Intra-bunch motion:
a simple theoretical approach for impedance-induced TMCI

E. Métral (many thanks to all the HSC team!)

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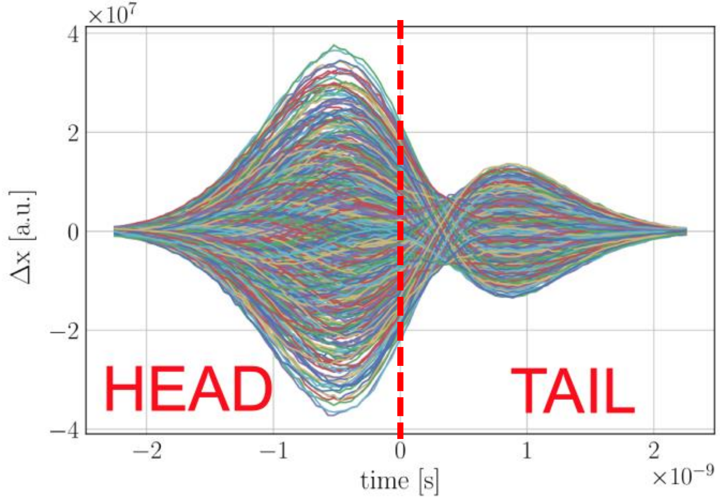
- **Motivation**: Can I understand (theoretically) such a picture?



M. Beck et al., IPAC18 (simulation from PyHEADTAIL tracking code, 1 bunch, SPS impedance model)

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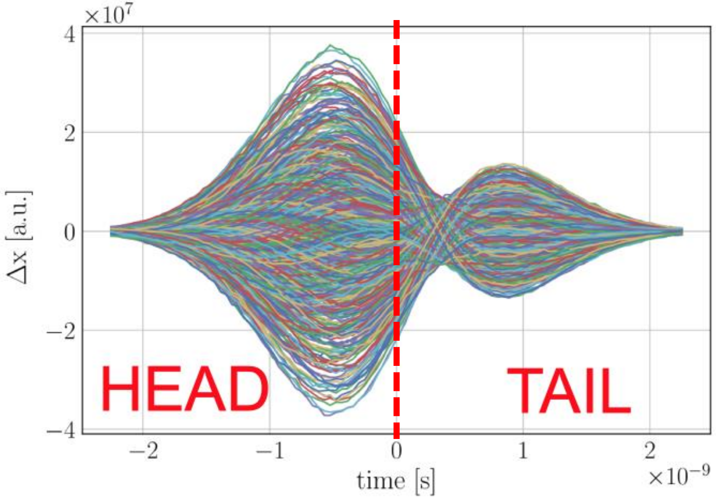
=> See https://cds.cern.ch/record/2714322/files/CERN-ACC-NOTE-2020-0018.pdf



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Contents

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- Reminder from D. Amorim's result with DELPHI Vlasov solver
- General approach with GALACTIC Vlasov solver
- Simple analytical model
- Conclusion and outlook

Laclare & Sacherer result << TMCI

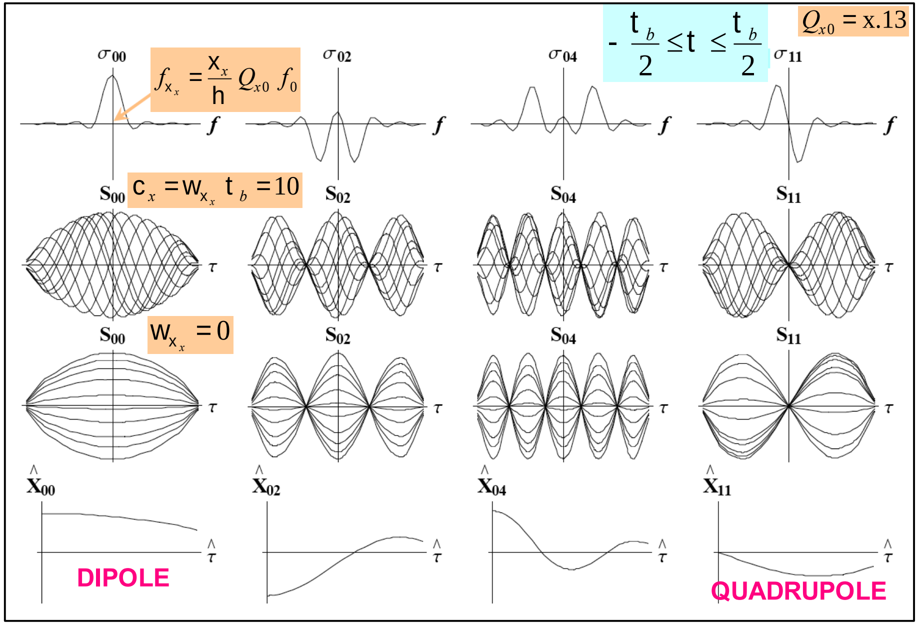
- Linearized Vlasov equation => Eigenvalue system to solve

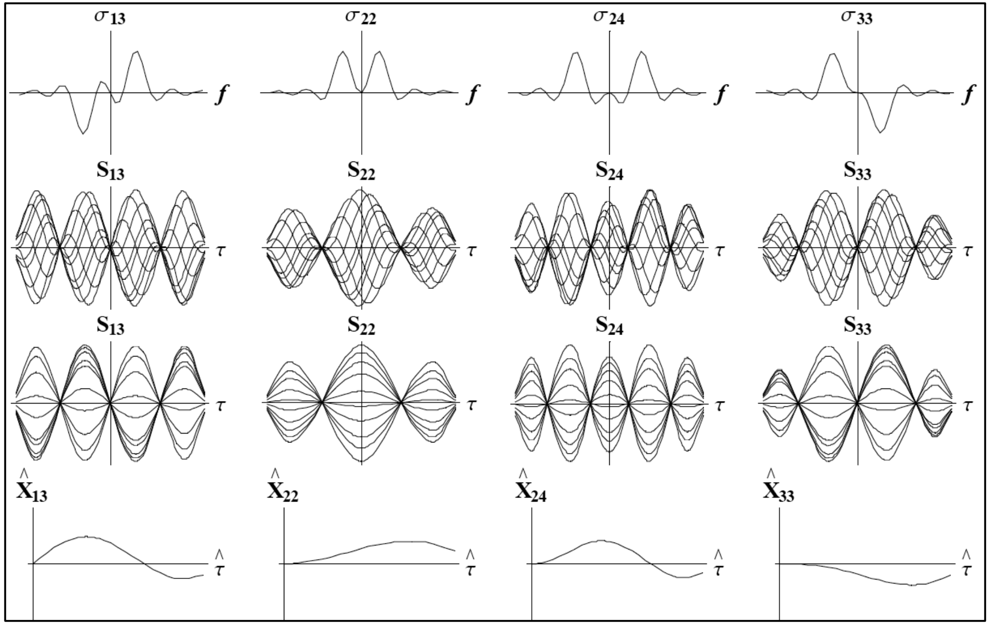
  - Result = infinite number of modes of oscillation $\boldsymbol{mq}$
    - $\boldsymbol{m}$ = azimuthal mode number
    - $\boldsymbol{q = |m| + 2k}$ = radial mode number

  - Eigenvalues describe the beam oscillation mode-frequency shifts
  - Eigenvectors describe the intra-bunch motion

  - **When $\boldsymbol{q = |m|}$**, the mode $mq = m|m|$ is simply called **mode $\boldsymbol{m}$**

Laclare & Sacherer result << TMCI





Laclare & Sacherer result << TMCI

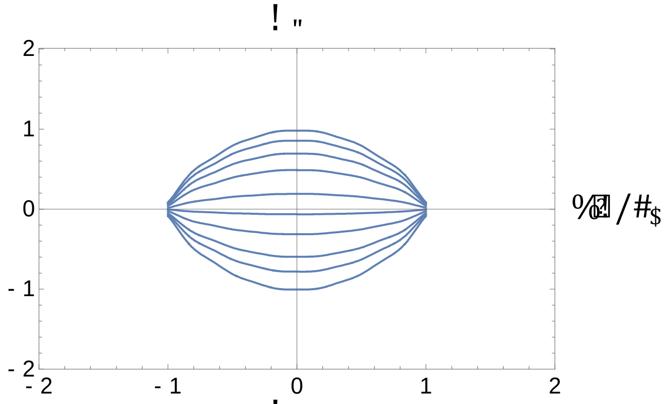
MODE 0

MODE -1

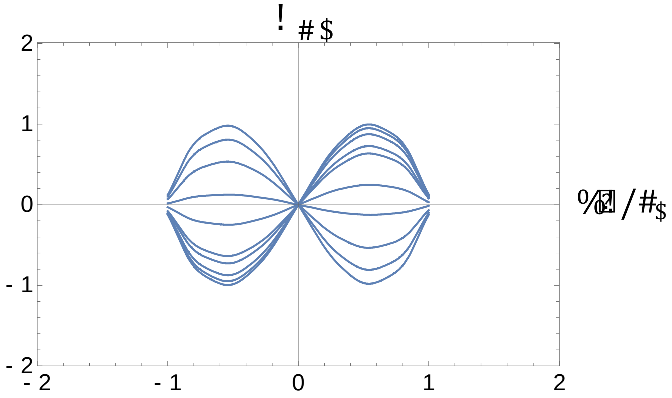
Laclare & Sacherer result << TMCI

Solutions of the Eigenvalue problem at low intensity (Laclare)

MODE 0



MODE -1

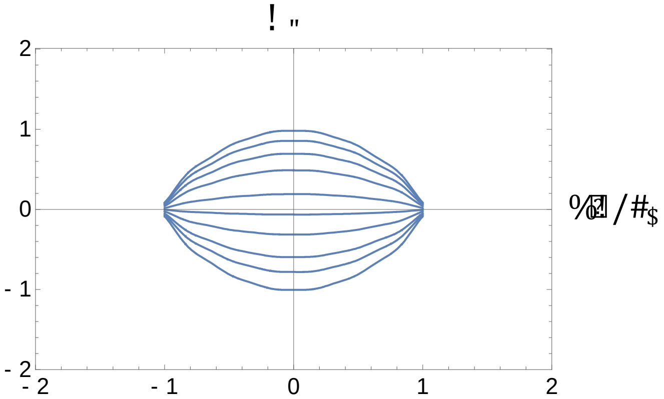


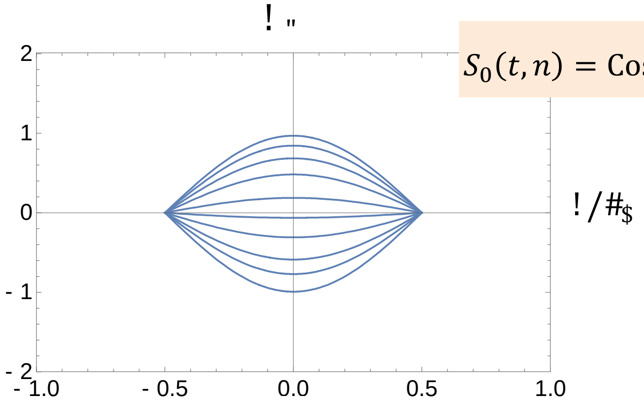
Laclare & Sacherer result << TMCI

Solutions of the Eigenvalue problem at low intensity (Laclare)

