THE FULL ANGLE DEPENDENCE OF THE FOUR-LOOP CUSP ANOMALOUS DIMENSION IN QED

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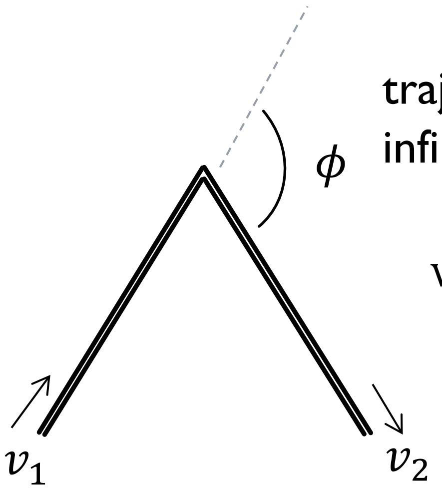
Based on 2002.02340, 2007.04851
with Brüser,Dlapa,Henn

The angle-dependent cusp anomalous dimension governs infinite-mass limit of physical processes involving heavy-quark scattering/production:

Heavy quark scattering off an external potential:
heavy meson decay.
cross channel: top quark pair production.

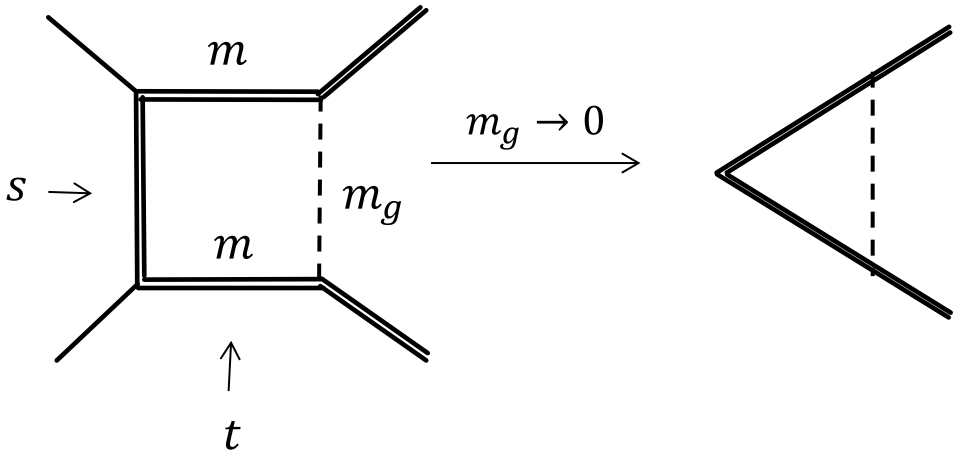
$m_Q \to \infty$, Form factor is a function of a single variable $v_1 \cdot v_2 = \cosh\phi_M$ (Minkowski recoiling angle).



trajectory of classical source: semi-infinite Wilson-line with a cusp

velocity $v_i^\mu = \frac{p_i^\mu}{m_Q}$

$v_1^2 = 1, v_2^2 = 1,$
$v_1 \cdot v_2 := \cosh\phi_M := \cos\phi$



Cusp anomalous dimension governs the infrared divergences of massive scattering amplitudes.

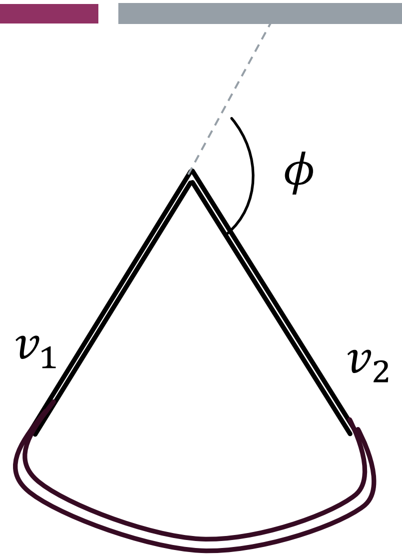
$$\log A_4(s,t,m^2;m_g) \sim \log m_g \;\; \Gamma_{cusp}(\phi) \qquad s = 4m^2 \sin^2\frac{\phi}{2}$$

Plays a role in study of IR structure of massive scattering amplitudes;
connects to high-energy (Regge) limit (e.g. via dual conformal symmetry in massive planar N=4 sYM)

Anomalous dimension of cusped Wilson loop

$$\langle W_C \rangle = \frac{1}{N_R} \langle tr_R P \exp \oint_C dt\, \dot{x}_\mu \cdot A^\mu(x) \rangle$$

Local UV divergence: $$\frac{d}{d \ln \mu} \log\langle W_C \rangle = \Gamma_{cusp}(\phi)$$



$$\frac{v_1 \cdot v_2}{\sqrt{v_1^2}\sqrt{v_2^2}} = \cos\phi$$

- Expectation value of Wilson loop studied in N=4 sYM in particular in its planar limit.

Integrability methods. [Beisert, Eden, Staudacher. J. Stat. Mech. 0701:P01021 (2007)] [Basso, Korchemsky. J. Phys. A 42:254005 (2009)]

ADS-CFT correspondce. [Maldacena, Phys. Rev. Lett. 80 (1998) 4859, hep-th/9803002]

- Full QCD result is computed at three-loop order .

Matter dependence exhibits remarkable universal iterative structure.
[Grozin, Henn, Korchemsky, Marquard, 2014].

$$\Gamma_{cusp}(\phi) = C_R \frac{a}{\pi}\left[\Omega(\phi) + C_A \Omega_A(\phi)\frac{a}{\pi} + C_A^2 \Omega_{AA}(\phi)\left(\frac{a}{\pi}\right)^2\right] + O(a^4)$$

$$\supset \left(\frac{\alpha_s}{\pi}\right)^4 n_l d_{RF} B(\phi)$$

Breaks Casimir scaling;
Breaks universal structure of matter dependence

$\Omega(\phi)$ independent of the particle content of the theory.
$n_f -$ depdence encoded in the effective coupling a.

OUTLINE

- The four-loop matter-dependent quartic casimir component of angle dependent cusp anomalous dimension in a generic U(N) gauge theory. [Brüser, Dlapa, Henn, Yan 2002.02340]

$$\Gamma_{cusp} \Big|_{d_{RF},\alpha^4} = \left(\frac{\alpha}{\pi}\right)^4 \frac{d_R d_F}{N_R} [n_f \, B(\phi) + n_s \, C(\phi)]$$

  – Methods of calculation and analytic results.
  – Properties of the four-loop formula, asymptotic behaviours.

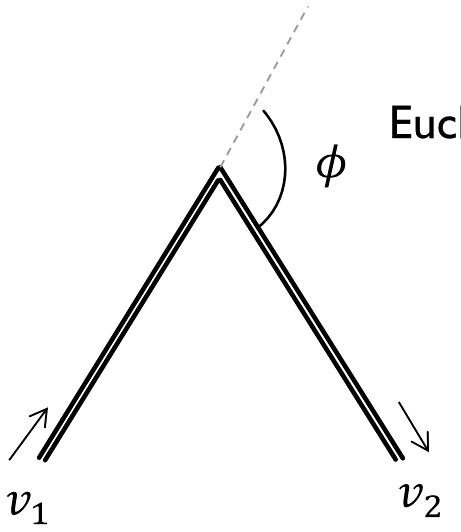
- A novel algorithm for solving coulpled sytem of differential system based on n-th order Picard-Fuchs equations. [Dlapa, Henn, Yan 2007.04851]

  – ideas for finding basis of uniform transcendental weight integral
  – Design of algorithm and work flow
  – Cutting-edge applications, efficiencies

Case of study

$$\log\langle W_C\rangle = \log\frac{V(\phi)}{V(0)}$$

$V(\phi)$ irreducible vetex corrections, $V(0)$ self-energy correction



Euclidean cusp angle

$$x := e^{i\,\phi}$$

$$\frac{v_1\cdot v_2}{\sqrt{v_1^2}\sqrt{v_2^2}} = \frac{1}{2}\left(x+\frac{1}{x}\right) = \cos\phi$$

$\phi\sim 0$ $(x\to 1)$: HQET field-strength anomalous dimension $\gamma_h = \frac{d\log V(0)}{d\ln\mu}$, $\Gamma_{cusp}\sim O(\phi^2)$.

