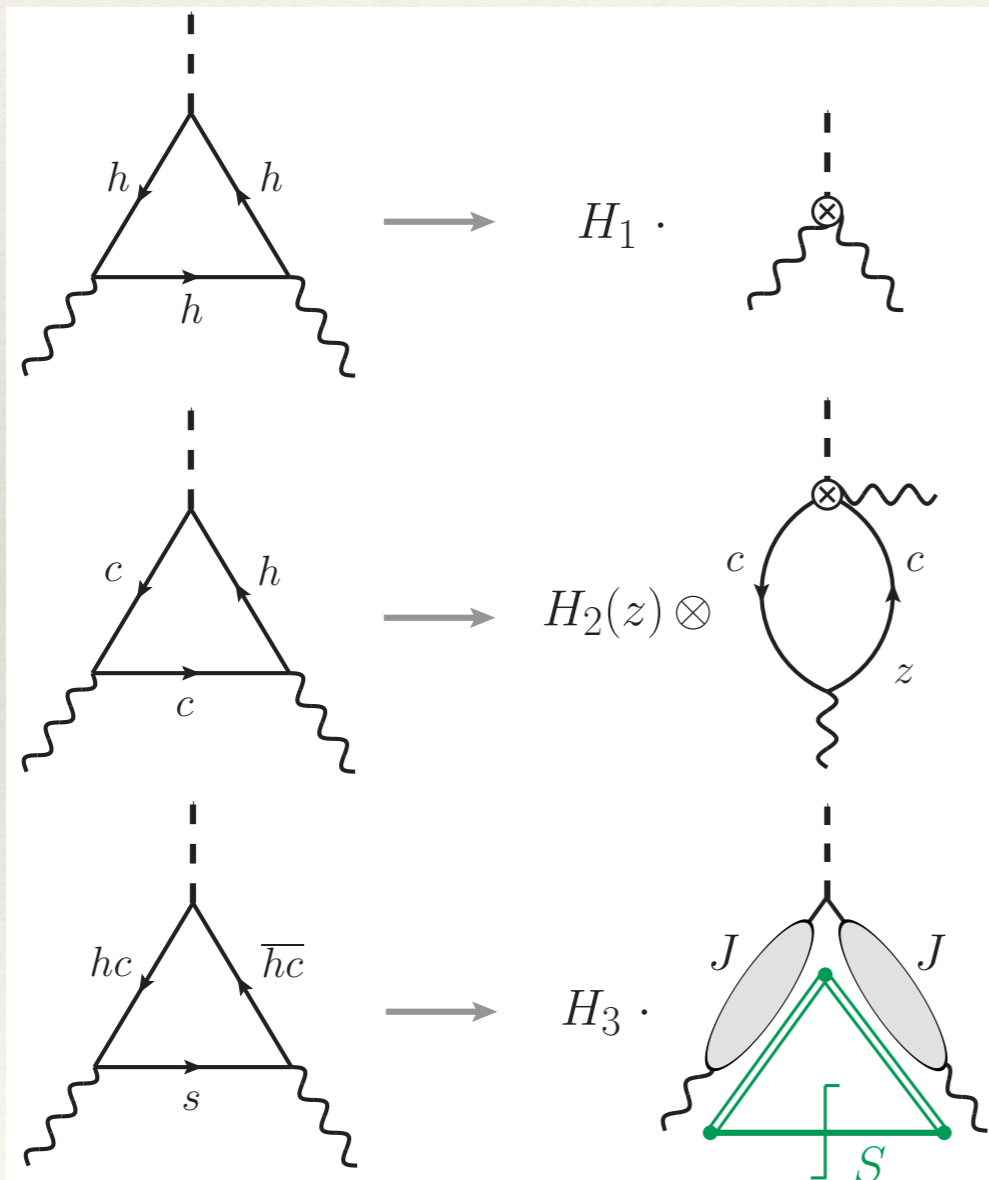
dressed collinear quark fields

$$O_3^{(0)} = T \left\{ h \bar{\mathcal{X}}_{n_1} \mathcal{X}_{n_2}, i \int d^D x \mathcal{L}_{q\xi_{n_1}}^{(1/2)}(x), i \int d^D y \mathcal{L}_{\xi_{n_2}q}^{(1/2)}(y) \right\} + \text{h.c.}$$

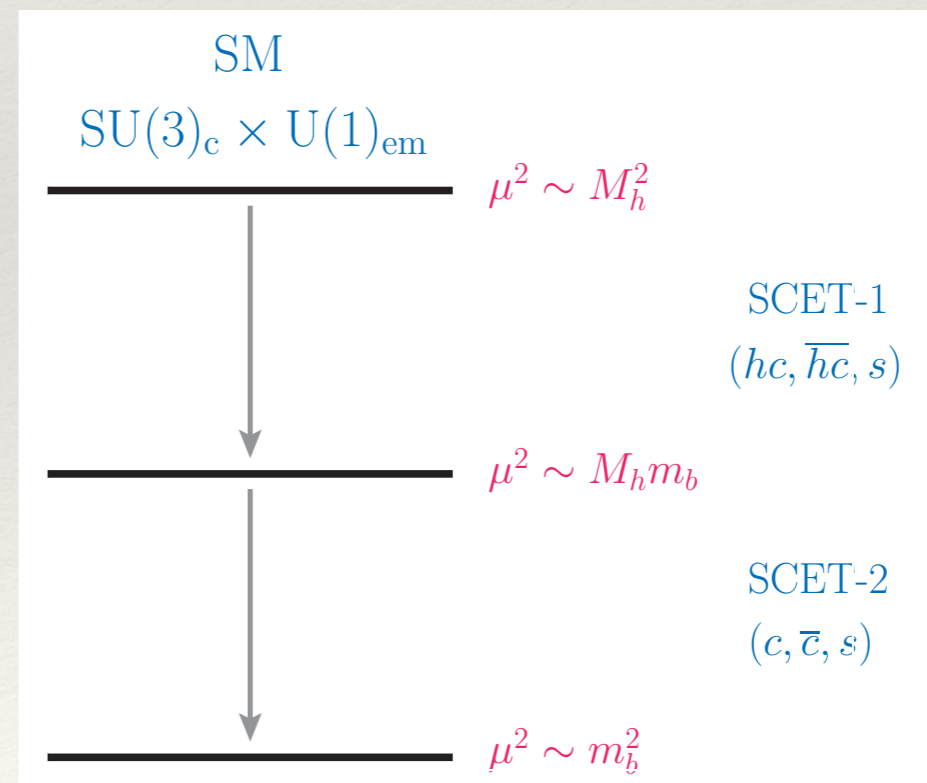
subleading SCET Lagrangian

A subleading-power observable

- ❖ Corresponding SCET operators at $O(\lambda^3)$:



Existence of only three SCET operators at $O(\lambda^3)$ ensures that these regions account for all higher-order loop graphs (see [Liu, MN 2019] for a 2-loop example)!