Approximation by sinusoidal modes (Sacherer)

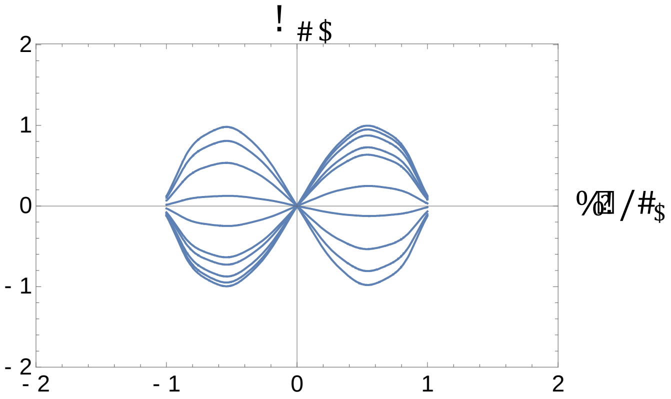
MODE 0

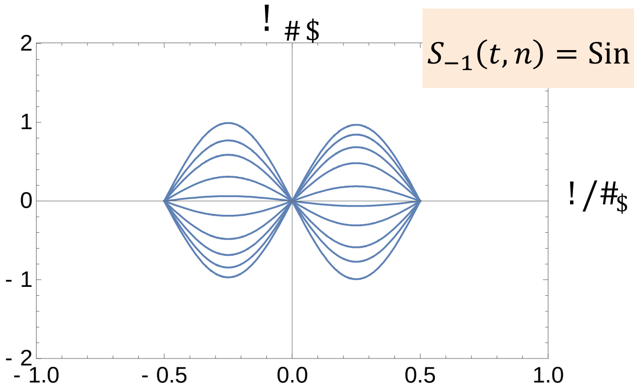




$$S_0(t,n) = \mathrm{Cos}\left(\frac{\pi\, t}{\tau_b}\right)\, \mathrm{Cos}(2\,\pi\, n\, Q)$$

MODE -1



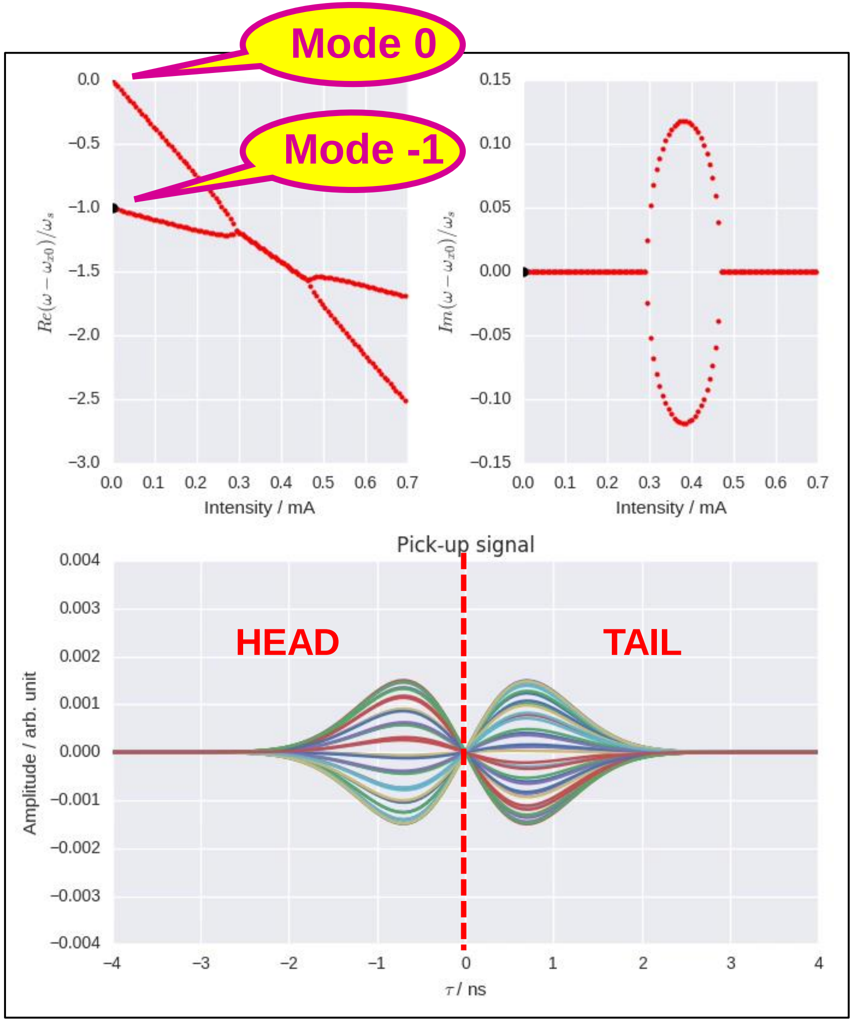


$$S_{-1}(t,n) = \mathrm{Sin}\left(\frac{2\,\pi\, t}{\tau_b}\right)\, \mathrm{Cos}(2\,\pi\, n\, Q)$$

D. Amorim's result with DELPHI Vlasov solver
(movie)

D. Amorim, PHD defence
(CERN, 07/10/2019, supervisor: N. Biancacci)

$$Q' = 0$$

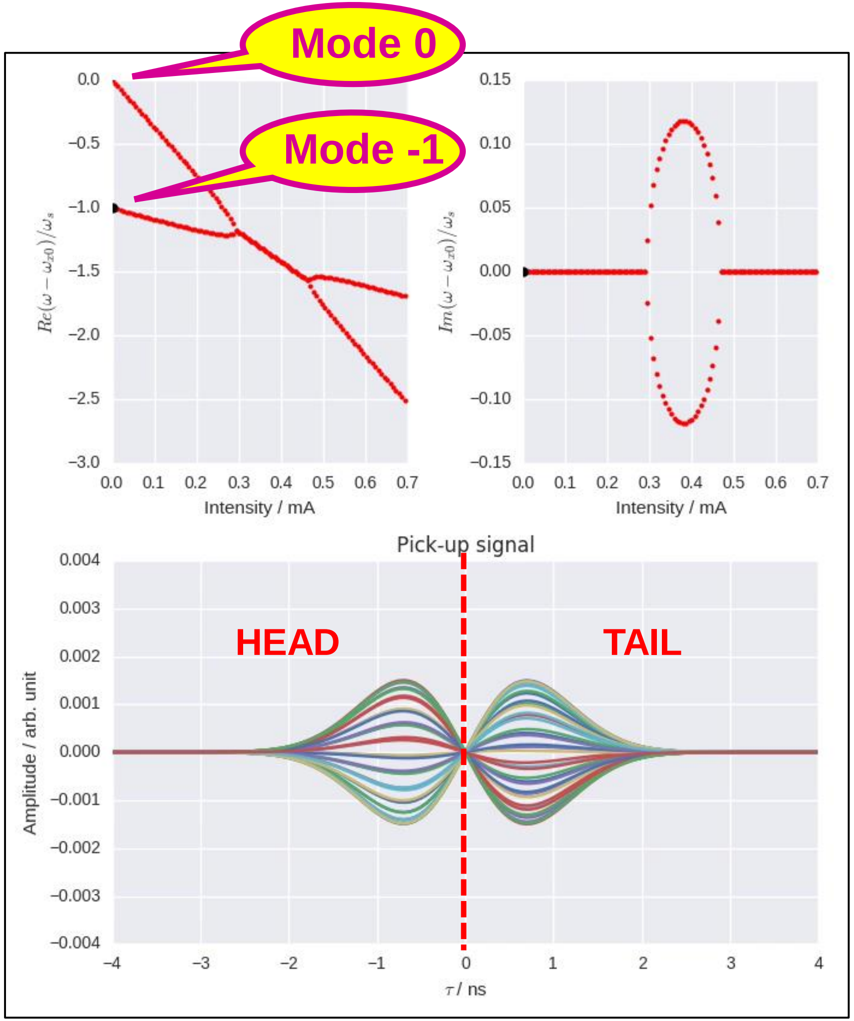


D. Amorim's result with DELPHI Vlasov solver
(movie)

D. Amorim, PHD defence
(CERN, 07/10/2019, supervisor: N. Biancacci)

$$Q' = 0$$

=> Question: Can I understand (theoretically) the asymmetries and fixed points?



General approach with GALACTIC Vlasov solver

$$\sigma(l) = \sum_{i,j=-\infty}^{\infty} a_{ij} \sigma_{ij}(l)$$

Low-intensity Eigenvectors

$$\frac{\Delta Q}{Q_s} a_{kl} = H a_{ij}$$

Matrix to be diagonalised:

1) Eigenvalues give the mode frequency shifts (Re and Im)

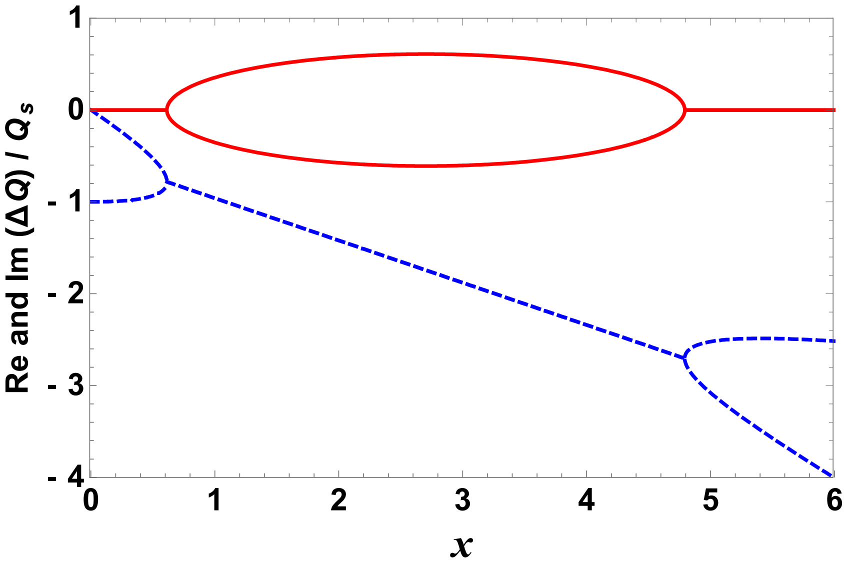
2) Eigenvectors give the coeffcients a_{ij} to be used in the equation on the left to be able to plot the intra-bunch signal

