BMS flux balance equations

and applications for Gravitational waves

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Based on JHEP 10 (2020) 116 with *Geoffrey Compère* and *Roberto Oliveri*

Belgian GW meeting
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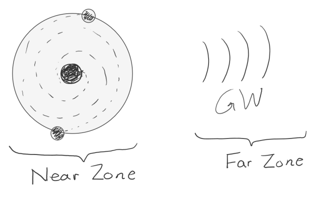
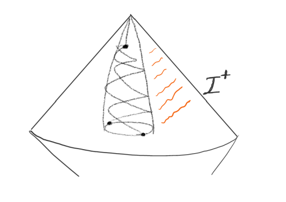
The context of this talk

Localized sources in flat spacetime.





We study the structure of gravitational field far from the source

Importance of the asymptotic structure of spacetime

Why this study is important?

- Nonlinear stability of flat spacetime
- Holographic principle
- Rigorous definition of gravitational charges
- Gravitational memory effects
- Radiation reaction effects

The Bondi-Sachs formalism

Asymptotically flat spacetime

Gravitational waves in general relativity
VII. Waves from axi-symmetric isolated systems

By H. Bondi, F.R.S., M. G. J. van der Burg and A. W. K. Metzner

(*Received* 8 *January* 1962—*Revised* 2 *April* 1962)

Gravitational waves in general relativity
VIII. Waves in asymptotically flat space-time

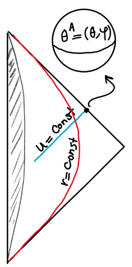
By R. K. Sachs*
King's College, University of London

(*Communicated by H. Bondi, F.R.S.—Received* 7 *May* 1962)

GR in Bondi gauge

Bondi gauge: Use coordinate system (u, r, θ^A) such that

- u is a lightlike coordinate
- r is the areal distance



To ensure these, we impose the gauge conditions

$$g_{rr} = g_{rA} = 0, \qquad \partial_r \det\left(r^{-2} g_{AB}\right) = 0$$

The metric then takes the form

$$ds^2 = -e^{2\beta}(U du^2 + 2dudr) + r^2 g_{AB}(d\theta^A - U^A du)(d\theta^B - U^B du)$$

- *Boundary conditions:* $\lim_{r\to\infty}(g_{\mu\nu} - \eta_{\mu\nu}) = 0$
- Einstein equations solve for (β, U, U^A, g_{AB})

Solving Einstein equations

Asymptotic expansion: $f = f_{(0)} + \frac{f_{(1)}}{r} + \frac{f_{(2)}}{r^2} + \cdots$

$$
\begin{aligned}
U &= \mathring{U} - 2m\, r^{-1} + \mathcal{O}(r^{-2}) \\
\beta &= \mathring{\beta}\, r^{-2} + \mathcal{O}(r^{-3}) \\
g_{AB} &= \gamma_{AB} + C_{AB}\, r^{-1} + D_{AB}\, r^{-2} + \mathcal{O}(r^{-3}) \\
U^A &= \mathring{U}^A\, r^{-2} - \frac{2}{3}\left[N^A - \frac{1}{2} C^{AB} D^C C_{BC}\right] r^{-3} + \mathcal{O}(r^{-4})
\end{aligned}
$$

Bondi shear $C_{AB}(u, \theta^A) = e^i_A e^j_B \left(\lim_{r\to\infty} r h^{TT}_{ij}\right)$

Mass aspect $m(u, \theta^A)$

Angular momentum aspect $N_A(u, \theta^A)$

Einstein equations

Free data: γ_{AB}, C_{AB}

Algebraic equations

$$\mathring{U} = -\frac{1}{2}R[\gamma], \quad \mathring{\beta} = -\frac{1}{32}C^{AB}C_{AB}, \quad \mathring{U}^A = -\frac{1}{2}D_B C^{AB}, \quad D_{AB} = -8\mathring{\beta}\,\gamma_{AB}$$

Balance equations

$$\partial_u m = -\frac{c^3}{8G}\dot{C}_{AB}\dot{C}^{AB} + \frac{c^4}{4G}D_A D_B \dot{C}^{AB},$$

$$\partial_u N_A = \partial_A m + \frac{c^3}{4G}\left(D_B(\dot{C}^{BC}C_{CA}) + 2D_B\dot{C}^{BC}C_{CA}\right) + \frac{c^4}{4G}D^B(D_A D^C C_{BC} - D_B D^C C_{AC}).$$

BMS charges

BMS charges

Generalized BMS charges

$$\boxed{\begin{aligned} \mathcal{P}_{\ell m}(u) &= \frac{1}{c}\oint_S Y_{\ell m}\, m\,, && \textit{Supermomentum} \\ \mathcal{J}_{\ell m}(u) &= \frac{1}{2}\oint_S \psi^A_{\ell m} N_A, && \textit{S-angular momentum} \\ \mathcal{K}_{\ell m}(u) &= \frac{1}{2c}\oint_S \phi^A_{\ell m} N_A && \textit{S-center of mass} \end{aligned}}$$

where the two Vector spherical harmonics on the sphere are

$$\phi^A_{\ell m} = \epsilon^{AB}\partial_B Y_{\ell m}, \qquad \psi^A_{\ell m} = \gamma^{AB}\partial_B Y_{\ell m}$$

These are indeed Noether charges associated to BMS symmetries.