“Bare factorization theorem”

- ❖ Adding up the three contributions we find:

$$\mathcal{M}_b(h \rightarrow \gamma\gamma) = H_1^{(0)} \langle \gamma\gamma | O_1^{(0)} | h \rangle + 2 \int_0^1 dz H_2^{(0)}(z) \langle \gamma\gamma | O_2^{(0)}(z) | h \rangle + H_3^{(0)} \langle \gamma\gamma | O_3^{(0)} | h \rangle$$

with:

$$\begin{aligned} \langle \gamma\gamma | O_3^{(0)} | h \rangle &= \frac{g_{\perp}^{\mu\nu}}{2} \int_0^{\infty} \frac{dl_+}{l_+} \int_0^{\infty} \frac{dl_-}{l_-} \\ &\quad \times \left[J^{(0)}(M_h l_+) J^{(0)}(-M_h l_-) + J^{(0)}(-M_h l_+) J^{(0)}(M_h l_-) \right] S^{(0)}(l_+ l_-) \end{aligned}$$

- ❖ Factorization formula accomplishes a naive scale separation, but all component functions are still unrenormalized!

“Bare factorization theorem”

- ❖ Adding up the three contributions we find:

$$\mathcal{M}_b(h \rightarrow \gamma\gamma) = H_1^{(0)} \langle \gamma\gamma | O_1^{(0)} | h \rangle + 2 \int_0^1 dz H_2^{(0)}(z) \langle \gamma\gamma | O_2^{(0)}(z) | h \rangle + H_3^{(0)} \langle \gamma\gamma | O_3^{(0)} | h \rangle$$

- ❖ Hard matching coefficients:

$$H_1^{(0)} = \frac{y_{b,0}}{\sqrt{2}} \frac{N_c \alpha_{b,0}}{\pi} (-M_h^2 - i0)^{-\epsilon} e^{\epsilon\gamma_E} (1 - 3\epsilon) \frac{2\Gamma(1 + \epsilon) \Gamma^2(-\epsilon)}{\Gamma(3 - 2\epsilon)}$$

$$\times \left\{ 1 - \frac{C_F \alpha_{s,0}}{4\pi} (-M_h^2 - i0)^{-\epsilon} e^{\epsilon\gamma_E} \frac{\Gamma(1 + 2\epsilon) \Gamma^2(-2\epsilon)}{\Gamma(2 - 3\epsilon)} \right.$$

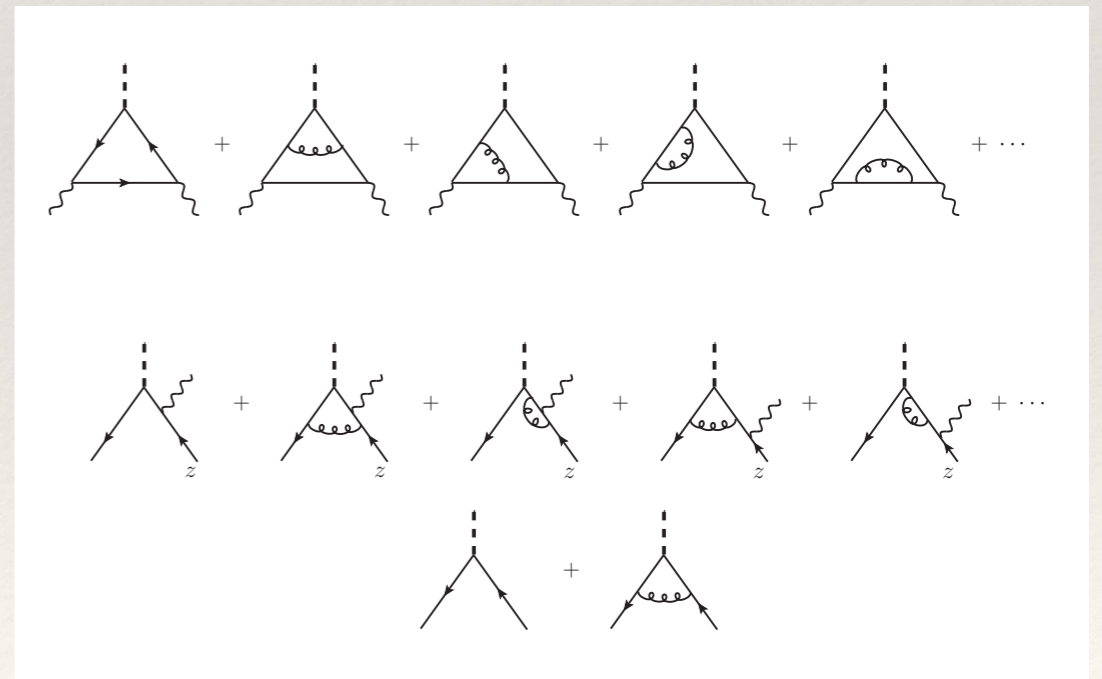
$$\times \left[\frac{2(1 - \epsilon)(3 - 12\epsilon + 9\epsilon^2 - 2\epsilon^3)}{1 - 3\epsilon} + \frac{8}{1 - 2\epsilon} \frac{\Gamma(1 + \epsilon) \Gamma^2(2 - \epsilon) \Gamma(2 - 3\epsilon)}{\Gamma(1 + 2\epsilon) \Gamma^3(1 - 2\epsilon)} \right.$$