Simple analytical model

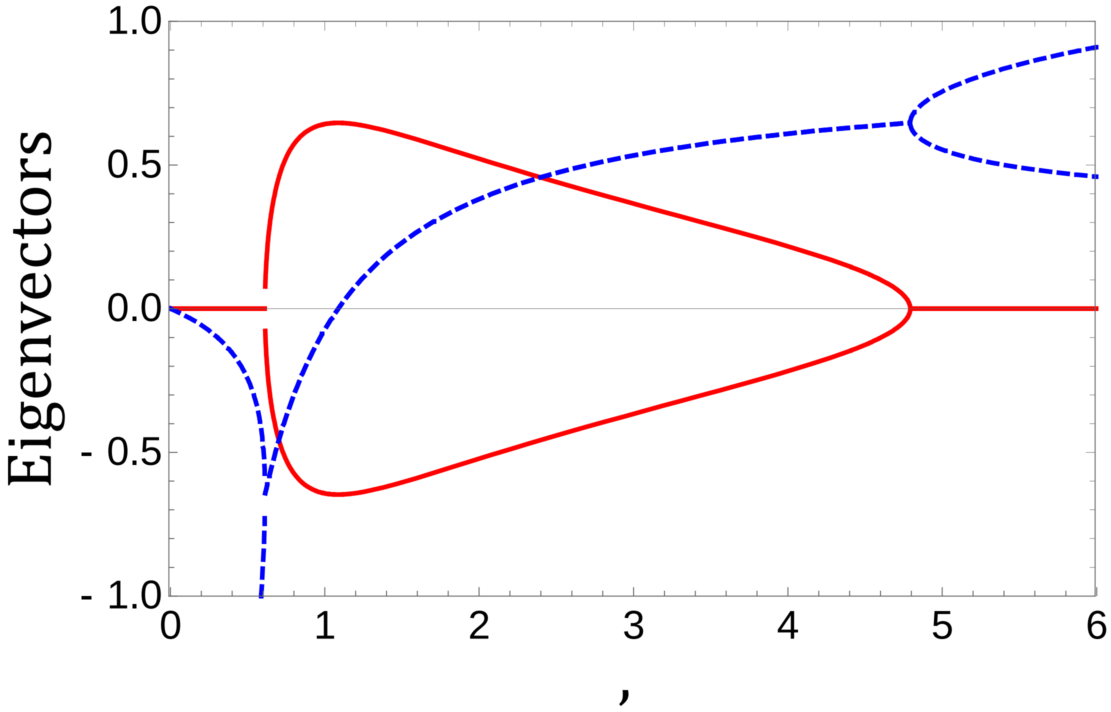
- If I use the simple model I used in the past to study the destabilising effect of the LHC transverse damper, i.e. the case of a short bunch interacting with a broad-band resonator with a quality factor of 1 and a resonance frequency such that $f_r \tau_b = 0.8$

$$H = \begin{bmatrix} -1 & -0.23jx \\ -0.55jx & -0.92x \end{bmatrix}$$

x **is a normalized parameter proportional to the bunch intensity**



Simple analytical model



Simple analytical model

=> Considering
the 2 modes together

Simple analytical model

- Below TMCI => $\text{Signal} \propto a_0 S_0 - a_{-1} S_{-1}$ (a_0 and a_{-1} reals)

- At TMCI threshold => $\text{Signal} \propto a\,(S_0 - S_{-1})$ as $a_0 = a_{-1} = a$ (real)
  - Signal is 0 at both bunch extremities
  - Signal is also 0 when $\text{Cos}\left(\frac{\pi\, t}{\tau_b}\right) - \text{Sin}\left(\frac{2\,\pi\, t}{\tau_b}\right) = 0 \Rightarrow\; t = \frac{\tau_b}{6}$

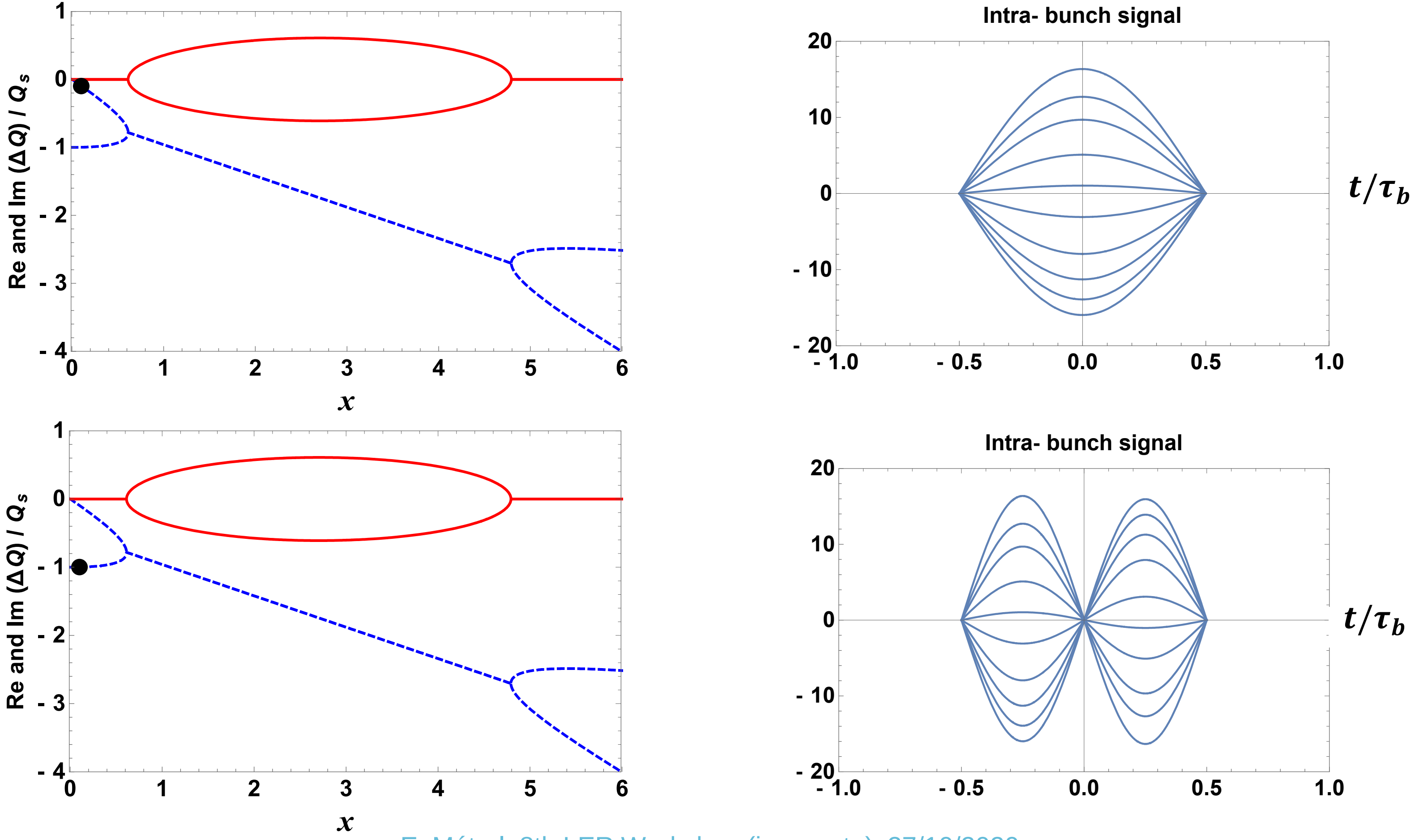
Simple analytical model

- Above TMCI => $\text{Signal} \propto (a + jb)\, S_0 - (a - jb)\, S_{-1}$ (a and b reals)

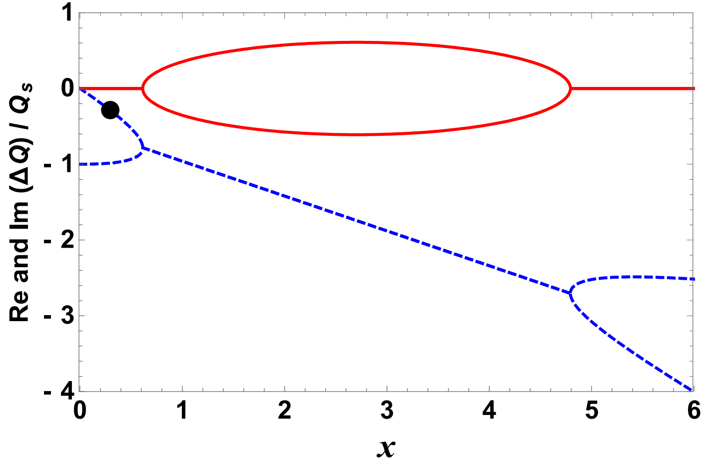
$$\propto \sqrt{\left\{a\left[\text{Cos}\left(\frac{\pi\, t}{\tau_b}\right) - \text{Sin}\left(\frac{2\,\pi\, t}{\tau_b}\right)\right]\right\}^2 + \left\{b\left[\text{Cos}\left(\frac{\pi\, t}{\tau_b}\right) + \text{Sin}\left(\frac{2\,\pi\, t}{\tau_b}\right)\right]\right\}^2}\ \text{Cos}[2\,\pi\, n\, Q + \varphi(t)$$

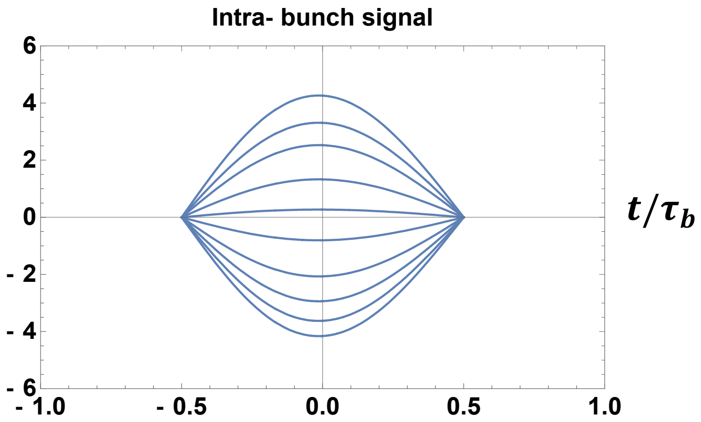
with $\varphi(t) = \text{ArcTan}\left\{\frac{b\left[\text{Cos}\left(\frac{\pi\, t}{\tau_b}\right) + \text{Sin}\left(\frac{2\,\pi\, t}{\tau_b}\right)\right]}{a\left[\text{Cos}\left(\frac{\pi\, t}{\tau_b}\right) - \text{Sin}\left(\frac{2\,\pi\, t}{\tau_b}\right)\right]}\right\}$

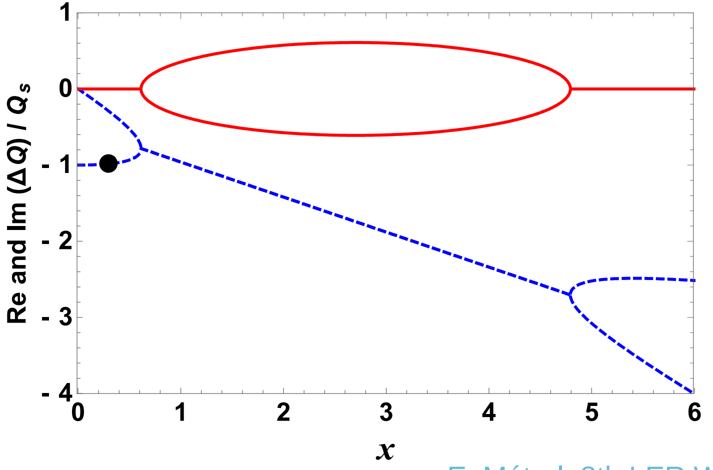
Simple analytical model ($x = 0.1$)