Full angle dependence of $\Gamma_{cusp}(\phi)$

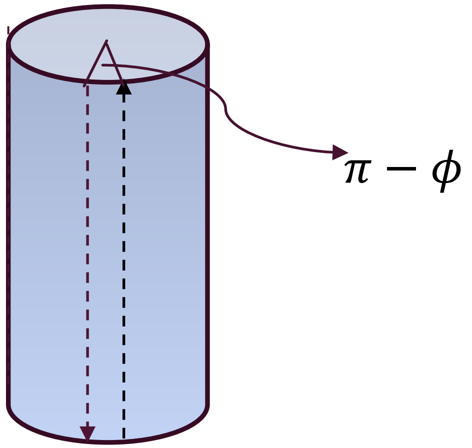
Cusped Wilon loop in Euclidean space mapped onto antipodal lines on $S^3\times R$

-i$\phi\to\infty$ $(x\to 0)$: light-like cusp anomlous dimension $K(\alpha)\ln\frac{1}{x}$. Recent 4-loop result from massless form factor; null-polygon Wilson loop.

[Henn, Korchemsky, Mistlberger.(2020)]
[von Manteuffel, Panzer, Schabinger. (2020)]

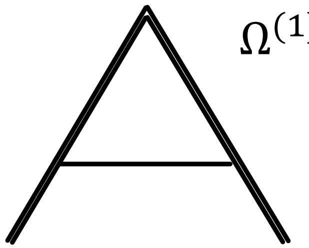
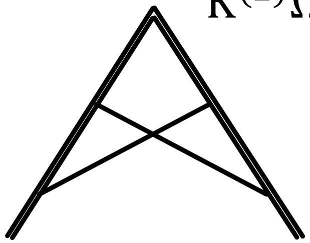
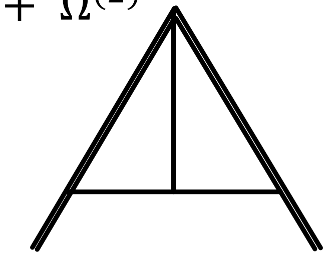
$\phi\sim\pi$ $(x\to -1)$: quark-antiquark static potential.

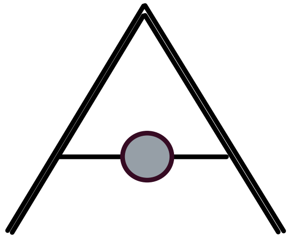
$$\Gamma_{cusp}(\pi-\delta) \sim -C_R\frac{\alpha}{\delta}\left[V_{Q\bar{Q}}+\beta(\alpha)C(\alpha)\right]$$



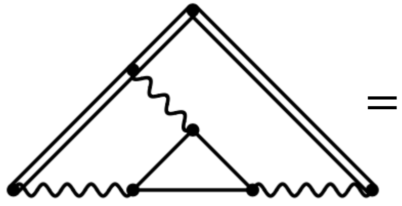
Matter dependence of Γ_{cusp}

$\Omega^{(1)}$ $\quad K^{(2)}\Omega^{(1)} + \Omega^{(2)}$

$$\log\langle W_C\rangle\Big|_{N=4\ sYM} = 1 + $$  $+$   $$+\ O\left(\alpha^3\right)$$

$$\log\langle W_C\rangle\Big|_{QCD} = \log\langle W_C\rangle\Big|_{N=4\ sYM} + $$  $$+\ O\left(\alpha^3\right)$$

matter dependence is proportional to the lower-loop formula $\sim B_l\Omega^{(1)}$
ratio fixed by the asymptotic behaviour in the light-like limit.



$$= C_R C_A T_F n_l\left[B_{Al}\Omega^{(1)} + B_l\Omega^{(2)}\right]$$

This pattern holds up to three-loop order in QCD.

$$\Gamma_{cusp}(\phi,\alpha) = C_R \sum_{k\geq 1}\left(\frac{K(\alpha)}{C_R}\right)^k \Omega^{(k)}(\phi)$$

$K(\alpha)$: light-like cusp anomalous dimension.

$$\Gamma_{cusp}(x,\alpha) \longrightarrow K(\alpha)\log\frac{1}{x}$$

$\Omega^{(k)}$ $(k > 1)$ vanishes in the light-like limit

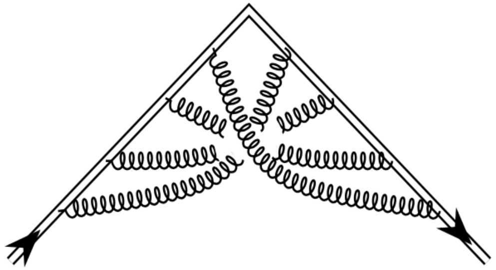
The conjecture breaks down at four loop order for certain new types of color structure.
[Brüser, Grozin, Henn, Stahlhofen 2019]

Four loop quartic casimir color structure

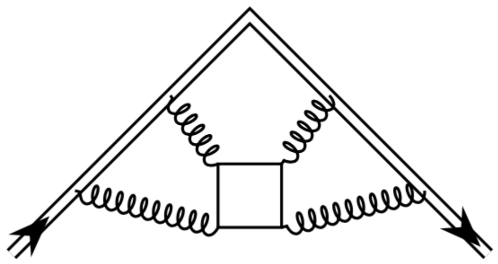
$$\mathrm{tr}_R\,[T^aT^bT^cT^d] = d_R$$

d^2 tems is the first non-planar contribution to Γ_{cusp}

4-loop planar contribution in N=4 obtained [Huber, Henn, 2013] thansks to the simplicity in massive planar amplitudes [Bern, Czakon, Dixon, Kosower, Smirnov 2007]. Less is known for non-planar contributions.



$\frac{d_R d_A}{N_R}$



$\frac{d_R d_F}{N_R}$

We are interested in the matter-dependent quartic casimir component

$$\Gamma_{cusp}\Big|_{d_{RF}} = \left(\frac{\alpha}{\pi}\right)^4 \frac{d_R d_F}{N_R}\left[n_f\, B(\phi) + n_s\, C(\phi)\right]$$

- Necesary input to obtain full QCD result: the gluon quartic Casimir term could be obtained from N=4 super Yang-Mills result for the bosonic Wilson loop !

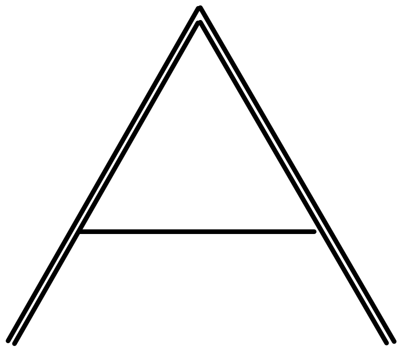
- First quantum correction in the U(1) abelian theory ($d_R = d_F = 1, N_R = 1.$)

Methodology

Compute HQET integrals through Integration-by-part (IBP) reduction + canonical differential equation (DE)

$log\langle W_C\rangle = \sum(w_i(\phi) - w_i(0))$ decomposed onto quasi-finite HQET Integrals

Bring Wilson lines offshell ($\delta = 1$), long-distance divergence regulated



$$= g^2 \int d^d k \frac{1}{(-2\,k\cdot v_1 + 1)(-2\,k\cdot v_2 + 1)\,k^2}$$

$$= g^2 \int ds\, dt\, e^{-\frac{s+t}{2}} \frac{1}{[(s\,v_1 - t\,v_2)^2]^{1+\epsilon}}$$

$$\rho := s + t,\ y := \frac{s}{s+t}, \qquad \int \frac{d\,\rho}{\rho^{1+2\epsilon}} e^{-\frac{\rho}{2}} \sim -\frac{1}{2\epsilon}$$

- Quasi-finiteness: integrals are free from subdivergence. Coefficent of the leading pole is determined by the properties of integrand in 4D.