$$\left. \left. - \frac{4(3 - 18\epsilon + 28\epsilon^2 - 10\epsilon^3 - 4\epsilon^4)}{1 - 3\epsilon} \frac{\Gamma(2 - \epsilon)}{\Gamma(1 + \epsilon) \Gamma(2 - 2\epsilon)} \right] \right\}$$

$$H_2^{(0)}(z) = \frac{y_{b,0}}{\sqrt{2}} \left\{ \frac{1}{z} + \frac{C_F \alpha_{s,0}}{4\pi} (-M_h^2 - i0)^{-\epsilon} e^{\epsilon\gamma_E} \frac{\Gamma(1 + \epsilon) \Gamma^2(-\epsilon)}{\Gamma(2 - 2\epsilon)} \right.$$

$$\times \left[\frac{2 - 4\epsilon - \epsilon^2}{z^{1+\epsilon}} - \frac{2(1 - \epsilon)^2}{z} - 2(1 - 2\epsilon - \epsilon^2) \frac{1 - z^{-\epsilon}}{1 - z} \right] \left. \right\} + (z \rightarrow 1 - z)$$

$$H_3^{(0)} = \frac{y_{b,0}}{\sqrt{2}} \left[-1 + \frac{C_F \alpha_{s,0}}{4\pi} (-M_h^2 - i0)^{-\epsilon} e^{\epsilon\gamma_E} 2(1 - \epsilon)^2 \frac{\Gamma(1 + \epsilon) \Gamma^2(-\epsilon)}{\Gamma(2 - 2\epsilon)} \right]$$



“Bare factorization theorem”

- ❖ Adding up the three contributions we find:

$$\mathcal{M}_b(h \rightarrow \gamma\gamma) = H_1^{(0)} \langle \gamma\gamma | O_1^{(0)} | h \rangle + 2 \int_0^1 dz H_2^{(0)}(z) \langle \gamma\gamma | O_2^{(0)}(z) | h \rangle + H_3^{(0)} \langle \gamma\gamma | O_3^{(0)} | h \rangle$$

- ❖ Operator matrix elements:

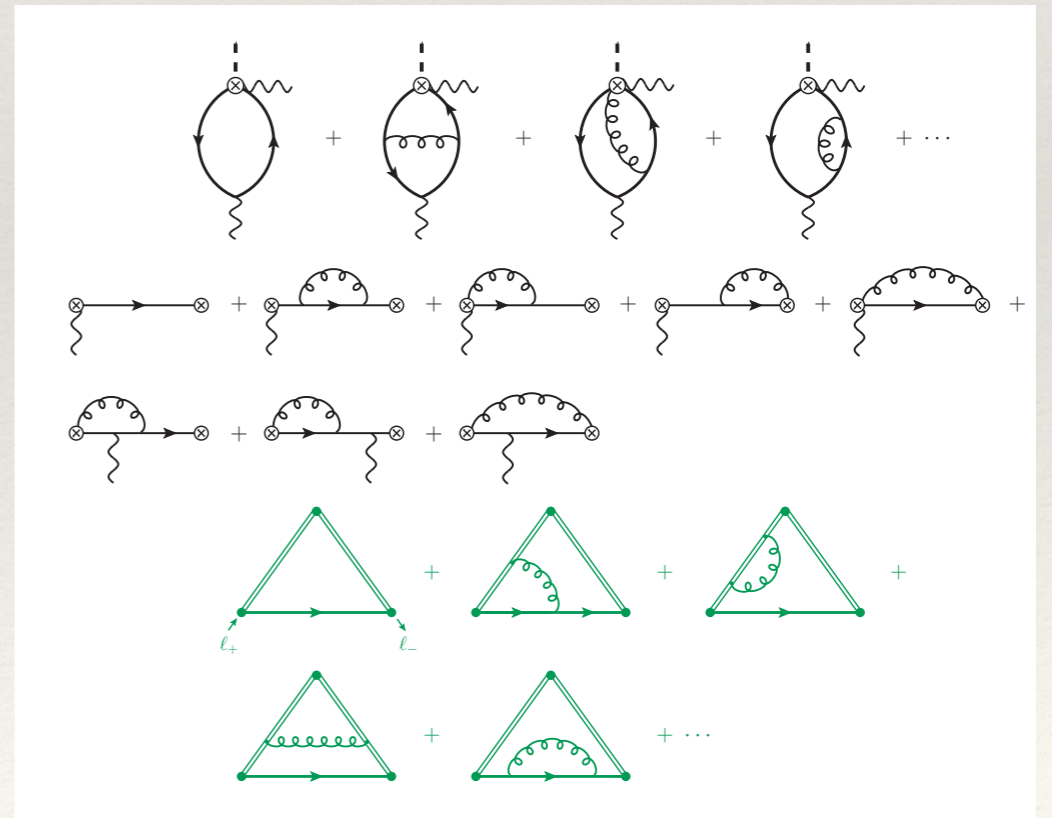
$$\langle \gamma\gamma | O_1^{(0)} | h \rangle = m_{b,0} g_{\perp}^{\mu\nu}$$

$$\langle \gamma\gamma | O_2^{(0)}(z) | h \rangle = \frac{N_c \alpha_{b,0}}{2\pi} m_{b,0} g_{\perp}^{\mu\nu} \left[e^{\epsilon\gamma_E} \Gamma(\epsilon) (m_{b,0}^2)^{-\epsilon} + \frac{C_F \alpha_{s,0}}{4\pi} (m_{b,0}^2)^{-2\epsilon} [K(z) + K(1-z)] \right]$$

$$J^{(0)}(p^2) = 1 + \frac{C_F \alpha_{s,0}}{4\pi} (-p^2 - i0)^{-\epsilon} e^{\epsilon\gamma_E} \frac{\Gamma(1+\epsilon) \Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)} (2 - 4\epsilon - \epsilon^2)$$