BMS symmetries

BMS symmetries generalize Poincaré symmetries

	$\mathcal{P}_{\ell m}$	$\mathcal{J}_{\ell m}$	$\mathcal{K}_{\ell m}$
$\ell = 0$	Energy	-	-
$\ell = 1$	Linear momentum	Angular momentum	center of mass

BMS balance equations

BMS charges are NOT conserved due to gravitational waves. Instead they obey certain balance equations

BMS balance equations

Balance equations in two body problem [*Peters, Mathews '63*]

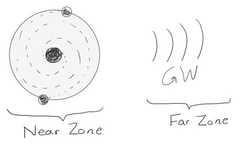
Newtonian gravity: Conservation laws enough to solve Kepler problem
Einstein GR: Balance equations replace conservation laws

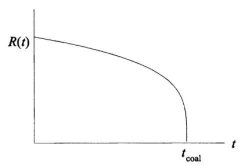
$$\frac{dE_{mech}}{dt} = -\mathcal{F}_{rad}$$

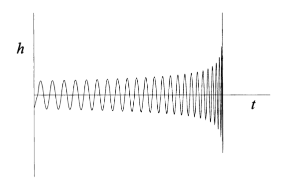
At lowest Post-Newtonian order

$$E_{mech} = -\frac{Gm_1 m_2}{2r}, \qquad \mathcal{F}_{rad} = \frac{G}{5c^5} \dddot{I}_{ij} \dddot{I}_{ij}$$

Solves the evolution of quasi-circular orbit [*Pictures: Maggiore*]







BMS flux balance equations

Flux balance equation for supermomentum (recall $\mathcal{P}_{\ell m} = \frac{1}{c} \oint_S Y_{\ell m}\, m$)

$$\boxed{\dot{\mathcal{P}}_{\ell m} = \frac{1}{c} \oint_S Y_{\ell m} \left(-\frac{c^3}{8G} \dot{C}_{AB} \dot{C}^{AB} + \frac{c^4}{4G} D_A D_B \dot{C}^{AB} \right)}$$

Similarly for superLorentz charges

Multipole expansion of radiation field $\left(n_i = \frac{x_i}{r}, N_L = n_{i_1} \cdots n_{i_\ell}\right)$

$$C_{ij} = \sum_{\ell=2}^{+\infty} \frac{4G}{c^{\ell+2} \ell!} \left(N_{L-2}\, \boldsymbol{U}_{ijL-2} - \frac{b_\ell}{c} N_{aL-2}\, \epsilon_{ab(i} \boldsymbol{V}_{j)bL-2} \right)^{TT}$$

in terms of mass and spin radiative multipole moments

Poincarè flux balance equations

Energy and angular momentum [*Blanchet, Faye '18*]

$$\dot{\mathcal{E}} = -\sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \mu_\ell \left\{ \dot{\boldsymbol{U}}_L \dot{\boldsymbol{U}}_L + \frac{b_\ell b_\ell}{c^2} \dot{\boldsymbol{V}}_L \dot{\boldsymbol{V}}_L \right\},$$

$$\dot{\mathcal{J}}_i = -\varepsilon_{ijk} \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \ell \mu_\ell \left\{ \boldsymbol{U}_{jL-1} \dot{\boldsymbol{U}}_{kL-1} + \frac{b_\ell b_\ell}{c^2} \boldsymbol{V}_{jL-1} \dot{\boldsymbol{V}}_{kL-1} \right\},$$

$$\dot{\mathcal{P}}_i = -\sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+3}} \left\{ 2(\ell+1)\mu_{\ell+1} \left(\dot{\boldsymbol{U}}_{iL} \dot{\boldsymbol{U}}_L + \frac{b_\ell b_{\ell+1}}{c^2} \dot{\boldsymbol{V}}_{iL} \dot{\boldsymbol{V}}_L \right) + \sigma_\ell \, \varepsilon_{ijk} \dot{\boldsymbol{U}}_{jL-1} \dot{\boldsymbol{V}}_{kL-1} \right\}$$

where $\mu_\ell = \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell\ell!(2\ell+1)!!}, \quad \sigma_\ell = \frac{8(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!}, \quad b_\ell = \frac{\ell}{\ell+1}$

Similar results for center of mass.