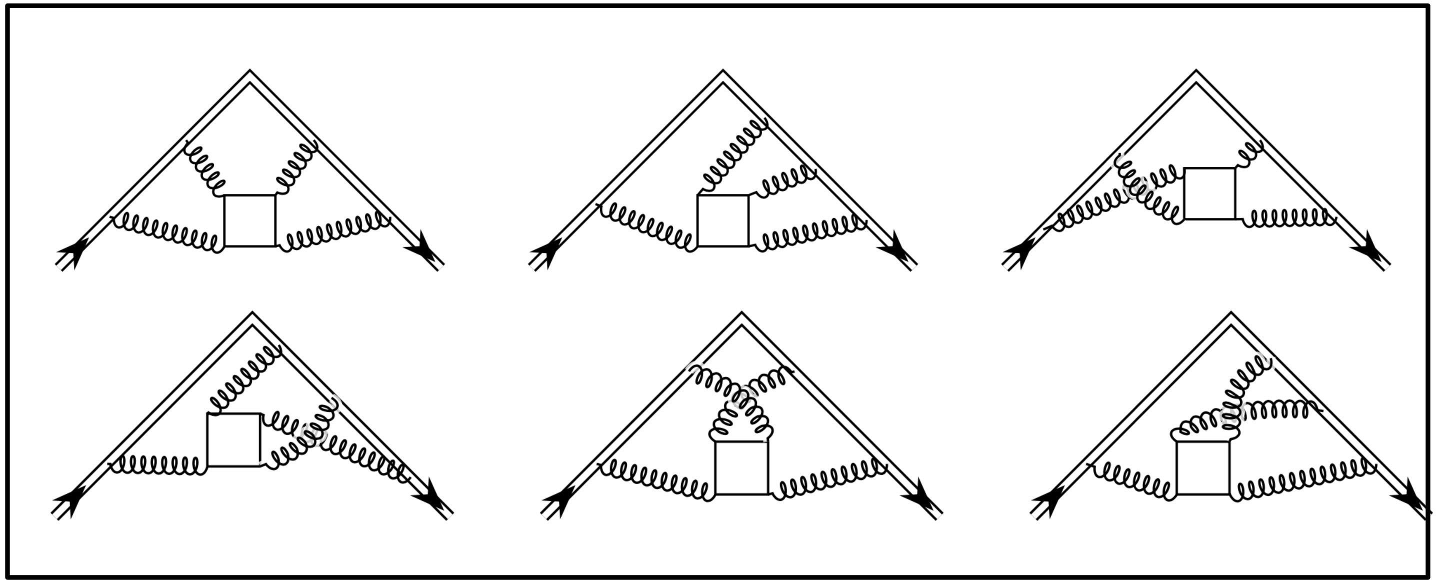
At higher-loop order, web diagrams only contain overall $\frac{1}{\epsilon}$ divergence, the coefficient is scheme independent.

Technical bottleneck

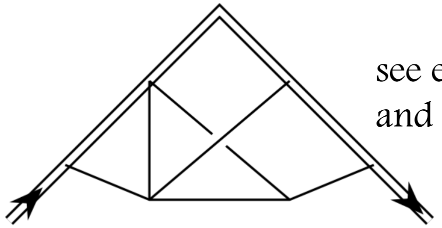
6 integral families, each involves ~500 integrals differential equation contains coupled system of size up to 17×17, denominators of up to degree-20 polynomials in x and D.

- New efficient tools needed for the automation of solving large complicated four-loop system.

IBP reduction [Smirnov 2019 (FIRE6)][Lee 2013 (LiteRed)][Peraro 2019 (FiniteFlow)]

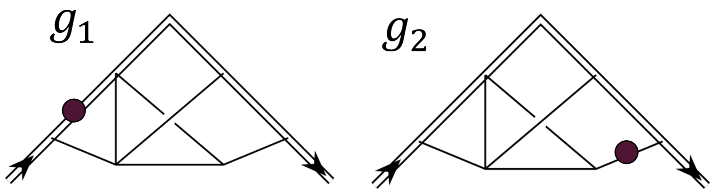


Possibly non-polylogarithmic integral sector



see e.g. [Lee, Talk at „Elliptics and Beyond 2020"]

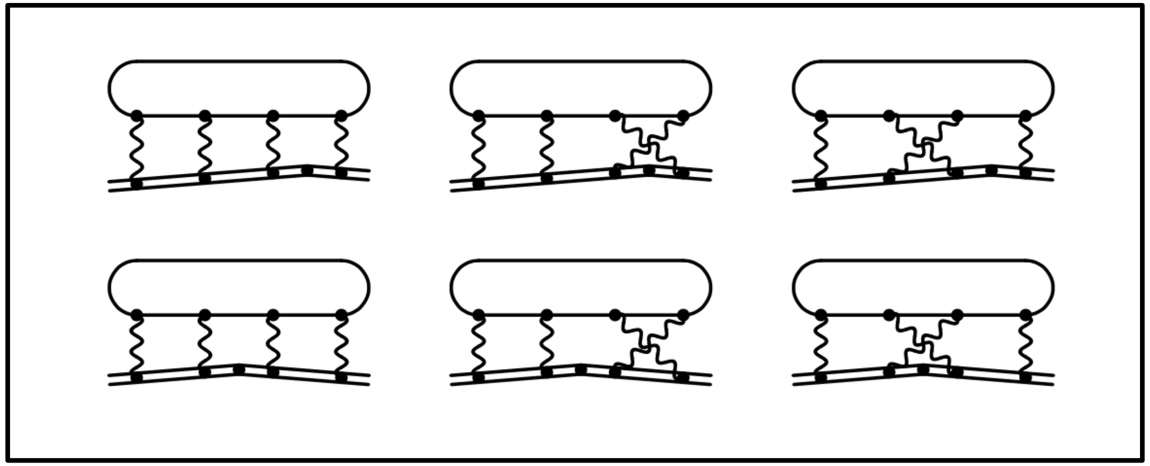
Coupled two-by-two sub-system. One of which is strictly finite, free from ϵ −poles. System decouples at $O(\epsilon^{-1})$



$O(\epsilon)$ terms may be scheme dependent, live in a more complicated function space (e.g. elliptic function)

- Information about finite and $O(\epsilon)$ contributions are mixed through IBP, due to the arbitrariness in the choice of integral basis.

Boundary constants computed from small angle expansions in the straight-line limit



[Brüser, Grozin, Henn, Stahlhofen 2019]

We develop a new method „Initial Algorithm“ based on n-th order Picard Fuchs equation and overcome these difficulties

- Transform DEs in polylogarithmic sectors into canonical form.

$$d\,\vec{f} = \epsilon \sum_a d\ln\alpha_a(x)\, m_a\, \vec{f} \qquad \vec{\alpha} = \{x, 1 \pm x, 1 + x^2, 1 - x + x^2, \frac{1-\sqrt{x}}{1+\sqrt{x}}, \frac{1-\sqrt{x}+x}{1+\sqrt{x}+x}\}$$

- Disentangle the finite and $O(\epsilon)$ contributions in the non-polylogarithmic sectors . Quasi-finite integrals relevant for Γ_{cusp} are all polylogarithmic.

Results for the four-loop quatic casimir terms

n_f −terms: $$\frac{d_R d_F}{N_R} n_f \left[\frac{1+x^2}{1-x^2} B_1 + \frac{x}{1-x^2} B_2 + \frac{1-x^2}{x} B_3 + B_4\right]$$

Knowlege on the function space provides valuable input for bootstrapping the gluonic quartic casimir terms.

Similar for the n_s −terms, $B_i \to C_i$

$B_i(C_i)$: Multiple polylogarithms of weight ranging from 3 to 7, symbol alphabet $\vec{\alpha} = \{x, 1+x, 1-x, 1+x^2\}$.

The first three rational structures appear in three-loop answer.
Symbol of $B_{1,4}$($B_{2,3}$), even (odd) under $x \leftrightarrow -x$.