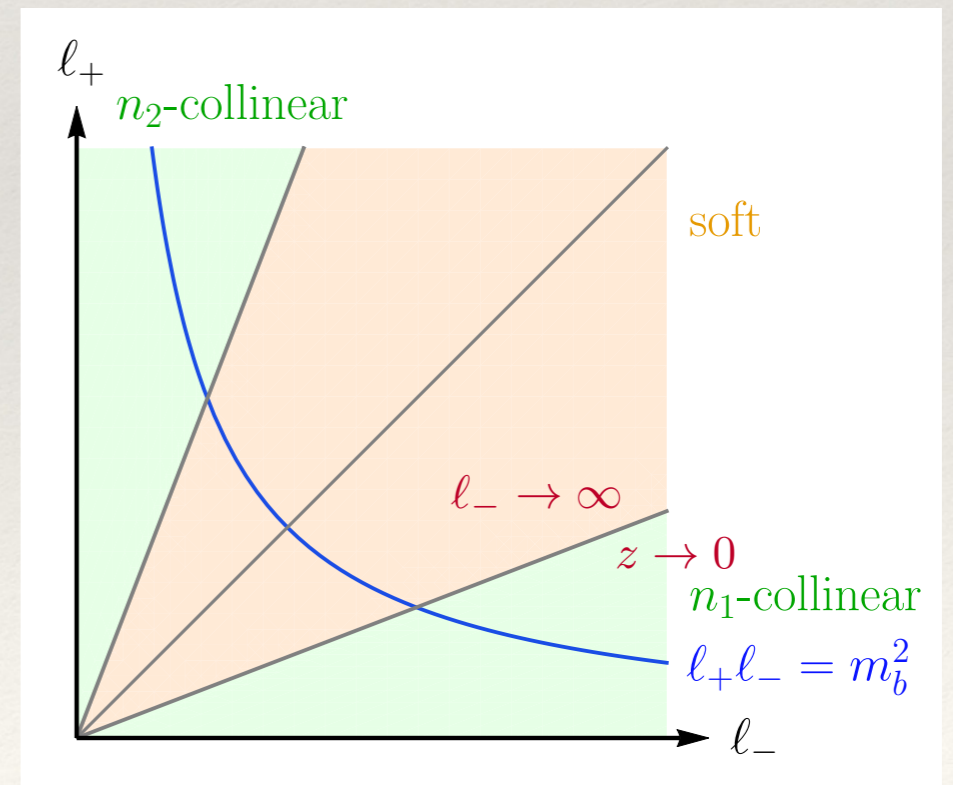
$$S^{(0)}(w) = -\frac{N_c \alpha_{b,0}}{\pi} m_{b,0} \left[S_a^{(0)}(w) \theta(w - m_{b,0}^2) + S_b^{(0)}(w) \theta(m_{b,0}^2 - w) \right]$$

...



Endpoint divergences

- ❖ Closer inspection shows that the convolution integrals in the factorization formula are divergent for $z \rightarrow 0, 1$ (second term) and $l_{\pm} \rightarrow \infty$ (third term)
- ❖ Second term is symmetric under $z \leftrightarrow (1 - z)$ and it suffices to study the singularity at $z \rightarrow 0$
- ❖ Physical origin: **overlap of soft and collinear regions**, whose boundaries are not separated by the dimensional regulator



Endpoint divergences

- ❖ In order to define the two convolutions properly one needs to introduce a **rapidity regulator** under the integrals:

$$\begin{aligned}
 \mathcal{M}_b(h \rightarrow \gamma\gamma) = & \lim_{\eta \rightarrow 0} H_1^{(0)} \langle \gamma\gamma | O_1^{(0)} | h \rangle + 4 \int_0^1 \frac{dz}{z} \left(\frac{-z M_h^2 - i0}{\nu^2} \right)^\eta \bar{H}_2^{(0)}(z) \langle \gamma\gamma | O_2^{(0)}(z) | h \rangle \\
 & + g_\perp^{\mu\nu} H_3^{(0)} \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^{\ell_-} \frac{d\ell_+}{\ell_+} S^{(0)}(\ell_+ \ell_-) \\
 & \times \left[\left(\frac{\bar{n}_2 \cdot k_2 \ell_- - i0}{\nu^2} \right)^\eta J^{(0)}(\bar{n}_1 \cdot k_1 \ell_+) J^{(0)}(-\bar{n}_2 \cdot k_2 \ell_-) \right. \\
 & \left. + \left(\frac{-\bar{n}_2 \cdot k_2 \ell_- - i0}{\nu^2} \right)^\eta J^{(0)}(-\bar{n}_1 \cdot k_1 \ell_+) J^{(0)}(\bar{n}_2 \cdot k_2 \ell_-) \right]
 \end{aligned}$$

- ❖ Endpoint divergences lead to $1/\eta$ poles, which cancel in the sum of all terms!

Endpoint divergences

- ❖ Things are, in fact, even more subtle. For example, in higher orders one finds that:

$$\bar{H}_2^{(0)}(z) \sim z^{-n\epsilon} \quad \text{but} \quad \langle O_2^{(0)}(z) \rangle \sim z^{+m\epsilon}$$

- ❖ Terms with $m = n$ require the rapidity regulator when integrated over $\int_0^1 \frac{dz}{z}$, while those with $m \neq n$ are regularized by the dimensional regulator
- ❖ In simpler examples based on SCET-1, the dimensional regulator regularizes all endpoint divergences, but this still leaves the problem of how to deal with the $1/\epsilon$ poles from the endpoint singularities, which spoil factorization

[Beneke et al., Moult et al. 2018-2020]

Endpoint divergences

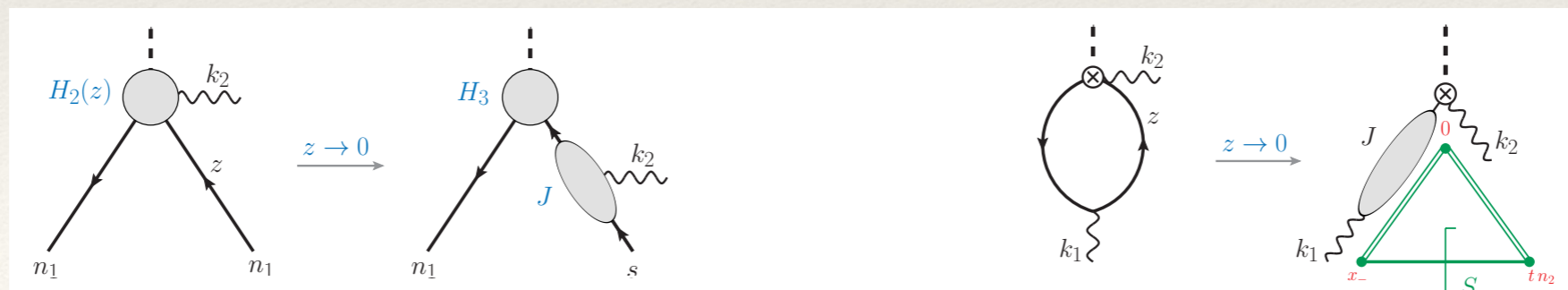
- ❖ All-order cancellation of $1/\eta$ poles requires that the integrands of the second and third term are the same when evaluated in the singular regions!