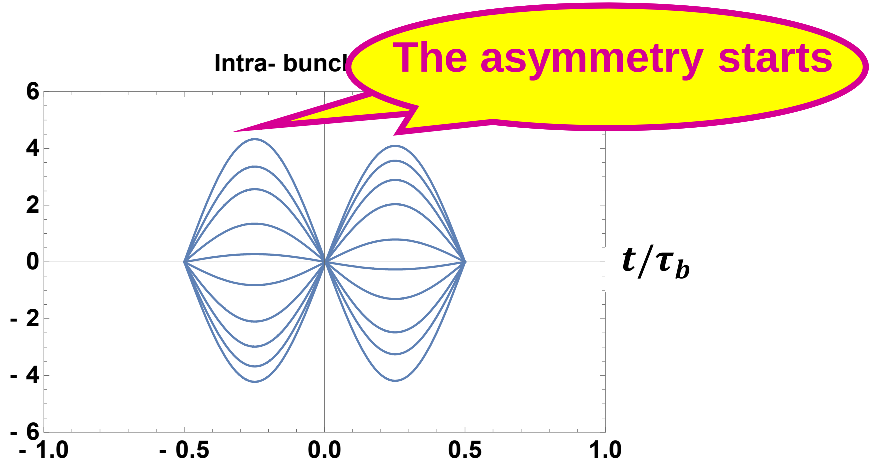


Simple analytical model ($x = 0.3$)

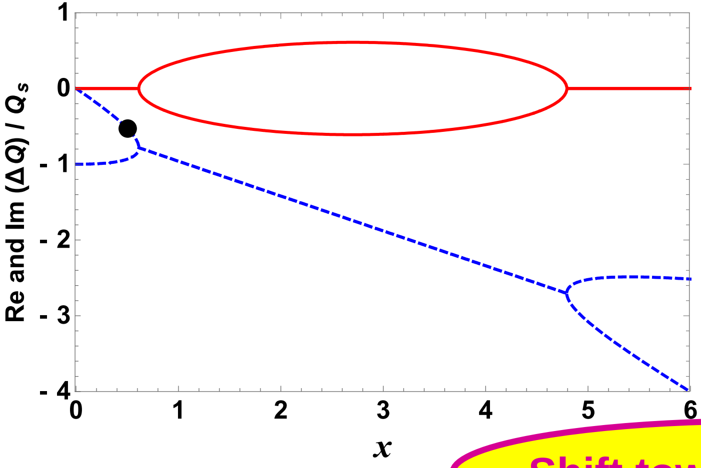


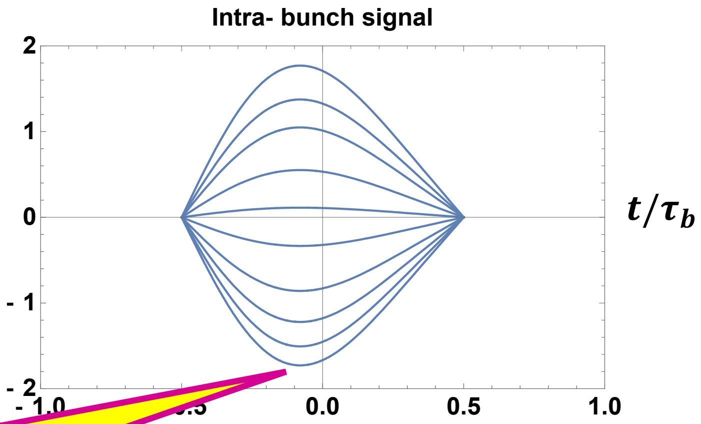




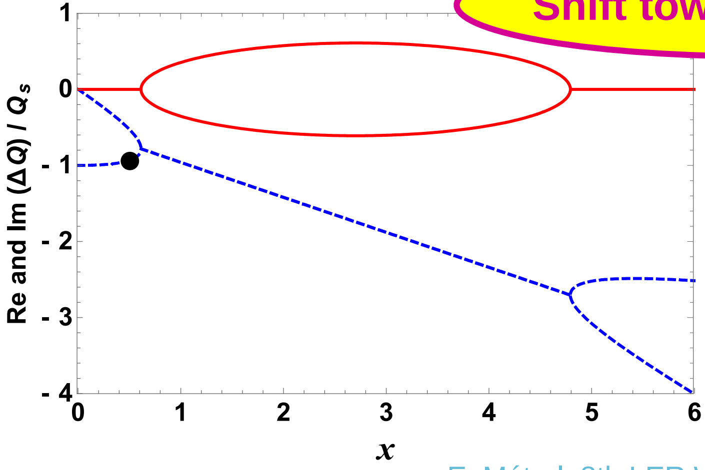


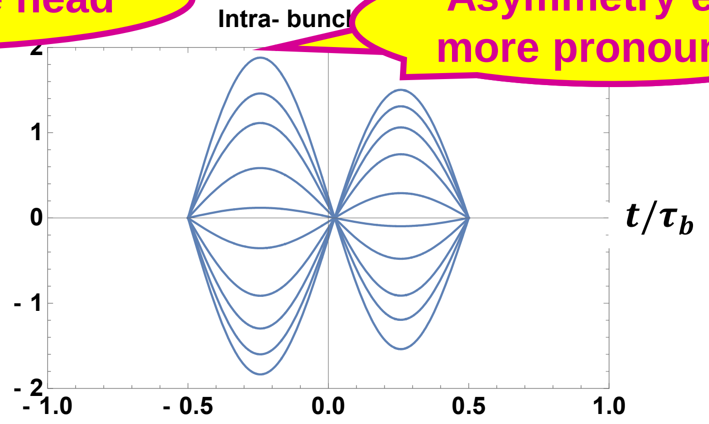
Simple analytical model ($x = 0.5$)





Shift towards the head

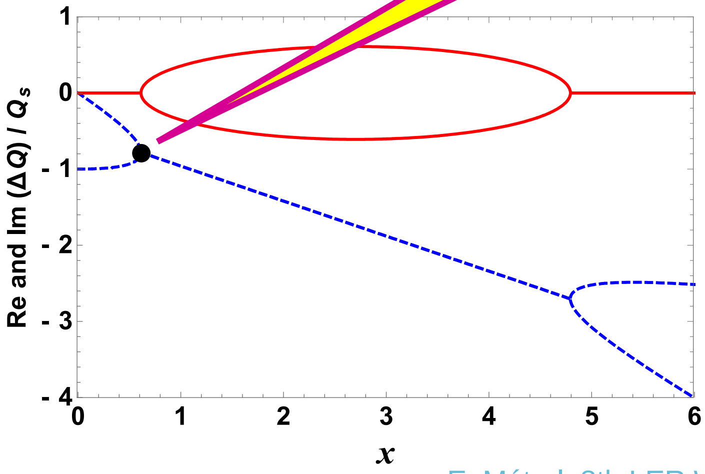


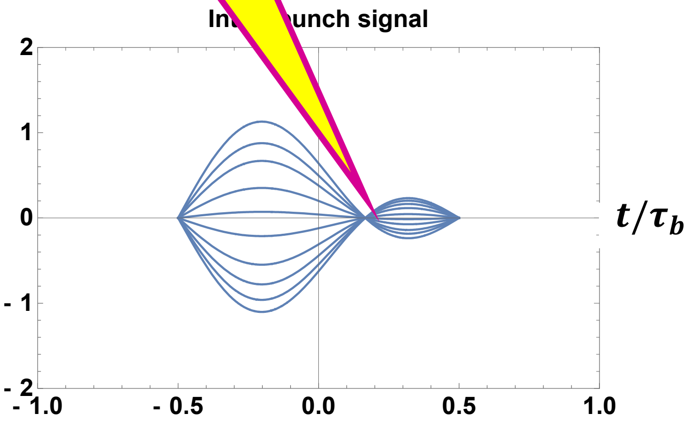


Asymmetry even more pronounced

Simple analytical model ($x = 0.613$)

This point is given by $\frac{\tau_b}{6}$





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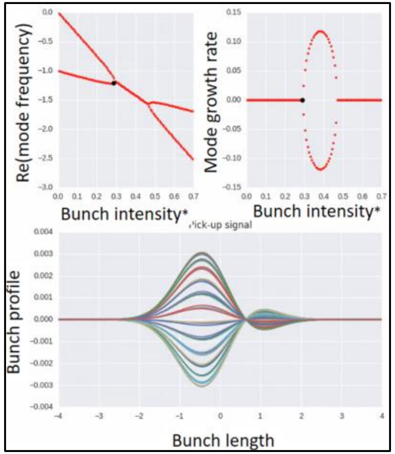
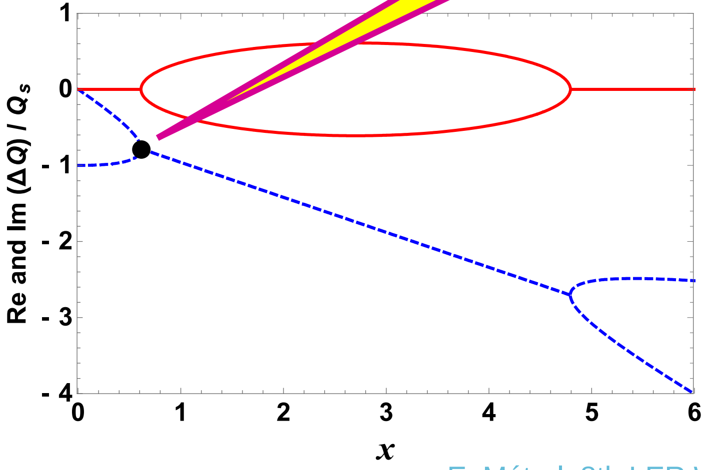
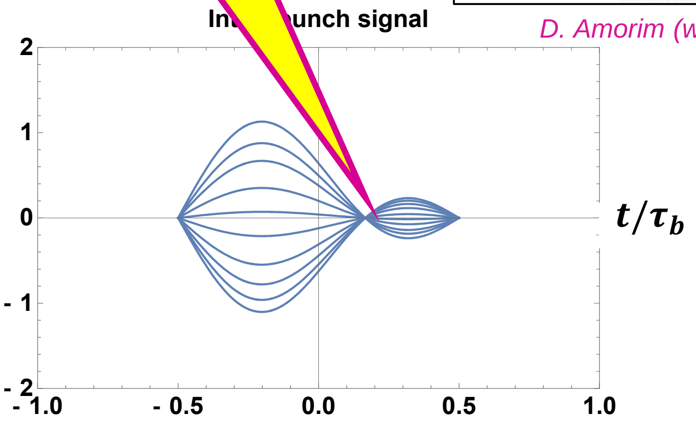