Asymptotic behaviours

- Small angle limit (x->1) [Grozin, Henn, Stahlhofen 2017] [Brüser, Grozin, Henn, Stahlhofen,2019]
  agree up to $O(\phi^4)$

Results for the four-loop quatic casimir terms

- 4-loop light-like cusp in QED : $K^{(4)} = \frac{\pi^2}{6} - \frac{\zeta_3}{3} - \frac{5\zeta_5}{3}$

[Lee, Smirnov^2, Steinhauser, 2019; Henn, Peraro, Stahlhofen,Wasser, 2019]

$$B(x) \to -\ln x\left(\frac{\pi^2}{6} - \frac{\zeta_3}{3} - \frac{5\zeta_5}{3}\right) + \frac{5\pi^2}{8} - \frac{11\pi^4}{36} + \frac{53\pi^6}{2835} - \frac{35\zeta_3}{12} - \frac{\pi^2\zeta_3}{6} + \frac{185\zeta_5}{12} - 3\zeta_3^2$$

Connects to collinear anomalous dimension of Wilson loop and twist-two anomalous dimensions of DGLAP kernels [Dixon, 2017]

- Quark-antiquark static potential in N=4 sYM at three-loop order for bosonic static charge

gluon and the fermion quartic Casimir terms known [Lee, Smirnov^2, Steinhauser 2016]

$$V_{sYM}\Big|_{d_{RA}} \to 7\pi^2 - \frac{47\pi^4}{24} + \frac{413\pi^6}{1440} + \frac{116\pi^2 l_2}{3} - \frac{89\pi^2\zeta_3}{4} + \frac{3\pi^4 l_2}{2} + \frac{2}{3}\pi^2 l_2^2 - 14\pi^2 l_2\zeta_3 - \frac{17}{12}\pi^2 l_2^4 - 34\pi^2 Li_4\left(\frac{1}{2}\right)$$

Depends on a set of constants propotional to π^2

$$\pi^2 \times \{1, l_2, \zeta_2, \zeta_2 l_2, \alpha_3, l_2\alpha_3, \zeta_2^2, \alpha_4\}$$

$$l_2 := \log 2,\ \alpha_n := Li_n\left(\frac{1}{2}\right) + \frac{1}{n!}\log^n\frac{1}{2}$$

Full four-loop result in QED

- In QED, first quantum correction to the one-loop formula

$$\Gamma_{cusp} = \gamma(\alpha)A(x) + \left(\frac{\alpha}{\pi}\right)^4 n_f\, B(x) + O(\alpha^5) \qquad A = -\frac{1+x^2}{1-x^2}\ln x - 1$$

corrections from propagator-type diagrams

- Breaks the conjecture that matter dependence can be associated with lower-loop function

$$\Gamma_{cusp}(\phi,\alpha) = \sum_{k\geq 1}(K(\alpha))^k\Omega^{(k)}(\phi) \qquad K(\alpha) = \gamma(\alpha) + \left(\frac{\alpha}{\pi}\right)^4 n_f K^{(4)}$$

$$= K(\alpha)A(x) + \left(\frac{\alpha}{\pi}\right)^4 n_f\left(B(x) - B_c(x)\right) + O(\alpha^5) \qquad B_c = K^{(4)}A = \left(\frac{\pi^2}{6} - \frac{\zeta_3}{3} - \frac{5\zeta_5}{3}\right)A$$

$\Omega^{(4)}$ has n_f dependence, through the light-by-light scattering diagrams.

How big is the deviation from the conjecture?

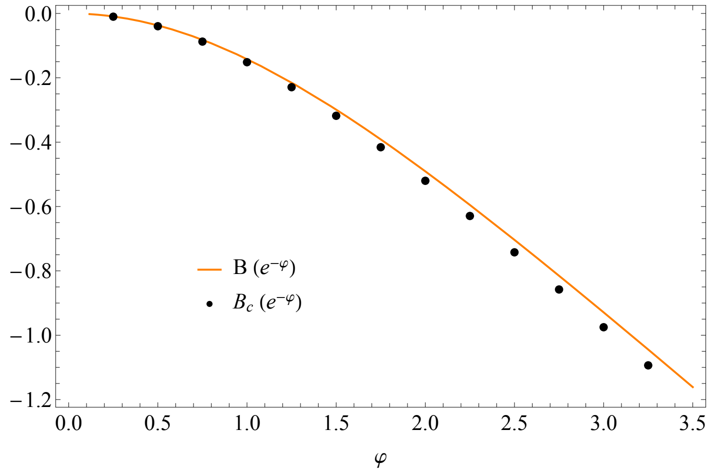
- Quantitative study of the deviation from the conjectured formula:

$$A = -\frac{1+x^2}{1-x^2}\ln x - 1,$$

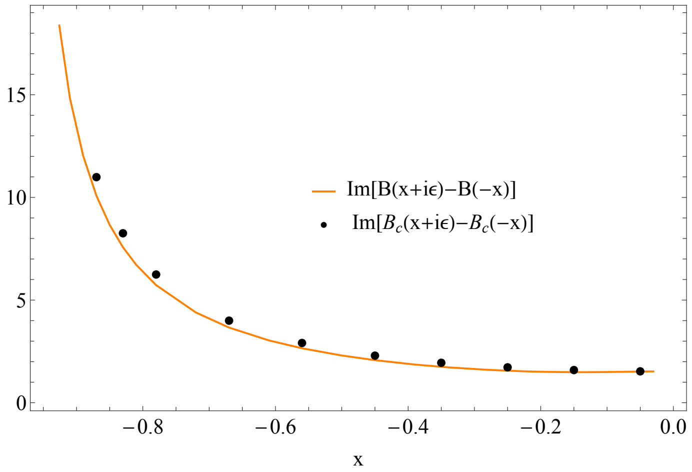
$$B_c = \left(\frac{\pi^2}{6} - \frac{\zeta_3}{3} - \frac{5\zeta_5}{3}\right)A = -0.484A$$

Deviation from the rescaled one-loop function is small in all kinematic regions

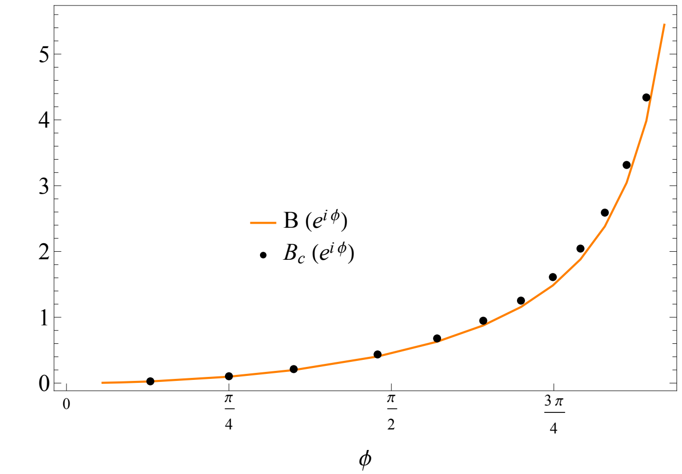
$0 < x = e^{-\varphi} < 1$



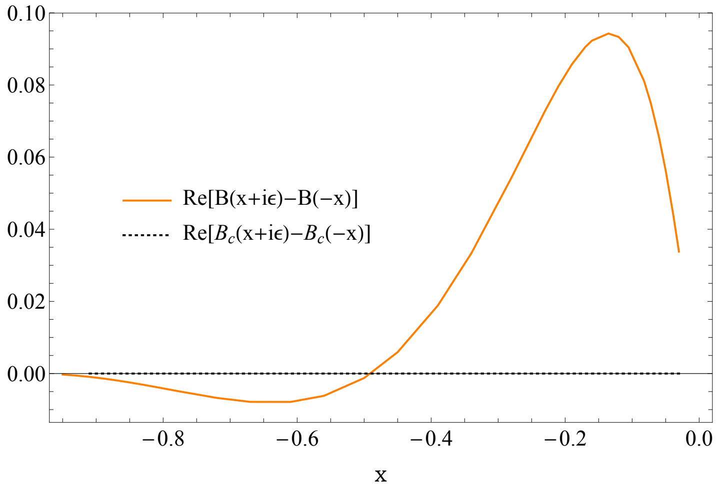
$-1 < x < 0$, imaginary part



$x = e^{-i\phi}$



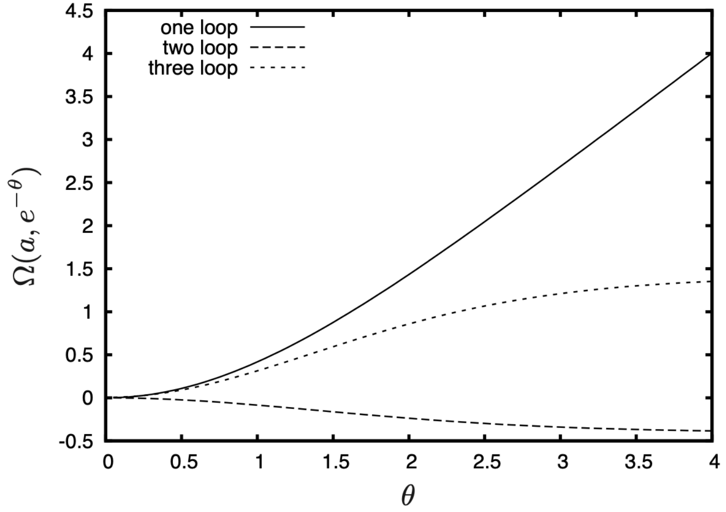
$-1 < x < 0$, real part



- Previous study on the three-loop Γ_{cusp} shows $K(\alpha)$ is a better expansion parameter than the gauge coupling.