- ❖ This is ensured by the D -dim. refactorization conditions:

$$\begin{aligned} \llbracket \bar{H}_2^{(0)}(z) \rrbracket &= -H_3^{(0)} J^{(0)}(zM_h^2) \\ \llbracket \langle \gamma\gamma | O_2^{(0)}(z) | h \rangle \rrbracket &= -\frac{g_\perp^{\mu\nu}}{2} \int_0^\infty \frac{d\ell_+}{\ell_+} J^{(0)}(-M_h\ell_+) S^{(0)}(zM_h\ell_+) \end{aligned}$$

[Liu, MN 2019]

- ❖ We have recently proved these relations using SCET tools:



[2009.06779]

Removing endpoint divergences

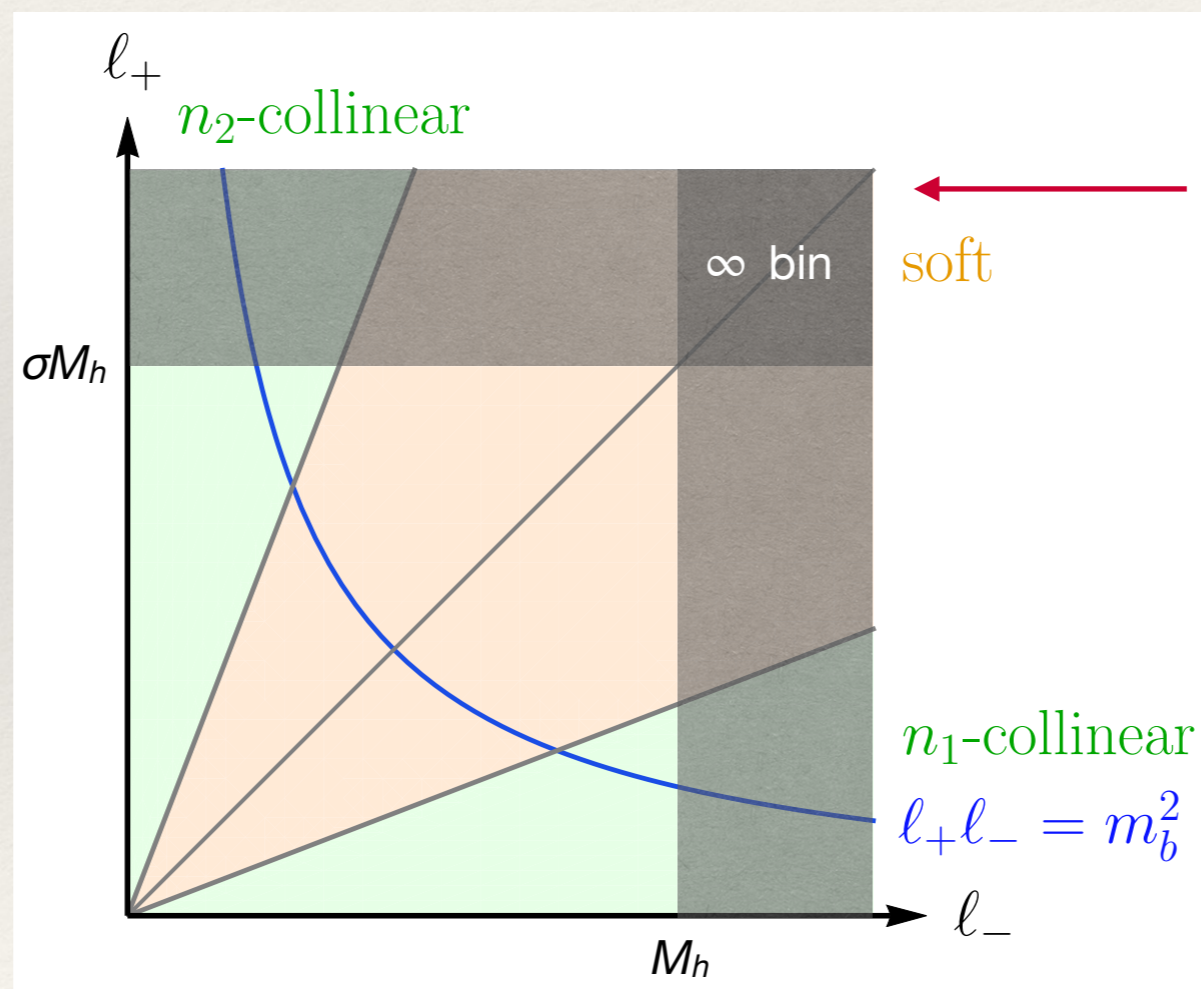
- Using these relations, the bare factorization formula can be rearranged in such a way that all endpoint divergences are removed and the limit $\eta \rightarrow 0$ can be taken. We find:

$$\begin{aligned}
 \mathcal{M}_b = & \left(H_1^{(0)} + \Delta H_1^{(0)} \right) \langle \gamma\gamma | O_1^{(0)} | h \rangle \\
 & + 2 \lim_{\delta \rightarrow 0} \int_{\delta}^{1-\delta} dz \left[H_2^{(0)}(z) \langle \gamma\gamma | O_2^{(0)}(z) | h \rangle - \frac{[\bar{H}_2^{(0)}(z)]}{z} \left[\langle \gamma\gamma | O_2^{(0)}(z) | h \rangle \right] \right. \\
 & \quad \left. - \frac{[\bar{H}_2^{(0)}(1-z)]}{1-z} \left[\langle \gamma\gamma | O_2^{(0)}(1-z) | h \rangle \right] \right] \\
 & + g_{\perp}^{\mu\nu} \lim_{\sigma \rightarrow -1} H_3^{(0)} \int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+} J^{(0)}(M_h \ell_-) J^{(0)}(-M_h \ell_+) S^{(0)}(\ell_+ \ell_-) \Big|_{\text{leading power}}
 \end{aligned}$$

integrand for $z \rightarrow 0$

Removing endpoint divergences

- ❖ In the space of momentum modes, this amounts to the following subtractions in the third term:



“infinity bin” is subtracted twice and must be added back as a hard contribution $\Delta H_1^{(0)}$ to the coefficient of the first term

Renormalized factorization theorem

- ❖ So far, the factorization formula is still expressed in terms of bare quantities, but we wish to establish a corresponding renormalized formula:

$$\begin{aligned}
 \mathcal{M}_b = & H_1(\mu) \langle O_1(\mu) \rangle \\
 & + 2 \int_0^1 dz \left[H_2(z, \mu) \langle O_2(z, \mu) \rangle - \frac{[\bar{H}_2(z, \mu)]}{z} [\langle O_2(z, \mu) \rangle] - \frac{[\bar{H}_2(\bar{z}, \mu)]}{\bar{z}} [\langle O_2(\bar{z}, \mu) \rangle] \right] \\
 & + g_{\perp}^{\mu\nu} H_3(\mu) \lim_{\sigma \rightarrow -1} \int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+} J(M_h \ell_-, \mu) J(-M_h \ell_+, \mu) S(\ell_+ \ell_-, \mu) \Big|_{\text{leading power}}
 \end{aligned}$$

- ❖ This is non-trivial, because the presence of cutoffs does not commute with renormalization!

Renormalized factorization theorem

- ❖ Renormalization conditions for the operators:

$$O_1(\mu) = Z_{11} O_1^{(0)}$$

$$O_2(z, \mu) = \int_0^1 dz' Z_{22}(z, z') O_2^{(0)}(z') + Z_{21}(z) O_1^{(0)}$$

$$[[O_2(z, \mu)]] = \int_0^\infty dz' [[Z_{22}(z, z')]] [[O_2^{(0)}(z')]] + [[Z_{21}(z)]] O_1^{(0)}$$

$$J(p^2, \mu) = \frac{1}{p^2} \int_0^\infty dp'^2 Z_J(p^2, p'^2; \mu) J^{(0)}(p'^2)$$

$$S(w, \mu) = \int_0^\infty dw' Z_S(w, w'; \mu) S^{(0)}(w')$$

[2009.06779]

[2003.03393]

[2005.03013]

with complicated Z factors containing plus distributions

Renormalized factorization theorem

- ❖ When the cutoffs are moved from the bare over to the renormalized functions, some **left-over terms** remain, which individually have a rather complicated structure and depend both on the hard scale M_h and the soft scale m_b
- ❖ The most non-trivial part of the derivation of the renormalized factorization theorem was to show that, to all orders of perturbation theory, the sum of the left-over terms takes the form of an **additional hard subtraction** $\delta H_1^{(0)}$ of the Wilson coefficient of the operator $O_1^{(0)}$ [2009.06779]

Renormalized factorization theorem

- ❖ After this crucial step had been accomplished, we could derive the renormalization conditions for the matching coefficients:

$$H_1(\mu) = \left(H_1^{(0)} + \Delta H_1^{(0)} - \delta H_1^{(0), \text{tot}} \right) Z_{11}^{-1} \\ + 2 \lim_{\delta \rightarrow 0} \int_{\delta}^{1-\delta} dz \left[H_2^{(0)}(z) Z_{21}^{-1}(z) - \frac{[[\bar{H}_2^{(0)}(z)]]}{z} [[Z_{21}^{-1}(z)]] - \frac{[[\bar{H}_2^{(0)}(\bar{z})]]}{\bar{z}} [[Z_{21}^{-1}(\bar{z})]] \right]$$

$$H_2(z, \mu) = \int_0^1 dz' H_2^{(0)}(z') Z_{22}^{-1}(z', z)$$

$$\frac{[[\bar{H}_2(z, \mu)]]}{z} = \int_0^\infty dz' \frac{[[\bar{H}_2^{(0)}(z')]]}{z'} [[Z_{22}^{-1}(z', z)]]$$

$$H_3(\mu) = H_3^{(0)} Z_{33}^{-1}$$

Renormalized factorization theorem

❖ Renormalized matrix elements, with $L_m = \ln(m_b^2/\mu^2)$:

$$\langle O_1(\mu) \rangle = m_b(\mu) g_\perp^{\mu\nu}$$

$$\langle O_2(z, \mu) \rangle = \frac{N_c \alpha_b}{2\pi} m_b(\mu) g_\perp^{\mu\nu} \left\{ -L_m + \frac{C_F \alpha_s}{4\pi} \left[L_m^2 \left(\ln z + \ln(1-z) + 3 \right) - L_m \left(\ln^2 z + \ln^2(1-z) - 4 \ln z \ln(1-z) + 11 - \frac{2\pi^2}{3} \right) + F(z) + F(1-z) \right] + \mathcal{O}(\alpha_s^2) \right\}$$

$$J(p^2, \mu) = 1 + \frac{C_F \alpha_s}{4\pi} \left[\ln^2 \left(\frac{-p^2 - i0}{\mu^2} \right) - 1 - \frac{\pi^2}{6} \right] + \mathcal{O}(\alpha_s^2)$$

$$S(w, \mu) = -\frac{N_c \alpha_b}{\pi} m_b(\mu) \left[S_a(w, \mu) \theta(w - m_b^2) + S_b(w, \mu) \theta(m_b^2 - w) \right]$$

$$S_a(w, \mu) = 1 + \frac{C_F \alpha_s}{4\pi} \left[-L_w^2 - 6L_w + 12 - \frac{\pi^2}{2} + 2 \text{Li}_2 \left(\frac{1}{\hat{w}} \right) \right]$$

$$L_w = \ln(w/\mu^2)$$

$$\hat{w} = w/m_b^2$$

$$- 4 \ln \left(1 - \frac{1}{\hat{w}} \right) \left(L_m + 1 + \ln \left(1 - \frac{1}{\hat{w}} \right) + \frac{3}{2} \ln \hat{w} \right) \right] + \mathcal{O}(\alpha_s^2)$$