Exact equations

Leading terms

Radiative vs. source multipoles

$$\boldsymbol{U}_L = I_L^{(\ell)} + \mathcal{O}\left(\frac{G}{c^3}\right), \qquad \boldsymbol{V}_L = J_L^{(\ell)} + \mathcal{O}\left(\frac{G}{c^3}\right)$$

Leading terms

$$\dot{\mathcal{E}} = -\frac{G}{c^5}\left(\frac{1}{5}I_{ij}^{(3)}I_{ij}^{(3)}\right) - \frac{G}{c^7}\left(\frac{1}{189}I_{ijk}^{(4)}I_{ijk}^{(4)} + \frac{16}{45}J_{ij}^{(3)}J_{ij}^{(3)}\right) + O(c^{-9})$$

$$\dot{\mathcal{J}}_i = -\frac{G}{c^5}\left(\frac{2}{5}\epsilon_{ijk}I_{jl}^{(2)}I_{kl}^{(3)}\right) - \frac{G}{c^7}\epsilon_{ijk}\left(\frac{1}{63}I_{jlm}^{(3)}I_{klm}^{(4)} + \frac{32}{45}J_{jl}^{(2)}J_{kl}^{(3)}\right) + O(c^{-9})$$

$$\dot{\mathcal{P}}_i = -\frac{G}{c^7}\left(\frac{2}{63}I_{ijk}^{(4)}I_{jk}^{(3)} + \frac{16}{45}\epsilon_{ijk}I_{jl}^{(3)}J_{kl}^{(3)}\right) + O\left(c^{-9}\right).$$

$$\dot{\mathcal{G}}_i = \mathcal{P}_i - \frac{G}{c^7}\left[\frac{1}{21}\left(I_{jk}^{(3)}I_{ijk}^{(3)} - I_{jk}^{(2)}I_{ijk}^{(4)}\right)\right] + O\left(c^{-9}\right).$$

Matches exactly with standard results[*Einstein '18, Epstein, Wagoner '75, Thorne '80, Kozameh '16, Blanchet, Faye '18*]

PN analysis of BMS fluxes

Example: Octupolar super angular momentum

$$\dot{\mathcal{J}}_{ijk} - \frac{2}{7c^2}\, u\, \dot{\boldsymbol{V}}_{ijk} = -\frac{G}{c^5}\left(\frac{6}{35}\,\epsilon_{pq\langle i}\,\dot{\boldsymbol{U}}_{j|p|}\boldsymbol{U}_{k\rangle q}\right) + \mathcal{O}(c^{-7})$$

PN order of quadratic fluxes

Supermomentum	
ℓ-pole	PN order
0,2, 4	3
1, 3, 5	3.5

s-ang.momemtum	
ℓ-pole	PN order
1, 3	2.5
2, 4	3

s-center of mass	
ℓ-pole	PN order
1,3, 5	3.5
2, 4, 6	4

Conclusion

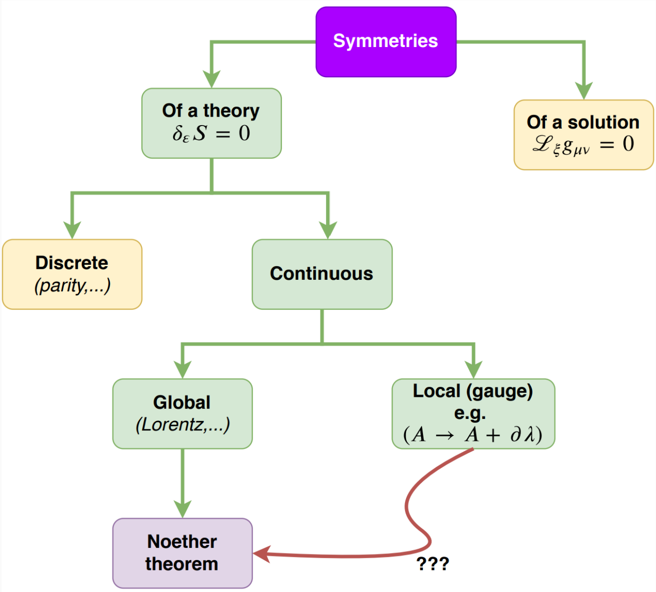
Outlook Use BMS balance equations to

- Constrain non-circular orbits
- Classify radiation reaction effects

Thank you very much

Back up

Noether's theorem



Adiabatic modes

Adiabatic modes *[Weinberg'05]*

Adiabatic modes are perturbations that in the long wavelength limit take the form of a gauge transformation

- Example: Maxwell theory in temporal gauge *[Mirbabayi,Simonović '16- AS,van den Bleeken '17]*

$$A_0 = 0 \implies A_i \to A_i + \partial_i \lambda(\boldsymbol{x})$$

- Gauss equation

$$\nabla \cdot E = \partial_i \dot{A}_i = 0 \qquad \text{trivially solved by} \qquad A_i = \partial_i \lambda(\boldsymbol{x})$$

- Introducing slow time dependence $A_i = \partial_i \lambda(\epsilon t, \boldsymbol{x})$ and requiring e.o.m= $\mathcal{O}(\epsilon^2)$ implies

$$A_i = \partial_i(\epsilon t\, \lambda(\boldsymbol{x})), \qquad \nabla^2 \lambda(\boldsymbol{x}) = 0$$

- Remark. Regular solutions cannot vanish at the boundary. Correspondence between adiabatic modes and asymptotic symmetries.