Perturbative expansions converges better over a wide range of kinematic regions.

Our four loop result in QED is another evidence of this statement.



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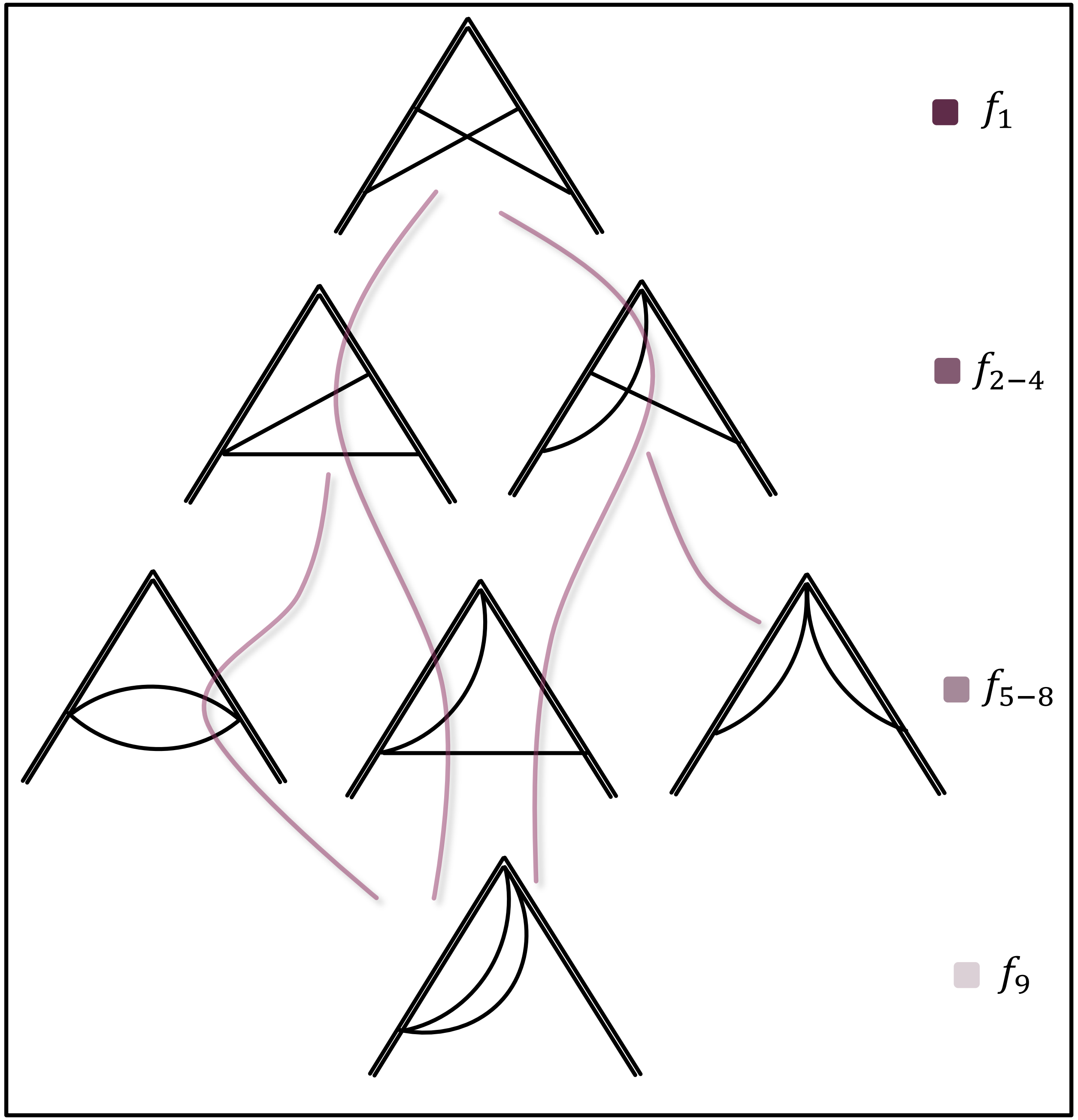
$$\Gamma_{cusp}(\pi - \delta) \sim \pi[\frac{c_1}{\delta} + c_2 + c_3\,\delta + \; ....] \qquad \delta = -\frac{1}{i}\ln(-x)$$

- $Re[B(x)]$ vanishes at x= -1, similarly for the scalar contributions $C(x)$. No sub-leading power corrections to the quark-antiquark static potential.

THE ‚INITIAL INTEGRAL ALGORITHM ‘

HTTPS://GITHUB.COM/UT-TEAM/INITIAL

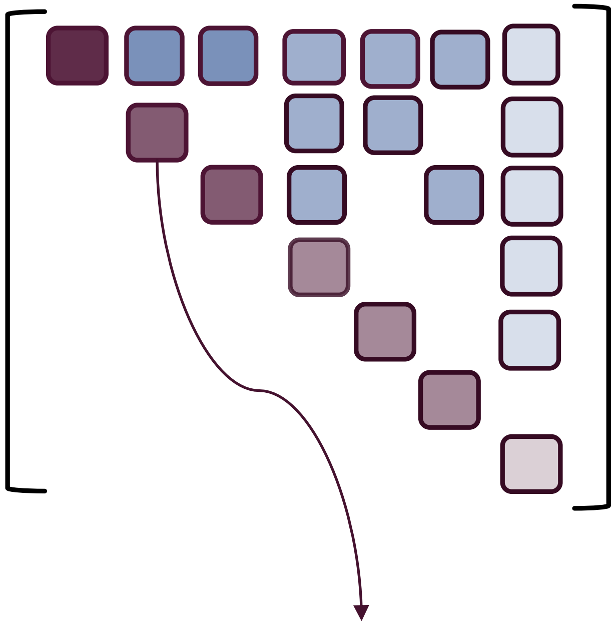


Coupled differential system in d-dimension

$$\frac{d}{dx}\,\vec{f} = A(d,x)\,\vec{f}$$

$\vec{f}$: basis of master integrals. Chosen in arbitrary way. IBP and DE are derived in d-dimension.

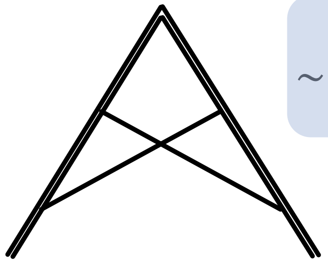
$A(d,x) =$



Coupled sub-system, typically complicated. Unphysical(apparent) singularites

$$\begin{bmatrix} -\frac{1+x^2}{(-1+x)x(1+x)} & \frac{-1+x}{x(1+x)} \\ \frac{(-4+d)(-7+2d)}{(-1+x)(1+x)(3d+5x)} & -\left(\frac{-8+3d+24x-6dx-8x^2+3dx^2}{2(-1+x)x(1+x)(3d+5x)}\right) \end{bmatrix}$$

dLog integrals and UT basis

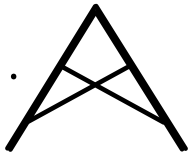


$$\sim \left(\frac{x}{x^2-1}\right)^2 \int \left[\frac{x}{(s_1+x\,t_1)(s_1+x\,t_1)}\,\frac{x}{(s_2+x\,t_2)(s_2+x\,t_2)}\right]^{-\epsilon}$$

$$d\log(s_1+x\,t_1) \wedge d\log(x\,s_1+t_1) \wedge d\log(s_2+x\,t_2) \wedge d\log(x\,s_2+t_2)$$

Leading singularity

dlog form (integrand has no double poles)

Normalised integral $g_1 := \left(x-\frac{1}{x}\right)^2 \cdot$ 

is purely logarithmic function of uniform transcendental weight (UT), at each order in ϵ.