Renormalized factorization theorem

❖ Renormalized matrix elements, with $L_m = \ln(m_b^2/\mu^2)$:

$$\langle O_1(\mu) \rangle = m_b(\mu) g_\perp^{\mu\nu}$$

$$\langle O_2(z, \mu) \rangle = \frac{N_c \alpha_b}{2\pi} m_b(\mu) g_\perp^{\mu\nu} \left\{ -L_m + \frac{C_F \alpha_s}{4\pi} \left[L_m^2 \left(\ln z + \ln(1-z) + 3 \right) - L_m \left(\ln^2 z + \ln^2(1-z) - 4 \ln z \ln(1-z) + 11 - \frac{2\pi^2}{3} \right) + F(z) + F(1-z) \right] + \mathcal{O}(\alpha_s^2) \right\}$$

$$J(p^2, \mu) = 1 + \frac{C_F \alpha_s}{4\pi} \left[\ln^2 \left(\frac{-p^2 - i0}{\mu^2} \right) - 1 - \frac{\pi^2}{6} \right] + \mathcal{O}(\alpha_s^2)$$

$$S(w, \mu) = -\frac{N_c \alpha_b}{\pi} m_b(\mu) \left[S_a(w, \mu) \theta(w - m_b^2) + S_b(w, \mu) \theta(m_b^2 - w) \right]$$

$$L_w = \ln(w/\mu^2)$$

$$\hat{w} = w/m_b^2$$

$$S_b(w, \mu) = \frac{C_F \alpha_s}{\pi} \ln(1 - \hat{w}) \left[L_m + \ln(1 - \hat{w}) \right] + \mathcal{O}(\alpha_s^2)$$

Renormalized factorization theorem

- ❖ Renormalized matching coefficients, with $L_h = \ln(-M_h^2/\mu^2)$:

$$H_1(\mu) = \frac{N_c \alpha_b}{\pi} \frac{y_b(\mu)}{\sqrt{2}} \left\{ -2 + \frac{C_F \alpha_s}{4\pi} \left[-\frac{\pi^2}{3} L_h^2 + (12 + 8\zeta_3) L_h - 36 - \frac{2\pi^2}{3} - \frac{11\pi^4}{45} \right] + \mathcal{O}(\alpha_s^2) \right\}$$

$$H_2(z, \mu) = \frac{y_b(\mu)}{\sqrt{2}} \frac{1}{z(1-z)} \left\{ 1 + \frac{C_F \alpha_s}{4\pi} \left[2L_h(\ln z + \ln(1-z)) + \ln^2 z + \ln^2(1-z) - 3 \right] + \mathcal{O}(\alpha_s^2) \right\}$$

$$H_3(\mu) = \frac{y_b(\mu)}{\sqrt{2}} \left[-1 + \frac{C_F \alpha_s}{4\pi} \left(L_h^2 + 2 - \frac{\pi^2}{6} \right) + \mathcal{O}(\alpha_s^2) \right]$$

Resummation of large logarithms

Liu, Mecaj, MN, Yang: 2009.04456 & 2009.06779

Liu, MN: 2003.03393 (JHEP)

Liu, Mecaj, MN, Yang, Fleming: 2005.03013 (JHEP)



Resummation of large logs

- ❖ The renormalized factorization formula

$$\begin{aligned}
 \mathcal{M}_b = & H_1(\mu) \langle O_1(\mu) \rangle \\
 & + 2 \int_0^1 dz \left[H_2(z, \mu) \langle O_2(z, \mu) \rangle - \frac{[\bar{H}_2(z, \mu)]}{z} [[\langle O_2(z, \mu) \rangle]] - \frac{[\bar{H}_2(\bar{z}, \mu)]}{\bar{z}} [[\langle O_2(\bar{z}, \mu) \rangle]] \right] \\
 & + g_{\perp}^{\mu\nu} H_3(\mu) \lim_{\sigma \rightarrow -1} \int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+} J(M_h \ell_-, \mu) J(-M_h \ell_+, \mu) S(\ell_+ \ell_-, \mu) \Big|_{\text{leading power}}
 \end{aligned}$$

provides a complete scale separation and allows us to resum large logarithms in the decay amplitude to all orders of perturbation theory!

Resummation of large logs

❖ RG equations for matrix elements:

$$\frac{d}{d \ln \mu} \langle O_1(\mu) \rangle = -\gamma_{11} \langle O_1(\mu) \rangle$$

$$\frac{d}{d \ln \mu} \langle O_2(z, \mu) \rangle = - \int_0^1 dz' \gamma_{22}(z, z') \langle O_2(z', \mu) \rangle - \gamma_{21}(z) \langle O_1(\mu) \rangle$$

$$\frac{d}{d \ln \mu} J(p^2, \mu) = - \int_0^\infty dx \gamma_J(p^2, xp^2) J(xp^2, \mu)$$

$$\frac{d}{d \ln \mu} S(w, \mu) = - \int_0^\infty dx \gamma_S(w, w/x) S(w/x, \mu)$$

Resummation of large logs

- ❖ RG equations for matching coefficients:

$$\frac{d}{d \ln \mu} H_1(\mu) = D_{\text{cut}}(\mu) + \gamma_{11} H_1(\mu) + 2 \int_0^1 dz \left[H_2(z, \mu) \gamma_{21}(z) - \frac{[\bar{H}_2(z, \mu)]}{z} [\gamma_{21}(z)] - \frac{[\bar{H}_2(\bar{z}, \mu)]}{\bar{z}} [\gamma_{21}(\bar{z})] \right]$$

inhomogeneous contribution due to cutoffs

$$\frac{d}{d \ln \mu} H_2(z, \mu) = \int_0^1 dz' H_2(z', \mu) \gamma_{22}(z', z)$$

$$\frac{d}{d \ln \mu} H_3(\mu) = \gamma_{33} H_3(\mu)$$

- ❖ where:

$$D_{\text{cut}}(\mu) = -\frac{N_c \alpha_b}{\pi} \frac{y_b(\mu)}{\sqrt{2}} \left[\frac{C_F \alpha_s}{4\pi} 16\zeta_3 + \left(\frac{\alpha_s}{4\pi} \right)^2 d_{\text{cut},2} + \mathcal{O}(\alpha_s^3) \right] \ni \alpha_b (\alpha_s L_h)^n$$