D. Amorim (with DELPHI)

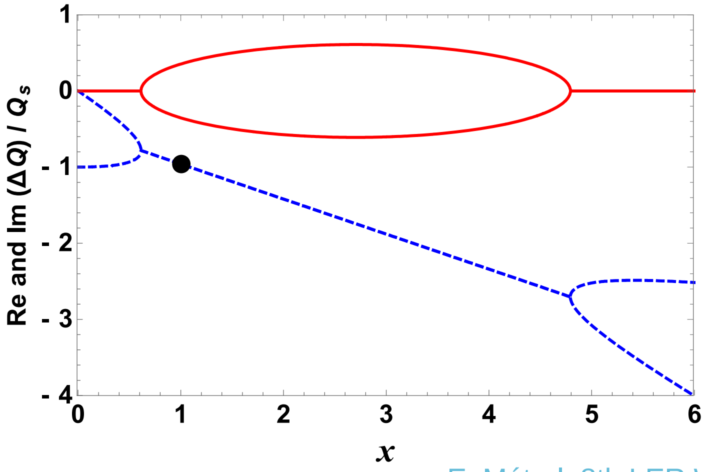
This point is given by $\frac{\tau_b}{6}$

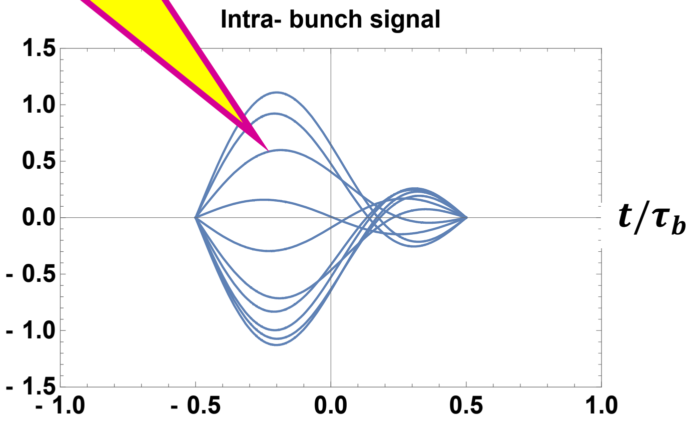




Simple analytical model ($x = 1.0$)

Travelling-wave comes from the fact that the eigenvectors have now both a Re and Im part => The phase $\varphi(t)$ appears





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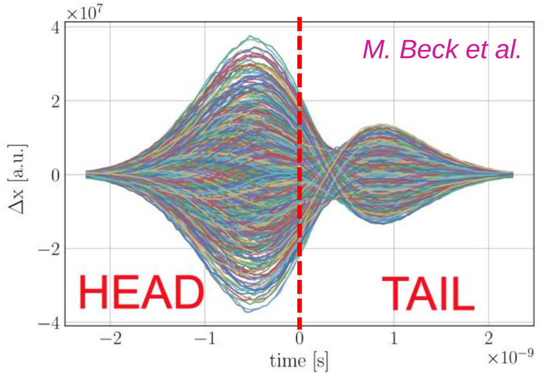
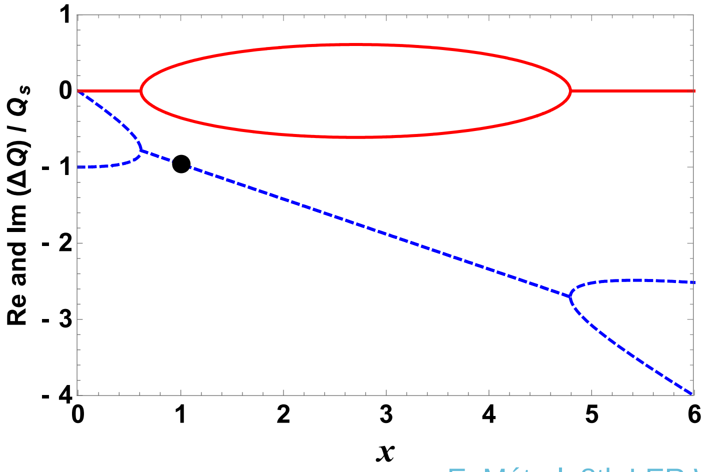
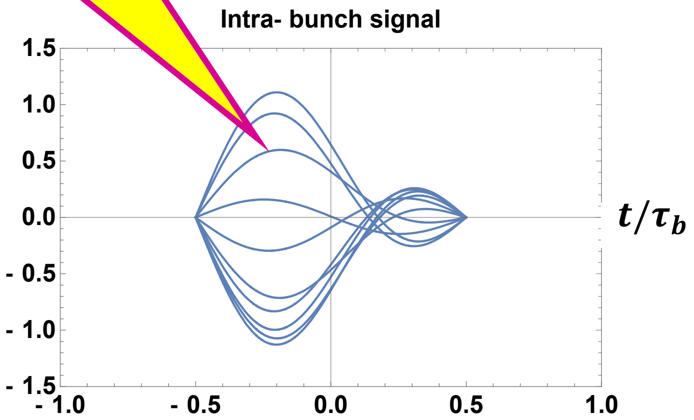
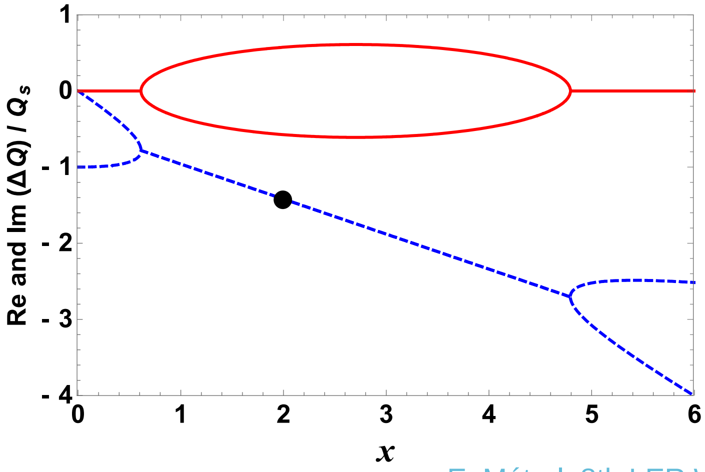
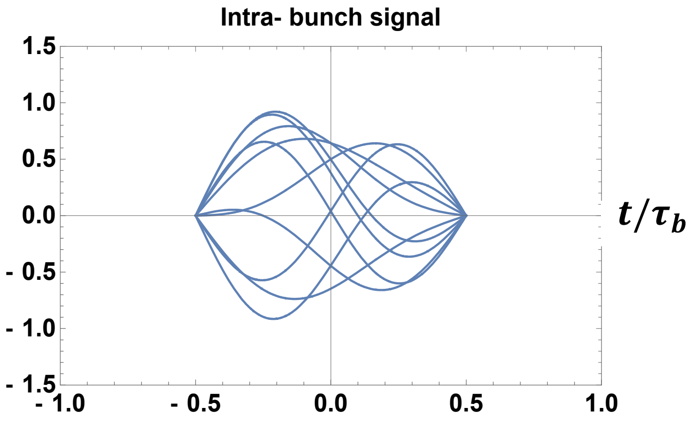




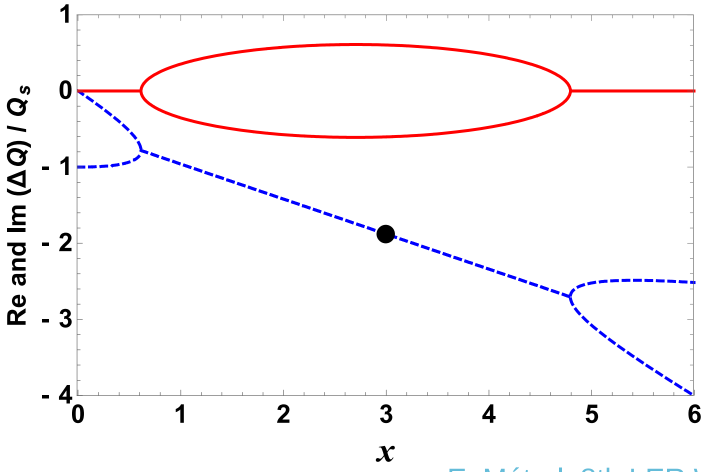


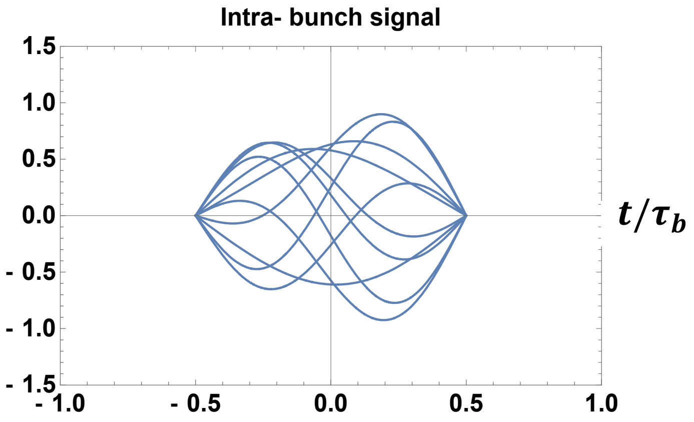
Simple analytical model ($x = 2.0$)



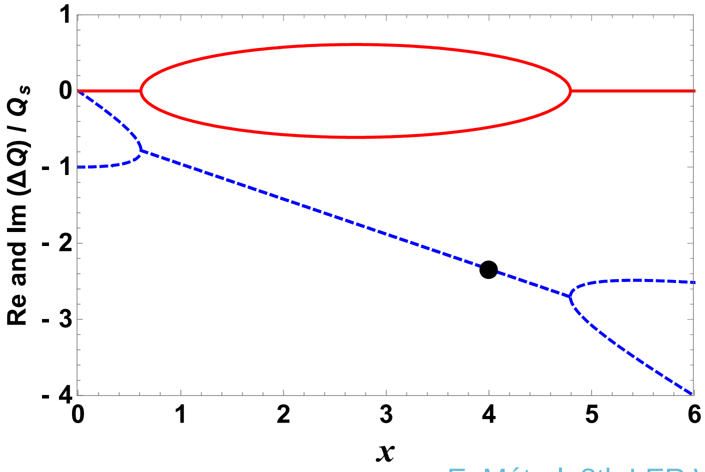


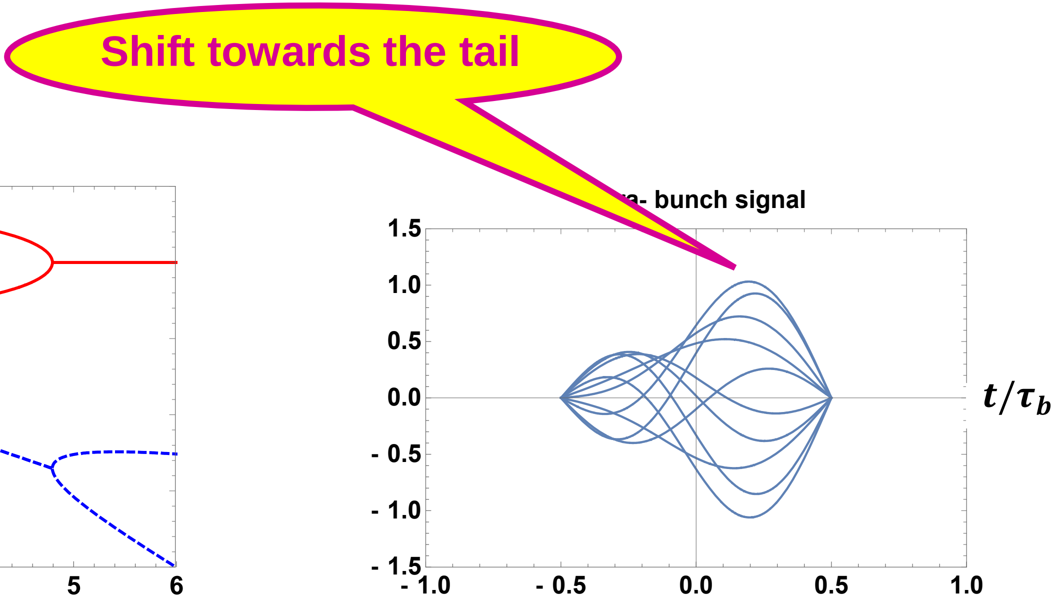
Simple analytical model ($x = 3.0$)