Canonical DEs [Henn 2013]

In case one find a basis of UT integrals $\vec{g}$, the DEs will simplify significantly.

ϵ assigned with weight -1

$$\frac{d}{d\,x}\,\vec{g} = \epsilon\, B(x)\,\vec{g}$$

$$\mathrm{B}(x) = \frac{1}{x}\,m_1 + \frac{1}{x-1}\,m_2 + \frac{1}{x+1}\,m_3$$

Fuchsian poles corresponding to physical singularities

m_a : constant matrices

Traditional methods to search for UT basis

DE in terms of arbitrary basis $\vec{f}$ (a) $\frac{d}{dx}\,\vec{f} = A(d,x)\,\vec{f}$

Search for UT basis $\vec{g}$, with transformation matrix T

$$\vec{f} \to T(d,x)\,\vec{g}$$

$$A \to T\,A\,T^{-1} - T^{-1}\partial_x T$$

DEs in canonical form: (b) $\frac{d}{d\,x}\,\vec{g} = \epsilon\,B(x)\,\vec{g}$

$$dB(x) = \sum_a d\ln\alpha_a(x)\,m_a$$

Iterative soluion in terms of multiple polylogarithms : $\vec{g} = \exp\,[\epsilon\int dB]\,\vec{g}_0$

- Methods based solely on dlog integrand analysis [Wasser 2016]:

  often easy to find a few UT integrals, but hard to find a complete UT basis.

- Mathematical tools based on the DE itself (Moser algorithms, Lee's algorithm):

  less efficient for large coupled systems, or multi-variable problems.

A new algorithm to search for UT basis based on n-th order Pichard-Fuchs equations

Dlapa, Henn,Yan 2002.02340

Start with a UT integral f_1, complete the basis in an arbitrary way

$$(a) \quad \frac{d}{dx}\,\vec{f} = A(d,x)\,\vec{f}$$

$$\left(f_1', f_1'', \dots f_1^{(n)}\right)^T = \Psi[A]\,\vec{f}$$

$$b_n(d,x)\,f_1^{(n)} + b_{n-1}\,f_1^{(n-1)} + \cdots + b_0 f_1 = 0 \qquad (c)$$

Derive nth-order **Picard-Fuchs** equation.
Plug in the ansatz and solve for constant matrices m_i iteratively (Gaussian ellimination)

$$\left(g_1', g_1'', \dots g_1^{(n)}\right)^T = \Psi[B]\,\vec{g}$$

Assume existence of a UT basis $\vec{g}$, s.t. $g_1 = f_1$. DE is in canonical form.

$$(b) \quad \frac{d}{d\,x}\,\vec{g} = \epsilon B(x)\,\vec{g}$$

Ansatz for the alphabet $\{\alpha_a\}$ (based on singularities) and hence for B:

$$dB(x) = \sum d \ln \alpha_a(x)\, m_a$$

- Solution for $\Psi[B]$ allows to construct a UT basis from higher derivatives of a single UT integral.

Matrix Formulation for the Picard-Fuchs equation

Given A, assume existence of its equivalent system B in terms of UT basis g: $g_1 = f_1$.

$$\frac{d}{dx}\vec{f} = A(d,x)\,\vec{f} \qquad\qquad \frac{d}{d\,x}\vec{g} = \epsilon B(x)\,\vec{g}$$

- Picard-Fuchs equation for a single integral is unique, and contains valuable information [Höschele, Hoff, Ueda 2014]
- We formulate the method in matrix form, and solve the equations systematically. Proven to be efficient cutting-edge problems. *Public code: https://github.com/UT-team/INITIAL*

Two basis are related through higher-order derivatives

$$\left(f_1', f_1'', \ldots f_1^{(n)}\right)^T = \Psi[A]\,\vec{f} = \Psi[\epsilon B]\,\vec{g}$$

$$\frac{d^k}{d\,x^k}\vec{f} = A^{[k]}\vec{f},\quad A^{[k]} := \frac{d}{dx}A^{[k-1]} + A^{[k-1]}A \qquad \Psi := \begin{pmatrix} \vec{v}_0 A^{[1]} \\ \vdots \\ \vec{v}_0 A^{[n]} \end{pmatrix}$$

Transformation matrix $\quad \vec{f} = T\,\vec{g},\quad T = \Psi^{-1}[A]\Psi[\epsilon\mathrm{B}]$

$$\boxed{v_0.\,\Psi^{-1}[A]\Psi[\epsilon B] = v_0. \quad v_0 := (1,0,..0)}$$

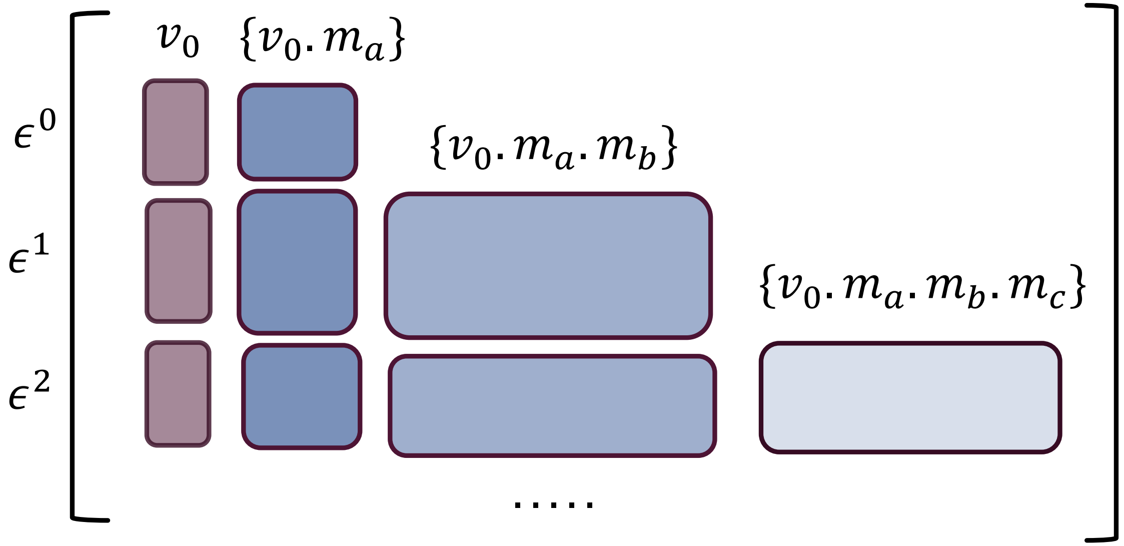
Linear equations, unknowns are row vectors parametrizing $\Psi[\epsilon B]$.

Plug in ansatz for B $\quad dB(x) = \sum d\ln\alpha_a(x)\,m_a\ , a = 1..L$

Goal is to solve for each row of the constant matrices m_a , up to a constant similarity transform.

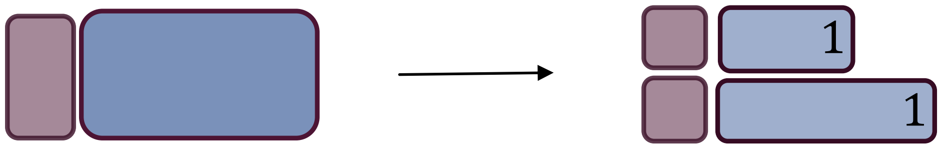
Solving for the L constant matrices $\{\boldsymbol{m_a}\}$: Recursive row reduction

Linear relations between row vectors
Expanded order by order in ϵ,
coefficients are rational function of x,
evaluated on finite fields



- Define a basis U of independent free row vectors. Initially, $U = \{v_0\}$.

- Row reduction on the top block (lowest order in ϵ)



- Removing the pivots, remaining columns reprensent free vectors in the linear equations, Redefine them as $\{v_1 \dots, v_{S_1}\}$, add to the Basis U.