Logarithms in the 3-loop amplitude

- ❖ From a perturbative solution of the RGEs, we have obtained predictions for the terms of order $\mathcal{O}(\alpha_s^2 L^k)$ with $k=6,5,4,3$ in the 3-loop decay amplitude in the on-shell scheme, finding:

$$\begin{aligned} \mathcal{M}_b &= \frac{N_c \alpha_b}{\pi} \frac{m_b^2}{v} \varepsilon_{\perp}^*(k_1) \cdot \varepsilon_{\perp}^*(k_2) \\ &\times \left\{ \frac{L^2}{2} - 2 + \frac{C_F \alpha_s(\hat{\mu}_h)}{4\pi} \left[-\frac{L^4}{12} - L^3 - \frac{2\pi^2}{3} L^2 + \left(12 + \frac{2\pi^2}{3} + 16\zeta_3 \right) L - 20 + 4\zeta_3 - \frac{\pi^4}{5} \right] \right. \\ &\quad \left. + C_F \left(\frac{\alpha_s(\hat{\mu}_h)}{4\pi} \right)^2 \left[\frac{C_F}{90} L^6 + \left(\frac{C_F}{10} - \frac{\beta_0}{30} \right) L^5 + d_4^{\text{OS}} L^4 + d_3^{\text{OS}} L^3 + \dots \right] \right\} \end{aligned}$$

- ❖ Find perfect agreement with recent numerical results!

[Czakon, Niggetiedt 2020]

Series of subleading logs

- ❖ We have reproduced the series of leading double logs (LL) and obtained a **new result** for the NLL logs to all orders in α_s :

$$\mathcal{M}_b^{\text{NLL}} = \frac{N_c \alpha_b}{\pi} \frac{y_b(M_h)}{\sqrt{2}} m_b \varepsilon_{\perp}^*(k_1) \cdot \varepsilon_{\perp}^*(k_2) \frac{L^2}{2} \sum_{n=0}^{\infty} (-\rho)^n \frac{2\Gamma(n+1)}{\Gamma(2n+3)} \times \left[1 + \frac{3\rho}{2L} \frac{2n+1}{2n+3} - \frac{\beta_0}{C_F} \frac{\rho^2}{4L} \frac{(n+1)^2}{(2n+3)(2n+5)} \right],$$

[Kotsky, Yakovlev 1997]

with $\rho = \frac{C_F \alpha_s(M_h)}{2\pi} L^2$

- ❖ The subleading terms disagree with earlier results in the literature!

[Akhoury, Wang, Yakovlev 2001; Anastasiou, Penin 2020]

Resummation in RG-improved PT

- ❖ Ultimate goal is to resum **all large logarithms** and exponentiate them (RG-improved perturbation theory)
- ❖ Particularly important for Sudakov problems, where leading logs are formally larger than $O(1)$
- ❖ In RG-improved perturbation theory one supplies the matching conditions for all component functions in the factorization theorem at matching scales where they are free of large logs; these functions are then evolved to a common scale solving their RG equations → **all large logs exponentiate!**

Resummation in RG-improved PT

- ❖ We have not yet performed a complete resummation, but we have resummed the most difficult contribution T_3 at LO in RG-improved perturbation theory, finding: [2009.04456]

$$T_3^{\text{LO}} = \frac{\alpha}{3\pi} \frac{y_b(\mu_h)}{\sqrt{2}} \int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{M_h} \frac{d\ell_+}{\ell_+} m_b(\mu_s) e^{2\mathcal{S}(\mu_s, \mu_h) - 2\mathcal{S}(\mu_-, \mu_h) - 2\mathcal{S}(\mu_+, \mu_h)} \left(\frac{-M_h \ell_-}{\mu_-^2} \right)^{a_\Gamma^-} \left(\frac{-M_h \ell_+}{\mu_+^2} \right)^{a_\Gamma^+} \left(\frac{-\ell_+ \ell_-}{\mu_s^2} \right)^{-a_\Gamma^s}$$

$$\times \left(\frac{\alpha_s(\mu_s)}{\alpha_s(\mu_h)} \right)^{\frac{12}{23}} e^{-2\gamma_E a_\Gamma^+} \frac{\Gamma(1 - a_\Gamma^+)}{\Gamma(1 + a_\Gamma^+)} e^{-2\gamma_E a_\Gamma^-} \frac{\Gamma(1 - a_\Gamma^-)}{\Gamma(1 + a_\Gamma^-)} e^{4\gamma_E a_\Gamma^s} G_{4,4}^{2,2} \left(\begin{matrix} -a_\Gamma^s, -a_\Gamma^s, 1 - a_\Gamma^s, 1 - a_\Gamma^s \\ 0, 1, 0, 0 \end{matrix} \middle| \frac{m_b^2}{-\ell_+ \ell_-} \right)$$

with:

$$a_\Gamma^i = -\frac{8}{23} \ln \frac{\alpha_s(\mu_i)}{\alpha_s(\mu_h)}, \quad \mathcal{S}(\mu_i, \mu_h) = \frac{12}{529} \left[\frac{4\pi}{\alpha_s(\mu_i)} \left(1 - \frac{1}{r} - \ln r \right) + \frac{58}{23} \ln^2 r + \left(\frac{2429}{207} - \pi^2 \right) (1 - r + \ln r) \right]$$

$$r = \alpha_s(\mu_h) / \alpha_s(\mu_i)$$

- ❖ dynamical matching scales:

$$\mu_s^2 \sim \ell_+ \ell_- \quad \mu_\pm^2 \sim M_h \ell_\pm \quad \mu_h \sim M_h$$

Conclusions

- ❖ We have derived the first SCET factorization theorem for an observable appearing at subleading order in power counting
- ❖ Generic features:
 - ▶ several SCET operators exist \rightarrow several terms in factorization formula
 - ▶ these operators mix under renormalization
 - ▶ endpoint divergences in convolutions cancel between the different terms; cancellation ensured by ***D*-dim. refactorization conditions**
 - ▶ endpoint divergences can be removed by performing subtractions and rearranging the various terms
- ❖ Our results are an important step towards establishing SCET as a complete EFT admitting a consistent power expansion!