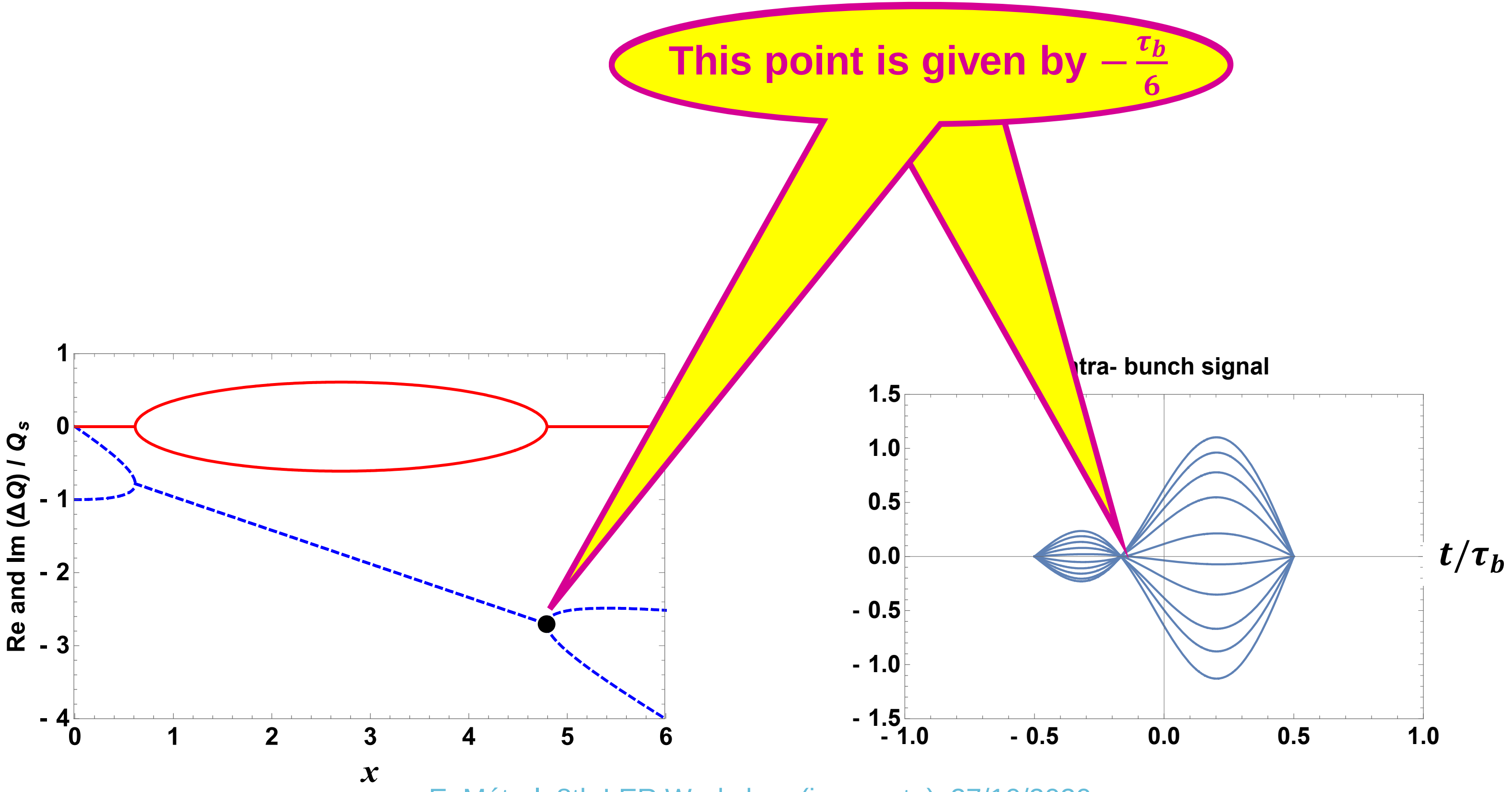


Simple analytical model ($x = 4.0$)



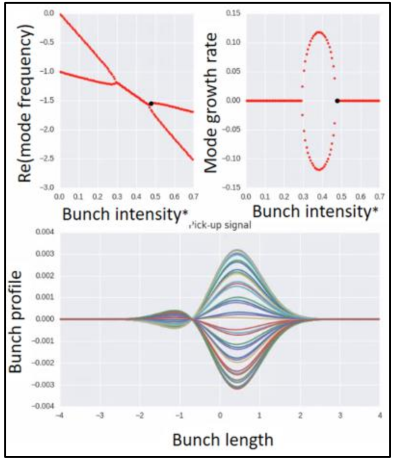


Simple analytical model ($x = 4.8$)

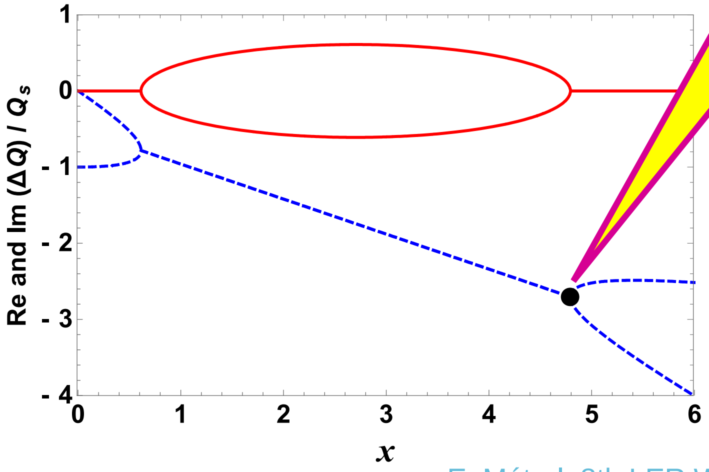


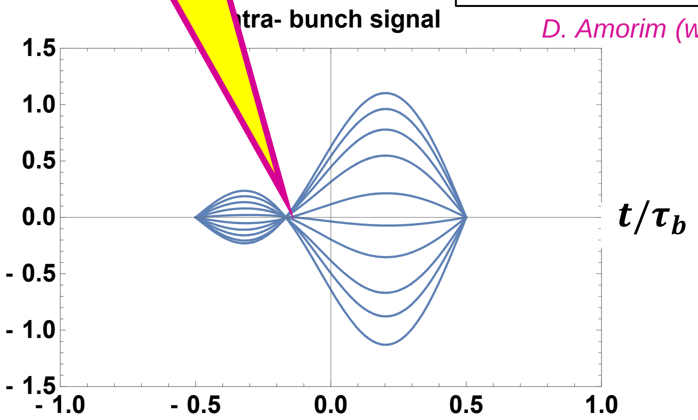
Simple analytical model ($x = 4.8$)

This point is given by $-\frac{\tau_b}{6}$

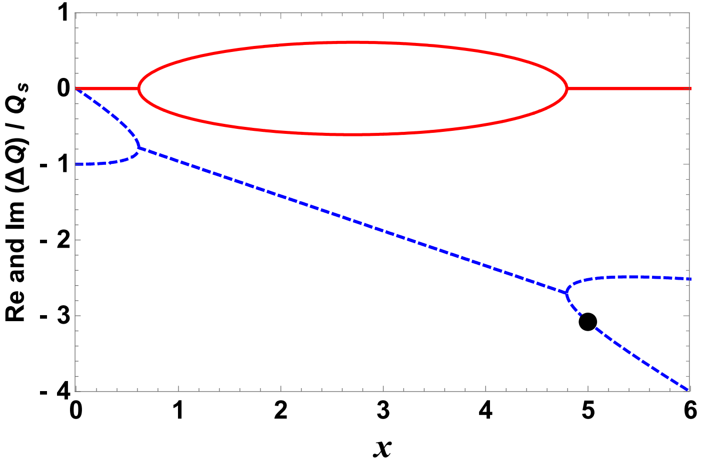


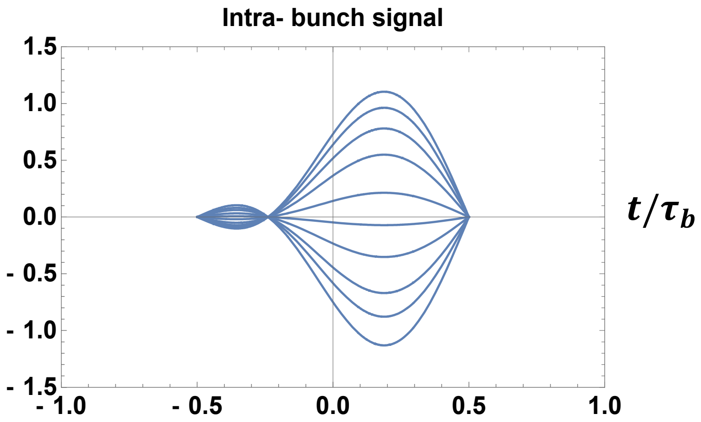
D. Amorim (with DELPHI)