Reduced linear relations

$$v_0.m_a = q^a_{0i}\, v_i \qquad i = 0, \dots, S_1$$

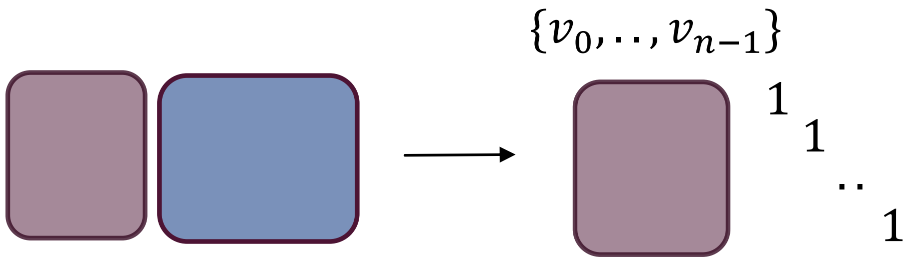
Recurrence relations define the first row in each m_a in terms of $v_0, v_1, \dots$.

$$v_0.m_a.m_b = q^a_{0i}\, v_i.m_b$$

Solving for the L constant matrices $\{m_a\}$: Recursive row reduction

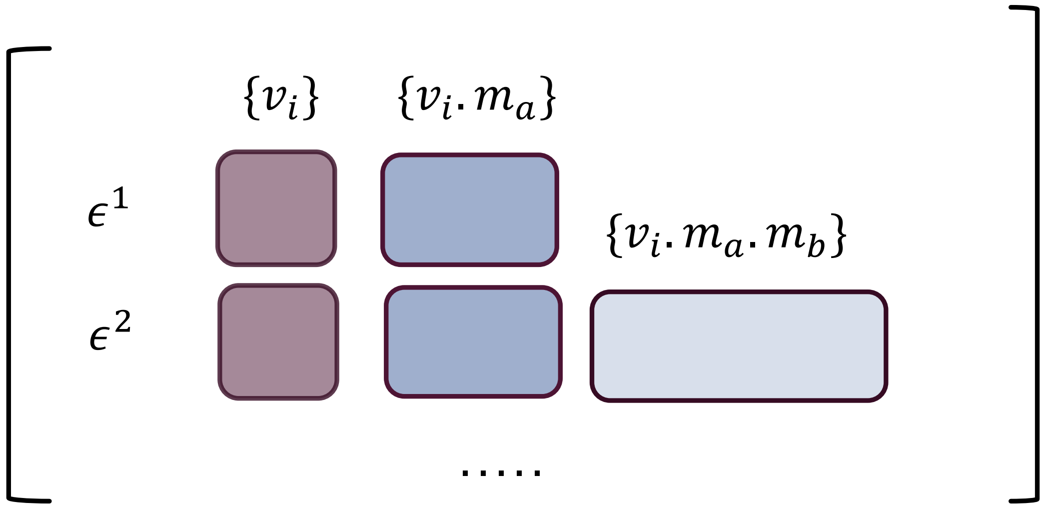
- Moving onto the next order, repeat the above procedure.

- Size of U keeps growing until a certain step



no new vectors added to the basis. In the next step, there will be no new unknowns.

1) All relations will be trivially satisfied. Solution found.
2) Certain vectors in U must vanish : Leads to contradiction.



all linear relations obtained

$$v_i . m_a = q^a_{ij}\, v_j \,, \qquad a = 1..L, i = 0.., n-1$$

Defines coefficients the m_a matrices, up to a constact similarity transform.

$$m_a = U^{-1} . q^a . U \,, \quad U \coloneqq \{v_0, v_1, .., v_{n-1}\}.$$

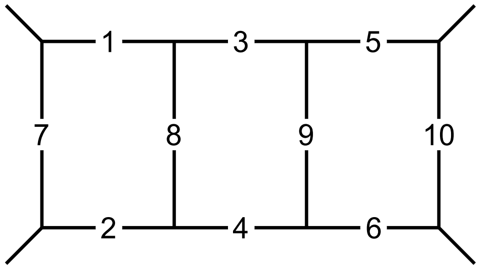
Our algorithm provides an efficient tool for solving large DE system automatically

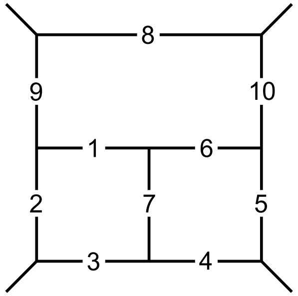
Application I:

Full diffenitial system for 3L planar ladder and tenis-court integral family

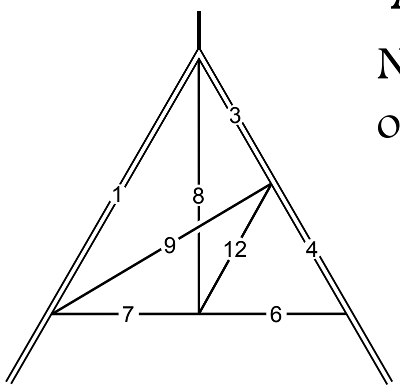
Size: 26×26, and 41×41.
alphabet $\{\frac{s}{t}, 1+\frac{s}{t}\}$

input: a single UT integral taken from planar amplitude in N=4 sYM .





No need to decompose sector by sector. A single UT integral from top sector is sufficient to derive the canonical DE in the family



Application II:

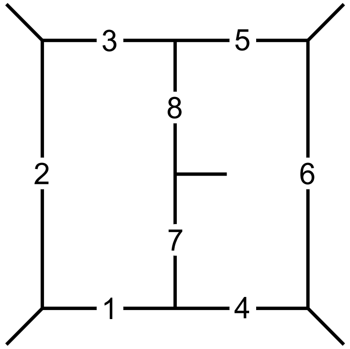
Non-planar four-loop HQET integral sector on maximal cut

Size: 17×17 on maximal cut.
alphabet $\{x, 1+x, 1-x\}$.

input: generate candidate UT integrals by power-counting and integrand analysis. Use our algorithm to test UT property.

$$\epsilon^6 \left(\frac{1-x^2}{x}\right)^2 G_{1,0,1,1,0,1,1,2,2,0,0,1,0,0,0,0,0,0}$$

Application III: Multi-variable differential system .
Non-planar double-pentagon integral family on maximal cut.



Size: 9×9 on maximal cut. Ansatz for alphabet $\{W_1 \ldots, W_{31}\}$. [Chicherin, Henn, Mitev, 2019]

Input: parity-even or odd UT integral given in literature. [Chicherin, Gehrmann, Henn, Wasser, Zhang ,Zoia, 2019]

$$s_{12} = b_1, \qquad s_{23} = b_1 b_4,$$
$$s_{45} = b_1 b_5, \qquad s_{15} = b_1 b_3 (b_2 - b_4 + b_5)$$
$$s_{34} = \frac{b_1(1+b_3)b_4}{b_2} - b_1 b_3 (1 - b_5).$$

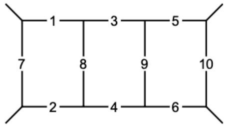
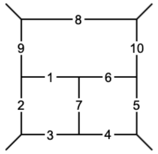
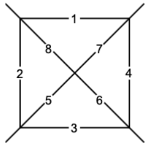
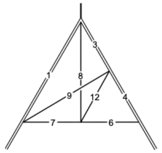
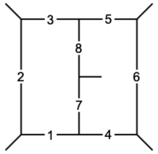
Methods: Set $b_{3,4,5}$ to constants. derive partial differential equation w.r.t. b_2.