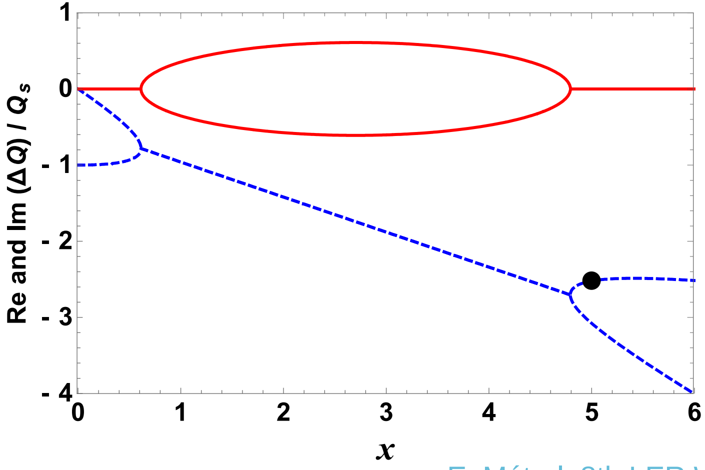


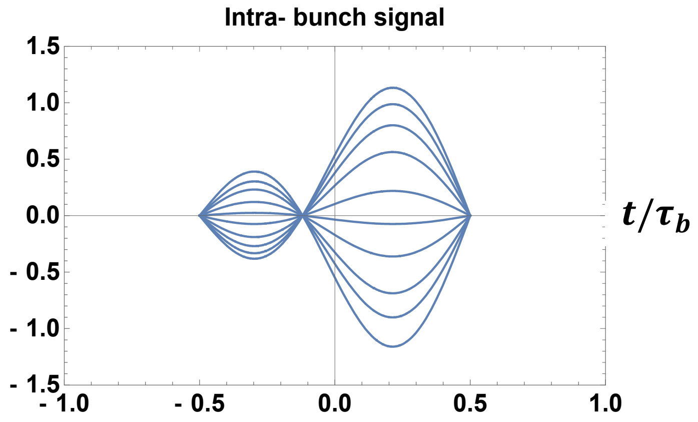


Simple analytical model ($x = 5.0$)

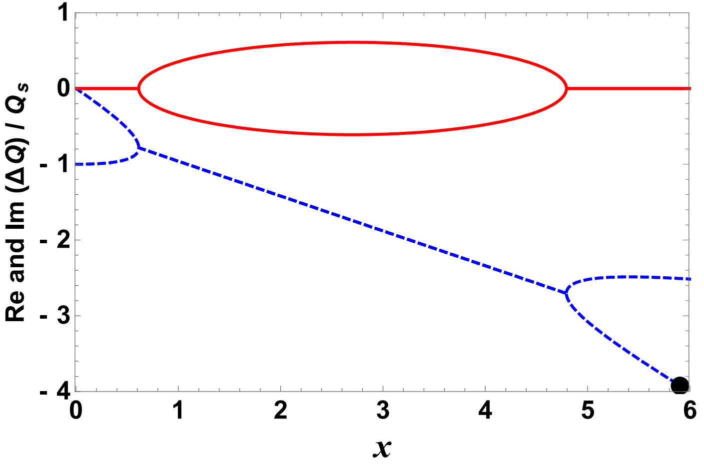


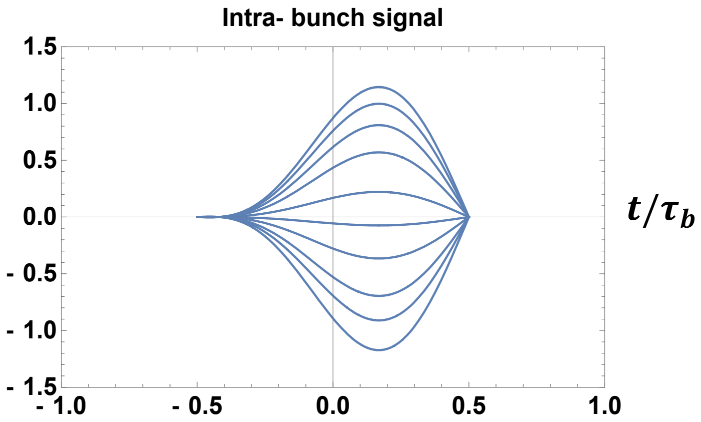


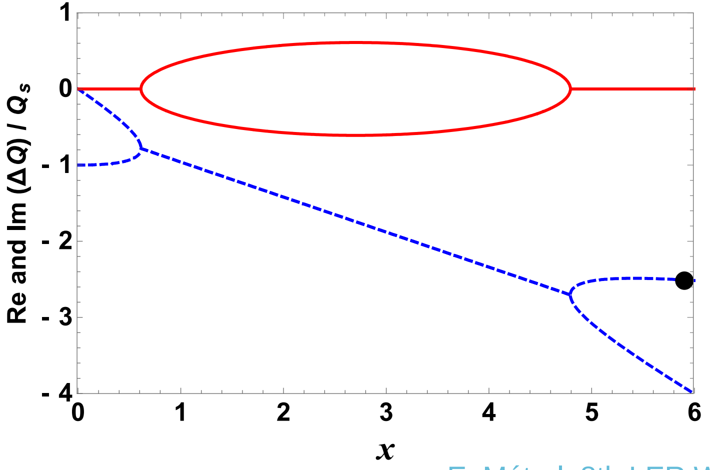


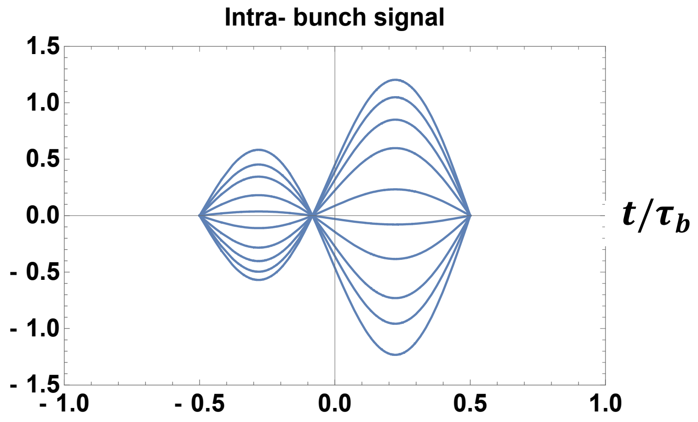


Simple analytical model ($x = 5.9$)



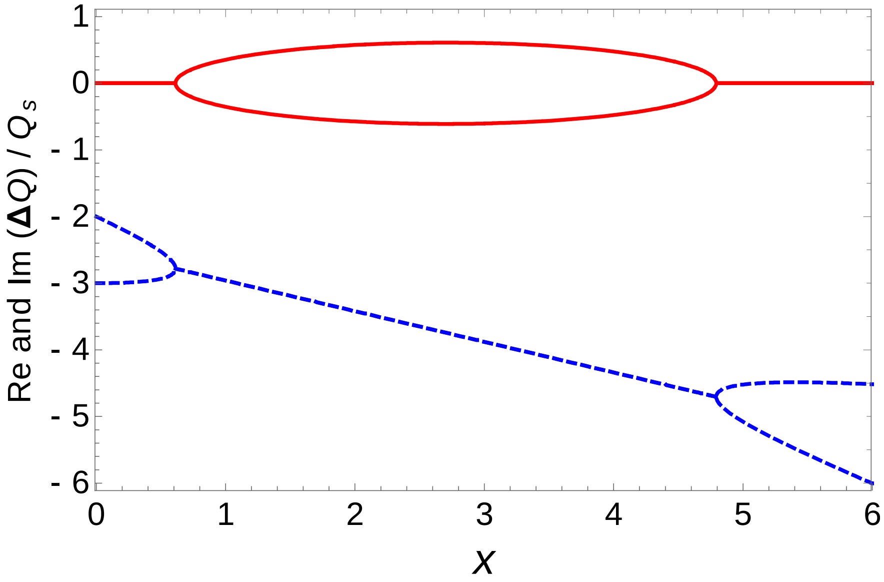






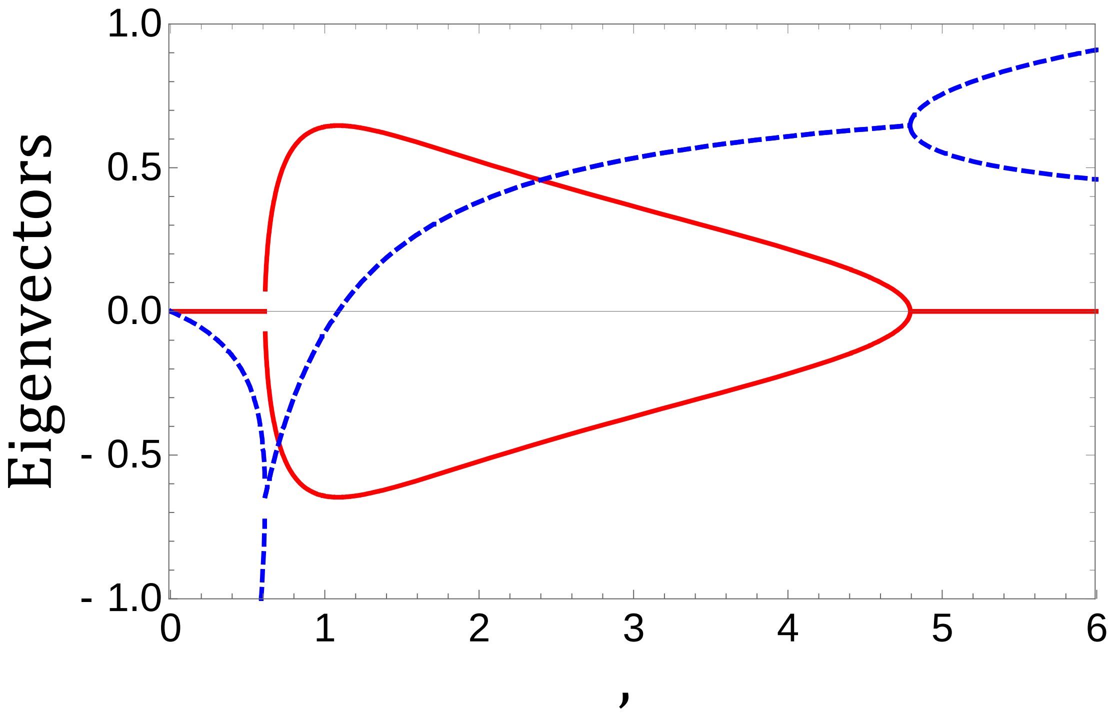
A similar approach can be done with higher-order modes (e.g. -2 and -3)

$$\begin{pmatrix} -2 - 1 & -0.23\, j\, x \\ -0.55\, j\, x & -0.92\, x - 2 \end{pmatrix}$$



A similar approach can be done with higher-order modes (e.g. -2 and -3)

- Eigenvectors (of the previous 2×2 matrix) => Same as before



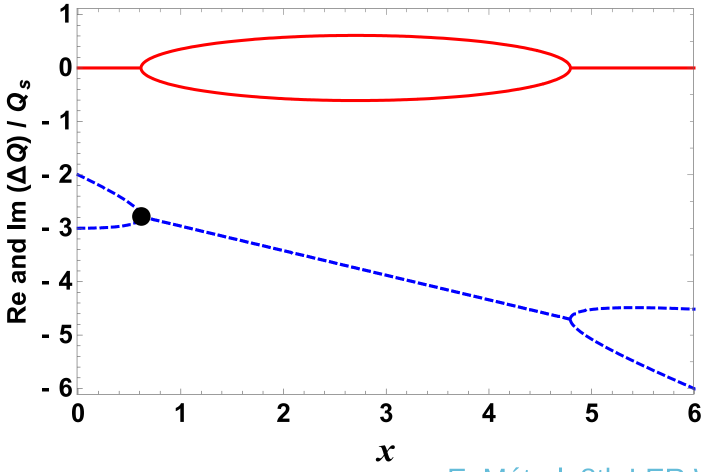
Simple analytical model ($x = 0.613$)