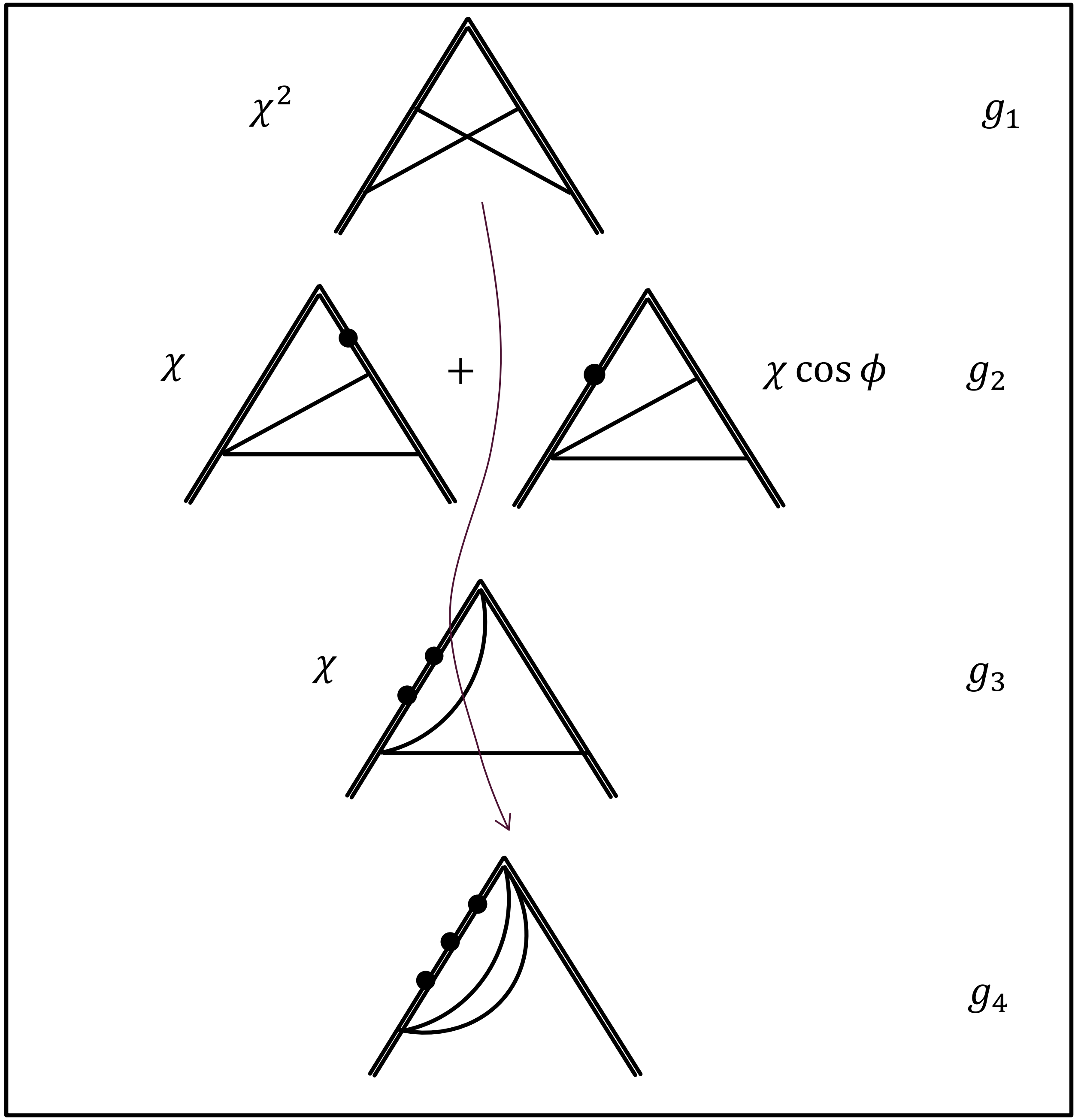
$$\partial_{b_2} g = \epsilon \sum \partial_{b_2} \ln \alpha_a(\{b_i\})\, m_a\, g \qquad a = 1..,12$$

Solve for constant matrices $m_1, \ldots m_{12}$.
Compute transformation matrix.
Reconstruct full analytic dependence on the other variables. Final canonical form dependends on 17 letters.

- Algorithm can be executed in the same way as the single-variable case.
- Requiries minimum input from the integrand analysis, compared with the methods in the literature [Abreu, Dixon, Herrmann, Page, Zeng, 2019] [Chicherin, Gehrmann, Henn, Wasser, Zhang ,Zoia, 2019]

Algorithm is efficient for many coupled integrals, and in multi-variable case.

Type of problem		#MI	#vars	#letters	time [min.]	Memory [MB]
Full three-loop DE		26 | 3	1	2	2	330
		41 | 3	1	2	34	1710
Full four-loop DE		19 | 12	1	2	1	240
HQET DE on cut		17 | 17	1	3	2	390
Five-point integrals DE on cut		9 | 9	4	17	5	510



Application IV: (quasi-)finite differential system

$$d\begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = \begin{bmatrix} 0 & d\ln x & & \\ & & d\ln\frac{x}{(x+1)(x-1)} & \\ & & & d\ln x \\ & & & 0 \end{bmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix}$$

The size of the coupled system in 4D is reduced compared with the d-dimensional system. Certain sectors completely decouple.[Caron-Huot, Henn `14]

These observations are crucial to solving for the four-loop HQET integrals.
The non-polylogarithmic integrals drop out of the differential system and do not contribute to Γ_{cusp} .

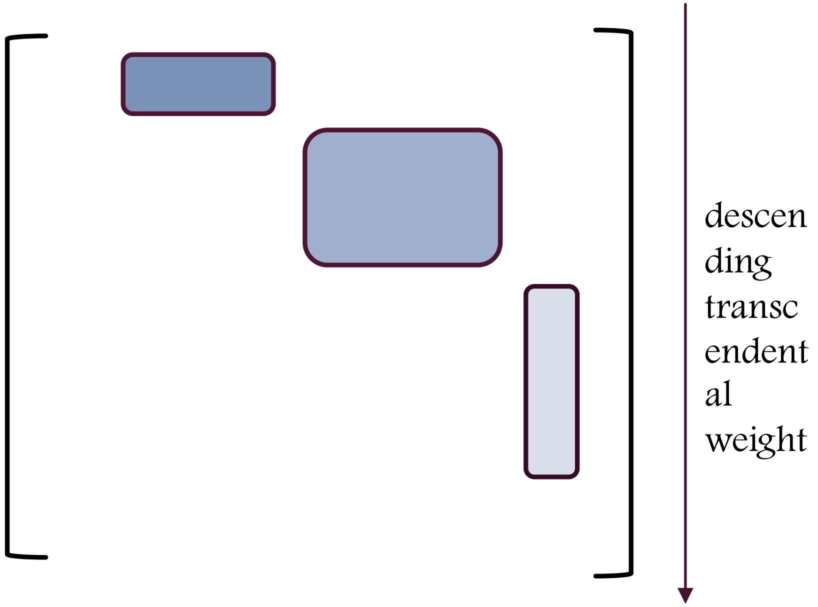
- Our algorithm is particular suited to dealing with finite sytem, without a priori knowledge of basis of finite integrals.
- Start from a single finite UT integral , directly set d=4 in the Picard-Fuchs equation.

$\vec{g}$: finte, purely logarithmic with uniform transcendental weight

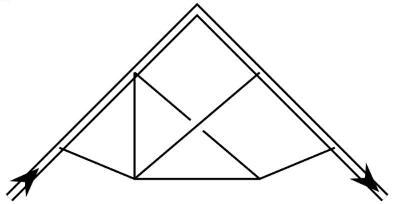
$$\frac{d}{d\,x}\,\vec{g} = B(x)\,\vec{g}$$

- The solution for B matrix and infomation on boundary values often suggests further relations between memebers of the basis. The size of finite differential system is typically smaller.

$B(x) =$



We find polylogarithmic solution for integrals needed for the cusp anomaloud dimension, in particular quasi-finite and dlog integral ,e.g.



For finite integrals, B is nilpotent. DE can be easily solved iteratively (bottom-up).

Conclusion and outlook

- Obtained full four-loop QED angle-dependent cusp anomalous dimension

Result is qualitatively well described by rescaled one-loop function

- Analytic result depends on relatively simple function alphabet.

Gives valuable input for bootstrap of soft anomalous dimension.

Future directions:

- Full 4loop non-planar contribution in QCD can be determined by considering bosonic Wilson loop in N=4 sYM .
- Systems of finite integrals has a smaller size, much simpler IBP relations and DEs. Our algorithm suggests one can build finite basis from higher-order derivatives. Shed new lights on novel methods for calculating finite loop integrals.

Thank you for your attention !