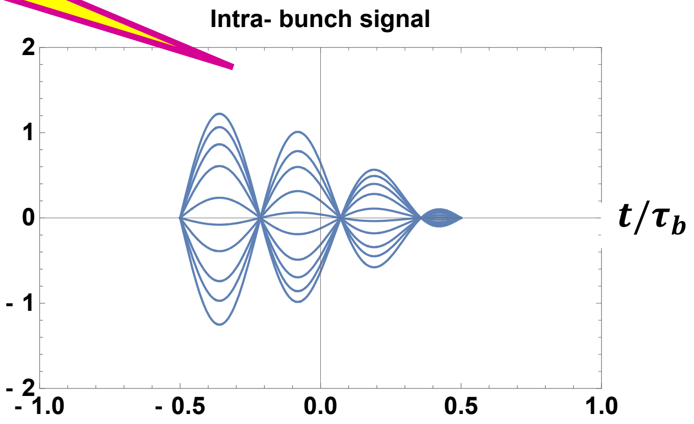
There are now 3 nodes inside the bunch which are given by the solutions of the equation on r.h.s

$$\mathrm{Cos}\left(\frac{3\,\pi\,t}{\tau_b}\right) - \mathrm{Sin}\left(\frac{4\,\pi\,t}{\tau_b}\right) = 0$$

$$\Rightarrow\ 1 - 4x - 4x^2 + 8x^3 = 0$$

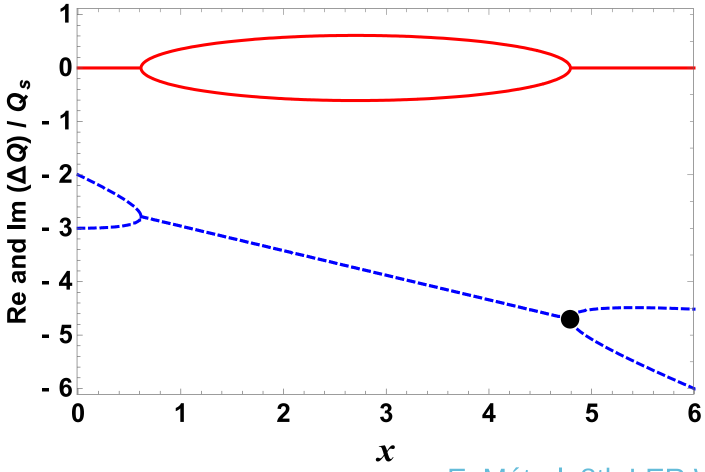
$$\text{with } x = \mathrm{Sin}\left(\frac{\pi\,t}{\tau_b}\right)$$

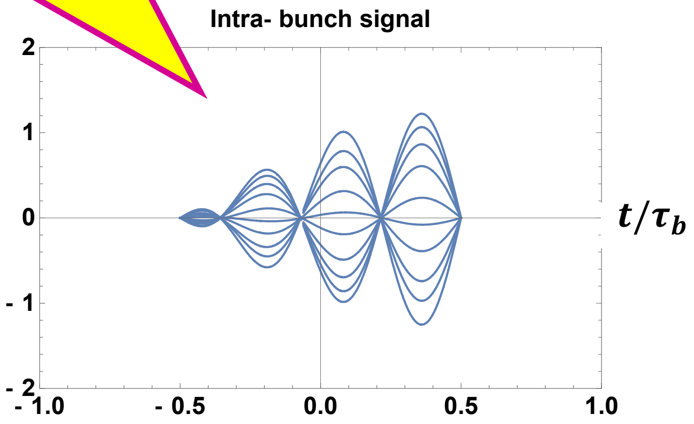




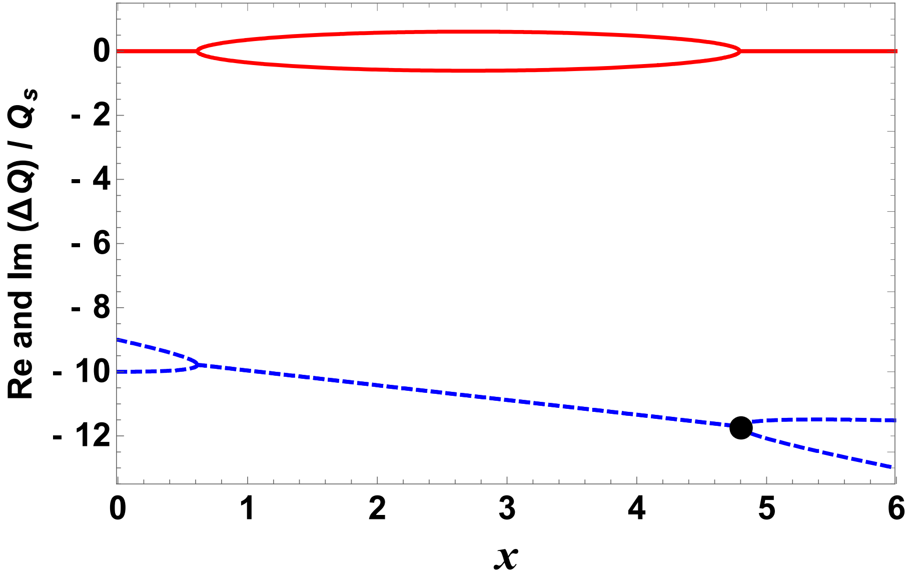
Simple analytical model ($x = 4.8$)

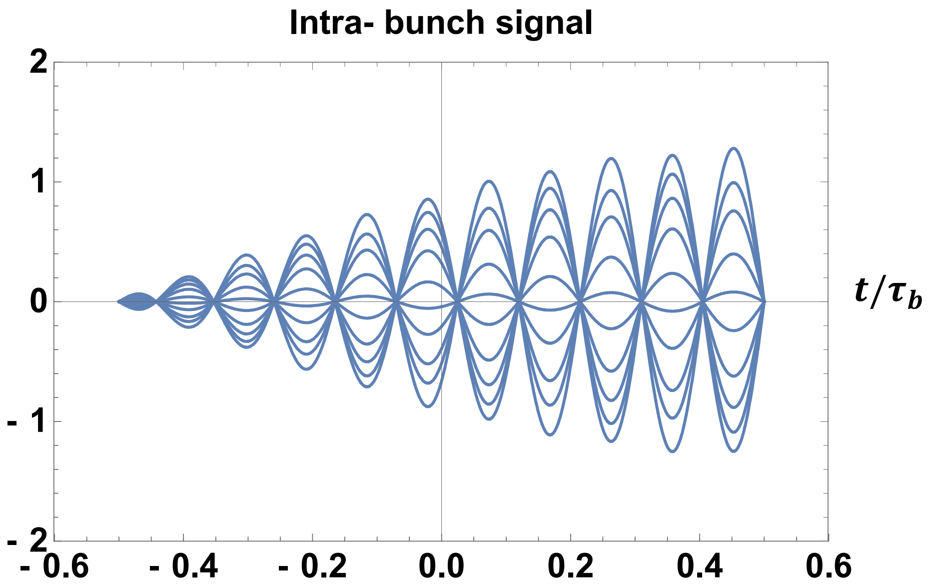
Huge amplification from Head to Tail with 0 growth rate! => Similar observations with the convective instabilities (in the presence of space charge) recently discussed by A. Burov





And similarly with e.g. modes -9 and -10





Conclusion and outlook

- The intra-bunch motion for TMCI (and its main features below-at-above TMCI threshold) can be explained with a simple analytical model, which helps to better understand what happens at each step

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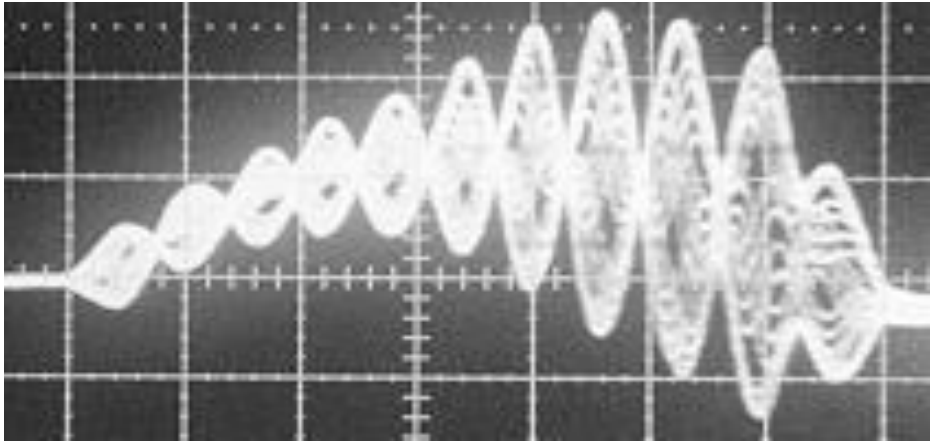
- Next
  - Perform a detailed comparison with the PyHEADTAIL tracking code and DELPHI Vlasov solver
  - Try and (better) explain all the observations of intra-bunch motion (from impedance only; with space charge; with e-cloud; etc.)

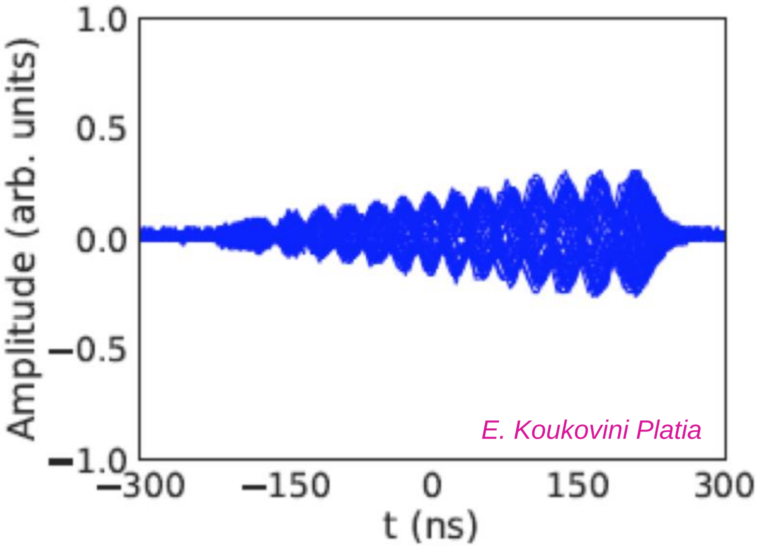
Can something like this explain some past measurements in CERN PS & PSB (in the presence of strong space charge)?

Past PS measurements



HEAD TAIL

Past PSB measurements



Can something like this explain some (parts of) simulations in the presence of e-cloud?